ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 2, pp. 247–262

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 2, pp. 247–262
Online ISSN: 1735-8515

EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS ON TIME SCALES

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(Communicated by Asadollah Aghajani)

Abstract. In this paper, we study the boundary-value problem of fractional order dynamic equations on time scales,

$$^{c}\Delta^{\alpha}u(t) = f(t, u(t)), \ \ t \in [0,1]_{\mathbb{T}^{\kappa^2}} := J, \ \ 1 < \alpha < 2,$$

$$u(0) + u^{\Delta}(0) = 0, \ \ u(1) + u^{\Delta}(1) = 0,$$

where $\mathbb{T}$ is a general time scale with $0, 1 \in \mathbb{T}$, $^{c}\Delta^{\alpha}$ is the Caputo Δ-fractional derivative. We investigate the existence and uniqueness of solution for the problem by Banach's fixed point theorem and Schaefer's fixed point theorem. We also discuss the existence of positive solutions of the problem by using the Krasnoselskii theorem.

Keywords: Fractional differential equation, time scales, boundary-value problem, fixed-point theorem.
MSC(2010): Primary 34A08; Secondary: 34N05, 34B05.

1. Introduction

Fractional differential equations have been of increasing importance in the past decades due to their diverse applications in science and engineering, such as the memory of a variety of materials, signal identification and image processing, optical systems, thermal system materials and mechanical systems, control system, etc., see [18, 20].

Many scholars paid much attention to it, and many interesting results on the existence of solutions of various classes of fractional differential equations have been obtained, see [15, 16, 21–24, 29–31, 33–37, 39], and the references therein. Recently, much attention has been focused on the study of the existence and multiplicity of solutions or positive solutions for boundary value problems of fractional differential equations with various boundary conditions by the use of

Article electronically published on April 30, 2016.
Received: 26 February 2014, Accepted: 16 December 2014.
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techniques of nonlinear analysis (fixed-point theorems, Leray–Schauder theory, the upper and lower solution method, *etc.*), see [2,9,12,15,16,21–23,27,33–38].

Fractional calculus, the study of integration and differentiation of fractional order, has recently been extended to include its discrete analogues of fractional difference calculus and fractional quantum calculus. Due to the similarities of the three theories there has been research on whether there exist a single theory that encapsulates them [7,13,14,26,28].

A time scale is any nonempty closed subset of real numbers $\mathbb{R}$. It is introduced to unify and extend the theory of differential equation, difference equations and other differential systems defined. The theory of time scale calculus is a fairly new area of research. Hilger proved the existence and uniqueness of initial value problems including differential equations on time scales in [17]. Some applications of this kind of problems can be found in [1,3,5].

A time scale of the form of a union of disjoint closed real intervals constitutes a good background for the study of population (of plants, insects, *etc.*) models. Such models appear, for example, when a plant population exhibits exponential growth during the months of Spring and Summer, and at the beginning of Autumn all plants die while the seeds remain in the ground. A book on the subject of time scale, by Bohner and Peterson [10] summarizes and organizes much of the time scale calculus, we refer also the last book by Bohner and Peterson [11] for advances in dynamic equations on time scales. For the notions used below we refer to the next section that provides some basic facts on time scales extracted from Bohner and Peterson [10].

In recent years, several attempts have been done to join the two subjects, developing a fractional calculus on time scales by G. A. Anastassiou, N. Bastos, D. Mozyrska, D. Torres, A. Ahmadkhanlu, M. Jahanshahi and P. A Williams [4, 6, 8, 25, 28, 32]. It was expected to establish a general definition of fractional derivative on an arbitrary time scale and unify the theories of fractional differential equations and discrete fractional equations finally.

There have been extensive study and application of fractional differential equation, but limited work has been done in the study of fractional differential equations on time scales. To the best of our knowledge, no any results devoted to the study of the boundary value problems for fractional differential equations on a general time scale. To fill this gap, we initiate to do this work.

Zhang [33] studied positive solutions for boundary value problems of nonlinear fractional differential equations

$$D_{0+}^{\alpha}u(t) = f(t, u(t)),\ 0 < t < 1,$$

$$u(0) + u'(0) = 0,\ \ u(1) + u'(1) = 0,$$

where $1 < \alpha \leq 2$ is a real number, and D_{0+}^{α} is the Caputo's fractional derivative, and $f : [0, 1] \times [0, +\infty] \to [0, +\infty]$ is continuous.

Ahmadkhanlu [6] *et al*. studied the initial value problem for the fractional differential equations on time scales

$$^{c}\Delta^{\alpha}y(t) = -f(t,y(t)),\ \ t \in [t_0, t_0+a] = J \subseteq \mathbb{T},\ 0<\alpha\leq 1,$$

$$y(t_0) = y_0,$$

where $^{c}\Delta^{\alpha}$ is Caputo fractional derivative operator and the function $f : J\times\mathbb{T} \to \mathbb{R}$ is a right-dense continuous function.

Motivated by all the works above, in this paper we consider the existence and uniqueness of solutions for the following boundary-value problem

$$(1.1)\qquad {}^{c}\Delta^{\alpha}u(t) = f(t,u(t)),\ \ t \in [0,1]_{\mathbb{T}^{\kappa^2}} := J,\ \ 1<\alpha<2,$$

$$(1.2)\qquad u(0)+u^{\Delta}(0) = 0,\ \ u(1)+u^{\Delta}(1) = 0,$$

where $\mathbb{T}$ is a general time scale with $0, 1 \in \mathbb{T}$, $^{c}\Delta^{\alpha}$ is the Caputo Δ-fractional derivative, $f : J \times [0,+\infty) \to [0,+\infty)$ is a right-dense continuous function. We present sufficient conditions for the existence and uniqueness of the problem (1.1)–(1.2) by some fixed point theorems. We claim that the results of this paper are a basic and important contribution to the theory of fractional differential equations on general time scales.

The rest of this paper is organized as follows. In Section 2, we shall introduce some definitions and lemmas to prove our main results. In Section 3, we discuss the existence and uniqueness of the problem. In Section 4, we get the existence of positive solutions of the problem.

2. **Preliminaries**

In this section, we introduce notations and definitions of fractional calculus on time scales, and we prove two lemmas before stating our main results.

A time scale is an arbitrary nonempty closed subset of $\mathbb{R}$ and is denoted by $\mathbb{T}$. Here we give some examples of sets that are time scales. The real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the non-negative integers $\mathbb{N}_0$, the h-numbers ($h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$ is a fixed real number), and the q-numbers ($kq = q^{\mathbb{Z}} \cup \{0\} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, where $q > 1$ is a fixed real number).

Definition 2.1. ([10]) The mapping $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ with $\inf\emptyset = \sup\mathbb{T}$ (i.e.,$\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) is called the forward jump operator. Accordingly, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ with $\sup\emptyset = \inf\mathbb{T}$ (i.e.,$\rho(m) = m$ if $\mathbb{T}$ has a minimum m). The symbol $\emptyset$ denotes the empty set.

Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. A point $t \in \mathbb{T}$ is called right-dense, right-scattered, left-dense or left-scattered if $\sigma(t) = t$, $\sigma(t) >$

t, $\rho(t) = t$, $\rho(t) < t$, respectively. Points that are right-scattered and left-scattered at the same time are called isolated. And points that are right-dense and left-dense at the same time are called dense.

Definition 2.2. (Delta Derivative) ([10]) Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}$. Then the delta derivative (or Δ-derivative) of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U of t in $\mathbb{T}$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \ \ for\ all\ s \in U.$$

Definition 2.3. (Delta Integral) ([10]) Let $[a, b]$ be a closed bounded interval in $\mathbb{T}$. A function $F : [a, b] \to \mathbb{R}$ is called a delta antiderivative of a function $f : [a, b) \to \mathbb{R}$ provided that F is continuous on $[a, b]$ and delta differentiable on $[a, b)$, and $F^{\Delta}(t) = f(t)$ for all $t \in [a, b)$. Then we define the Δ-integral from a to b of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

Definition 2.4. ([10]) A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (or rd-continuous) provided that it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limits exist(finite) at left-dense points of $\mathbb{T}$. The set of all right-dense continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$. Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous provided that it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points of $\mathbb{T}$. The set of all left-dense continuous functions on $\mathbb{T}$ is denoted by $C_{ld}(\mathbb{T})$.

Definition 2.5. [6] Suppose $\mathbb{T}$ is a time scale, $[a, b] \subseteq \mathbb{T}$ and the function $h(x)$ is an integrable function on $[a, b]$, then $\Delta-$fractional integral of h is defined by the following relation

$$^{\Delta}I^{\alpha}_{a+}h(t) = \int_a^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s,$$

where $\Gamma(\alpha)$ is the Euler Gamma function.

Definition 2.6. [6] Let $h : \mathbb{T} \to \mathbb{R}$. The Caputo $\Delta-$fractional derivative of h is defined by

$$^{c}\Delta^{\alpha}_{a+}h(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} h^{\Delta^n}(s)\Delta s, \tag{2.1}$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer of α.

Lemma 2.7. *Let $1 < \alpha < 2$, $t \in J$ and $f : J \times [0, +\infty) \to [0, +\infty)$ be a right-dense continuous function. Then the function $u(t)$ is a solution of problem (1.1)-(1.2) if and only if this function is a solution of the following integral* *equation:*

$$u(t) = \int_0^1 G(t,s)f(s,u(s))\Delta s + \tfrac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}f(1,u(1)) - \tfrac{(2-t)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}f(0,u(0)),$$

where

(2.2)

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}+(1-t)(1-s)^{\alpha-1}+(1-t)\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)}, & s \le t, \\ \frac{(1-t)(1-s)^{\alpha-1}+(1-t)\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)}, & t \le s. \end{cases}$$

Proof. For $u(t)$ from (2.1) we have

$${}^c\Delta^\alpha u(t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}u^{\Delta^2}(s)\Delta s = {}^\Delta I^{2-\alpha}u^{\Delta^2}(t).$$

Then it is easy to see that

$${}^\Delta I^\alpha\, {}^c\Delta^\alpha u(t) = {}^\Delta I^2 u^{\Delta^2}(t) = u(t) + c_1 + c_2 t,$$

and

$$u(t) = {}^\Delta I^\alpha f(t,u(t)) + c_1 + c_2 t,$$

for some $c_1,\ c_2 \in \mathbb{R}$. Considering

$$\begin{aligned} u^\Delta(t) &= \tfrac{1}{\Gamma(\alpha)}(\int_0^t (t-s)^{\alpha-1}f(s,u(s))\Delta s)^\Delta + c_1^\Delta + (c_2 t)^\Delta \\ &= \tfrac{1}{\Gamma(\alpha)}\int_0^t f(s,u(s))(\int_0^1(\alpha-1)(t-s+h\mu(t))^{\alpha-2}dh)\Delta s \\ &\quad + \tfrac{1}{\Gamma(\alpha)}(\sigma(t)-t)^{\alpha-1}f(t,u(t)) + c_2, \end{aligned}$$

and condition (1.2), we get

$$\begin{aligned} u(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,u(s))\Delta s + \tfrac{1-t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,u(s))\Delta s \\ &\quad + \tfrac{1-t}{\Gamma(\alpha)}\int_0^1 f(s,u(s))(\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh)\Delta s \\ &\quad + \tfrac{1-t}{\Gamma(\alpha)}(\sigma(1)-1)^{\alpha-1}f(1,u(1)) - \tfrac{2-t}{\Gamma(\alpha)}(\sigma(0))^{\alpha-1}f(0,u(0)) \\ &= \int_0^1 G(t,s)f(s,u(s))\Delta s + \tfrac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}f(1,u(1)) \\ &\quad - \tfrac{(2-t)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}f(0,u(0)). \end{aligned}$$

The proof is completed.□

Lemma 2.8. *Let $1 < \alpha < 2$. Then function $G(t,s)$ defined by (2.2) has the following properties*
(R1) $G(t,s) \geq 0$ for $t,s \in [0,1]\cap\mathbb{T}$.
(R2) There exists a positive function γ such that

$$\min_{\tau\leq t\leq\eta} G(t,s) \geq \gamma(s)M(s), \quad s\in[0,1]\cap\mathbb{T}, \quad \tau,\eta\in\mathbb{T},$$

$$\max_{t\in J} G(t,s) \leq M(s),$$

where

$$M(s) = \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)}, \quad s\in[0,1]\cap\mathbb{T}.$$

Proof. From the expression of $\mu(1)$ and $G(t,s)$, it is obvious that $\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh > 0$, and $G(t,s)\geq 0$ *for* $t,s\in J$. Next, we will prove (R2). Let

$$g_1(t) = \frac{(t-s)^{\alpha-1} + (1-t)((1-s)^{\alpha-1} + \int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh)}{\Gamma(\alpha)},$$

when $s\leq t$, and

$$g_2(t) = \frac{(1-t)(1-s)^{\alpha-1} + (1-t)\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)}, \ t\leq s,$$

That is, $g_2(t)$ is decreasing with respect to t. Hence, we have

$$\min_{\tau\leq t\leq\eta} g_1(t) \geq \frac{(1-\eta)((1-s)^{\alpha-1} + \int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh)}{\Gamma(\alpha)} = m(s),$$

$$\max_{t\in J} g_1(t) \leq \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)} = M(s).$$

$$\min_{\tau\leq t\leq\eta} g_2(t) = \frac{(1-\eta)(1-s)^{\alpha-1} + (1-\eta)\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)},$$

$$\max_{t\in J} g_2(t) = g_2(0) \leq \frac{2(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{\Gamma(\alpha)}.$$

Thus, we have

$$\min_{\tau\leq t\leq\eta} G(t,s) \geq \gamma(s)M(s), \quad s\in[0,1]\cap\mathbb{T}, \quad \tau,\eta\in\mathbb{T},$$

$$\max_{t\in J} G(t,s) \leq M(s),$$

where

$$\begin{aligned}\gamma(s) &= \frac{m(s)}{M(s)} = \frac{(1-\eta)(1-s)^{\alpha-1} + (1-\eta)\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh}{2(1-s)^{\alpha-1} + \int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh} \\ &\geq \frac{1-\eta}{2}, \quad s\in[0,1]\cap\mathbb{T}.\end{aligned}$$

The proof is completed.□

Lemma 2.9. *[19] Let E be a Banach space and K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \bigcap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*
(1) $\|Tu\| \leq \|u\|$ for $u \in K \bigcap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \bigcap \partial\Omega_2$, or
(2) $\|Tu\| \geq \|u\|$ for $u \in K \bigcap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in K \bigcap \partial\Omega_2$.
Then T has a fixed point in $K \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence and uniqueness

Now we are in the position to establish the main results. We begin with the existence and uniqueness of solution for the problem (1.1)-(1.2).

Theorem 3.1. *Assume that*
(H1) $f(t,u)$ is a right-dense continuous bounded function such that there exists $K > 0$, $|f(t,u)| < K$ on $J \times [0, +\infty)$.
(H2) There exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \leq L|x-y|$, for each $t \in J$, and $x, y \in [0, +\infty)$.
(H3) $L(\int_0^1 M(s)\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}) < 1$.
Then the boundary value problem (1.1)-(1.2) has a unique solution.

Proof. Let S be the set of rd-continuous functions. For $u \in S$ define

$$\|u\| = \sup_{t \in J} |u(t)|.$$

It is easy to see that S is a Banach space with this norm. The subset of $S(\varrho)$ and the operator T are defined by

$$S(\varrho) = \{u \in S : \|u\| \leq \varrho\}$$

and

$$\begin{aligned} Tu(t) &= \textstyle\int_0^1 G(t,s) f(s,u(s))\Delta s + \frac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)) \\ &\quad - \tfrac{(2-t)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)} f(0,u(0)). \end{aligned}$$

According to $(H1)$ and Lemma 2.2 we have

$$\begin{aligned} |Tu(t)| &\leq \textstyle\int_0^1 |G(t,s)||f(s,u(s))|\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} |f(1,u(1))| \\ &\quad + \tfrac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)} |f(0,u(0))| \\ &\leq K(\textstyle\int_0^1 M(s)\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}). \end{aligned}$$

Since $\int_0^1 M(s)\Delta s$ is a constant, let

$$\varrho = K(\int_0^1 M(s)\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}).$$

We conclude that T is an operator from $S(\varrho)$ to $S(\varrho)$. Moreover, considering $(H2)$, for $x, y \in S(\varrho)$, any $t \in J$, we have

$$|Tx(t) - Ty(t)|$$

$$\leq \int_0^1 |G(t,s)||f(s,x) - f(s,y)|\Delta s + \frac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}|f(1,x(1)) - f(1,y(1))|$$

$$+\frac{(2-t)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}|f(0,x(0)) - f(0,y(0))|$$

$$\leq L\int_0^1 M(s)\|x-y\|\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}L\|x-y\| + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}L\|x-y\|$$

$$\leq L\|x-y\|(\int_0^1 M(s)\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}).$$

Thus

$$\|Tx - Ty\| \leq L(\int_0^1 M(s)\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)})\|x-y\|.$$

Consequently T is a contraction by condition $(H3)$. As a consequence of Banach fixed-point theorem, we deduce that T has a fixed point which is a unique solution of the problem (1.1)-(1.2). The proof is completed.□

Next, we present a sufficient condition for the existence of solutions to the problem (1.1)-(1.2).

Theorem 3.2. *Suppose $f(t,u) : J \times [0,+\infty) \to [0,+\infty)$ is rd-continuous with respect to t and continuous with respect to u. And there exists $K > 0$ such that $|f(t,u)| \leq K$ on $J \times [0,+\infty)$. Then the BVP(1.1)-(1.2) has a solution on J.*

Proof. We shall use Schaefer fixed-point theorem to prove this result. We divide the proof into four steps.

First, we prove that T is continuous. Let u_n be a sequence such that $u_n \to u$ in $C(J,[0,+\infty))$. Then, for each $t \in J$,

$$|Tu_n(t) - Tu(t)|$$

$$\leq \left|\int_0^1 G(t,s)[f(s,u_n(s)) - f(s,u(s))]\Delta s\right|$$

$$+\frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}|f(1,u_n(1)) - f(1,u(1))| + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}|f(0,u_n(0)) - f(0,u(0))|$$

$$\leq \left|\int_0^1 M(s)[f(s,u_n(s)) - f(s,u(s))]\Delta s\right|$$

$$+\frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}|f(1,u_n(1)) - f(1,u(1))| + \frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}|f(0,u_n(0)) - f(0,u(0))|.$$

Since $f(t,u)$ is continuous function with respect to u, we conclude from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_0^1 M(s)[f(s,u_n(s))-f(s,u(s))]\Delta s=0.$$

Again, it's easy to see that

$$\lim_{n\to\infty}[f(1,u_n(1))-f(1,u(1))]=\lim_{n\to\infty}[f(0,u_n(0))-f(0,u(0))]=0.$$

Thus, we deduce

$$\|Tu_n-Tu\|\to 0 \ \ as\ n\to\infty,$$

which means that T is continuous.

In what follows we verify T is bounded. For each $t\in J$,

$$|Tu(t)|\le K\Big(\int_0^1 M(s)\Delta s+\frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}+\frac{2(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}\Big). \tag{3.1}$$

Thus T is bounded.

Next we show that T maps uniformly bounded sets into equicontinuous ones of $C(J,[0,+\infty))$. Letting $t_1,t_2\in J,\ t_1<t_2$, we have

$$\begin{aligned}&|Tu(t_2)-Tu(t_1)|\\&=|\textstyle\int_0^1 G(t_2,s)f(s,u(s))\Delta s-\int_0^1 G(t_1,s)f(s,u(s))\Delta s\\&+\tfrac{(t_1-t_2)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}f(1,u(1))+\tfrac{(t_2-t_1)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}f(0,u(0))|\\&\le|\textstyle\int_0^{t_2}\tfrac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\Delta s-\int_0^{t_1}\tfrac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s))\Delta s|\\&+\textstyle\int_0^1\tfrac{(t_2-t_1)((1-s)^{\alpha-1}+\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh)}{\Gamma(\alpha)}|f(s,u(s))|\Delta s\\&+\tfrac{(t_2-t_1)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}|f(1,u(1))|+\tfrac{(t_2-t_1)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)}|f(0,u(0))|,\end{aligned}$$

that is

$$\begin{aligned}&|Tu(t_2)-Tu(t_1)|\\&\le\tfrac{K}{\Gamma(\alpha)}\big(\textstyle\int_0^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\Delta s+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Delta s\big)\\&+\tfrac{K(t_2-t_1)}{\Gamma(\alpha)}\textstyle\int_0^1((1-s)^{\alpha-1}+\int_0^1(\alpha-1)(1-s+h\mu(1))^{\alpha-2}dh)\Delta s\\&+\tfrac{K}{\Gamma(\alpha)}(t_2-t_1)(\mu(1))^{\alpha-1}+\tfrac{K}{\Gamma(\alpha)}(t_2-t_1)(\sigma(0))^{\alpha-1}.\end{aligned}$$

Since $(t-s)^{\alpha-1}$ is continuous, it is easy to see the right-hand side of the above inequality tends to zero as $t_1\to t_2$. As a consequence of the first three steps above, together with the Arzela-Ascoli theorem, we get that T is completely continuous.

Now it remains to show that the set $\Omega = \{u \in C(J, [0, +\infty)) : u = \lambda T(u),\ 0 < \lambda < 1\}$ is a bounded set. Let $u \in \Omega$. Then $u = \lambda T(u),\ 0 < \lambda < 1$. Thus, for each $t \in J$,

$$\begin{aligned} u = \lambda Tu(t) &= \lambda(\textstyle\int_0^1 G(t,s)f(s,u(s))\Delta s + \frac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)) \\ &\quad - \frac{(2-t)(\sigma(0))^{\alpha-1}}{\Gamma(\alpha)} f(0,u(0))). \end{aligned}$$

This implies by (3.1) that Ω is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that T has a fixed point which is a solution of the problem (1.1)-(1.2). The proof is completed.□

4. **Existence of positive solutions**

In this section, we will use Krasnoselskii's fixed point theorem to investigate the existence of positive solutions to the problem (1.1)-(1.2).

Let $\sigma(0) = 0$. Then

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\Delta s + \frac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)).$$

Define cone $E \subset S$

$$E = \{u \in S \mid u(t) \geq 0, \quad \min_{\tau \leq t \leq \eta,\ t \in J} u(t) \geq \frac{1-\eta}{2}\|u\|\}.$$

Lemma 4.1. *Assume that $f(t,u) : J \times [0, +\infty) \to [0, +\infty)$ is rd-continuous with respect to t and continuous with respect to u. Assume also that f is bounded on $J \times [0, \infty)$. If $\sigma(0) = 0$. Then $T : E \to E$.*

Proof. According to Lemma 2.2, we can get $Tu(t) \geq 0$, and

$$\begin{aligned} &\min_{\tau \leq t \leq \eta,\ t \in J} Tu(t) \\ &= \min_{\tau \leq t \leq \eta,\ t \in J} \textstyle\int_0^1 G(t,s)f(s,u(s))\Delta s + \min_{\tau \leq t \leq \eta,\ t \in J} \frac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)) \\ &\geq \frac{1-\eta}{2} \textstyle\int_0^1 M(s)f(s,u(s))\Delta s + \frac{(1-\eta)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)). \end{aligned}$$

On the other hand

$$\begin{aligned} \|Tu\| &= \sup_{t \in J} \textstyle\int_0^1 G(t,s)f(s,u(s))\Delta s + \sup_{t \in J} \frac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)) \\ &\leq \textstyle\int_0^1 M(s)f(s,u(s))\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)). \end{aligned}$$

Thus, we obtain

$$\min_{\tau\le t\le\eta,\ t\in J} Tu(t) \ge \frac{1-\eta}{2}\|Tu\|,$$

which implies $T: E\to E$. The proof is completed.□

Denote

$$A=\left(\int_0^1 M(s)\Delta s+\frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}\right)^{-1}, B=\left(\int_\tau^\eta M(s)\Delta s\right)^{-1}.$$

Theorem 4.2. *Assume that* $f(t,u): J\times[0,+\infty)\to[0,+\infty)$ *is rd-continuous with respect to* t *and continuous with respect to* u. *Assume also that* f *is bounded on* $J\times[0,\infty)$ *and satisfies one of following conditions:*
(H4) There exist $0<a_1,b_1\le 1$ *such that*

$$\lim_{u\to\infty}\frac{f(t,u)}{u^{a_1}}=0,\ \ \lim_{u\to 0^+}\frac{f(t,u)}{u^{b_1}}=\infty,\ \textit{for all } t\in J.$$

(H5)There exist $a_2,b_2\ge 1$ *such that*

$$\lim_{u\to\infty}\frac{f(t,u)}{u^{a_2}}=\infty,\ \ \lim_{u\to 0^+}\frac{f(t,u)}{u^{b_2}}=0,\ \textit{for all } t\in J.$$

If $\sigma(0)=0$, *then the BVP(1.1)-(1.2) has one positive solution.*

Proof. According to Lemma 4.1, we have $T: E\to E$ is a completely continuous operator. Assume that (H4) holds, then for given ϵ and η satisfying $0<\epsilon<\frac{A}{2}$ and $\xi>(\frac{2}{1-\eta})^{b_1+1}B>0$, there exist $N_1>0,\ N_2>0$ such that

$$f(t,u)\le\epsilon u^{a_1}\quad for\ \ t\in J,\ u\ge N_1,$$
$$f(t,u)>\xi u^{b_1}\quad for\ \ t\in J,\ 0<u\le N_2.$$

So we have

$$f(t,u)\le\epsilon u^{a_1}+c\quad for\ \ t\in J,\ u\in[0,+\infty),$$

where

$$c=\max_{t\in J,\ 0\le u\le N_1}|f(t,u)|+1.$$

Let

$$\Omega_1=\{u\in E:\|u\|\le R_1\},$$

where $R_1>\{1,\ 2cA^{-1}\}$. For $u\in\partial\Omega_1$, from Lemma 2.2 we have

$$\begin{aligned}|Tu(t)| &= \int_0^1 G(t,s)f(s,u(s))\Delta s+\tfrac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}f(1,u(1))\\ &\le \int_0^1 M(s)(\epsilon||u||^{a_1}+c)\Delta s+\tfrac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}(\epsilon||u||^{a_1}+c)\\ &\le \epsilon R_1^{a_1}(\int_0^1 M(s)\Delta s+\tfrac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)})+c(\int_0^1 M(s)\Delta s+\tfrac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)})\\ &\le \tfrac{R_1}{2}+\tfrac{R_1}{2}=R_1,\end{aligned}$$

$\|Tu\| \leq R_1 = \|u\|$. Let

$$\Omega_2 = \{u \in E : \|u\| \leq R_2\},$$

where $0 < R_2 < \{1,\ N_2\}$, for $u \in \partial\Omega_2$, we obtain

$$\begin{aligned}
|Tu(t)| &= \int_0^1 G(t,s)f(s,u(s))\Delta s + \tfrac{(1-t)(\mu(1))^{\alpha-1}}{\Gamma(\alpha)} f(1,u(1)) \\
&\geq \int_\tau^\eta G(t,s)f(s,u(s))\Delta s \\
&\geq \tfrac{1-\eta}{2} \int_\tau^\eta M(s)f(s,u(s))\Delta s \\
&\geq \tfrac{1-\eta}{2}\xi \int_\tau^\eta M(s)u^{b_1}\Delta s \\
&\geq (\tfrac{1-\eta}{2})^{b_1+1}\xi R_2^{b_1} \int_\tau^\eta M(s)\Delta s \\
&\geq R_2^{b_1} = R_2 \cdot R_2^{b_1-1} > R_2 = \|u\|,
\end{aligned}$$

so $\|Tu\| \geq R_2 = \|u\|$. Hence the operator T has one fixed point $u^*(t) \in \overline{\Omega_1}\backslash\Omega_2$, and consequently $u^*(t)$ is one positive solution of the BVP(1.1)-(1.2).

For condition (H5), we can obtain the result in a similar way. Here we give a brief description. Assume that (H5) holds, then for given $0 < \epsilon < A$ and $\xi > (\frac{2}{1-\eta})^{a_2+1}B > 0$, there exist M_1 and M_2, such that

$$f(t,u) > \xi u^{a_2} \quad for\ \ t \in J,\ u \geq M_1,$$

$$f(t,u) < \epsilon u^{b_2} \quad for\ \ t \in J,\ 0 \leq u \leq M_2.$$

Let

$$\Omega_1 = \{u \in E : \|u\| < R_1\}, \quad \Omega_2 = \{u \in E : \|u\| < R_2\},$$

where $R_1 > \{1, \frac{2}{1-\eta}M_1\}$, $0 < R_2 < \{1, M_2\}$. Then for $u \in \partial\Omega_1$, for $\tau \leq t \leq \eta$, one has $u(t) \geq \min\limits_{\tau\leq t\leq\eta} u(t) \geq \frac{1-\eta}{2}\|u\| = \frac{1-\eta}{2}R_1 > M_1$. Thus, from Lemma 2.2, we have

$$\begin{aligned}
|Tu(t)| &\geq \int_\tau^\eta G(t,s)f(s,u(s))\Delta s \\
&\geq \tfrac{1-\eta}{2}\xi \int_\tau^\eta M(s)(u(s))^{a_2}\Delta s \\
&\geq (\tfrac{1-\eta}{2})^{a_2+1}\xi R_1^{a_2} \int_\tau^\eta M(s)\Delta s \\
&\geq R_1^{a_2} \geq R_1 = \|u\|.
\end{aligned}$$

For $u \in \partial\Omega_2$, we obtain

$$\begin{aligned} |Tu(t)| &\leq \int_0^1 M(s)\epsilon\|u\|^{b_2}\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}\epsilon\|u\|^{b_2} \\ &= \epsilon R_2^{b_2}\big(\int_0^1 M(s)\Delta s + \frac{(\mu(1))^{\alpha-1}}{\Gamma(\alpha)}\big) \\ &\leq R_2^{b_2} \leq R_2 = \|u\|. \end{aligned}$$

Hence, the operator T has one fixed point $u^*(t) \in \overline{\Omega_1}\backslash\Omega_2$. Then $u^*(t)$ is one positive solution of the BVP(1.1)-(1.2).□

Remark 4.3. If $\mathbb{T} = \mathbb{R}$, then $\sigma(0) = 0$. Hence Theorem 4.1 is true.

5. **Examples**

In this section, we will give some examples to illustrate our main results.

Example 5.1. Consider $\mathbb{T} = \mathbb{R}$,

$$D_{0+}^{1.5}u(t) = \frac{t}{10}\frac{u}{1+u}, \tag{5.1}$$

$$u(0) + u'(0) = 0, \;\; u(1) + u'(1) = 0, \tag{5.2}$$

where $f(t,u) = \frac{t}{10}\frac{u}{1+u}$, $\alpha = 1.5$, $t \in [0,1]$. Then $f(t,u)$ is a continuous function, and $|f(t,u)| < \frac{1}{10} = K$.

Let $t \in [0.1] = J$ and $x, y \in \mathbb{T}$. Then

$$|f(t,x) - f(t,y)| \leq \frac{1}{10}\big|x - y\big|.$$

Moreover, it is easy to verify that

$$L\int_0^1 M(s)\Delta s < \frac{1}{10}\cdot\int_0^1 \Big(\frac{2(1-s)^{0.5}}{\Gamma(1.5)} + \frac{0.5(1-s)^{-0.5}}{\Gamma(1.5)}\Big)ds < 0.34 < 1.$$

Therefore, all the conditions of Theorem 3.1 are satisfied and consequently the BVP(5.1)–(5.2) has a unique solution.

Example 5.2. Consider

$$D_{0+}^{1.5}u(t) = t\sin u, \tag{5.3}$$

$$u(0) + u^{\Delta}(0) = 0, \;\; u(1) + u^{\Delta}(1) = 0, \tag{5.4}$$

where $f(t,u) = t\sin u$, $\alpha = 1.5$, $t \in [0,1]_{\mathbb{T}^{\kappa^2}} = J$. Then $f(t,u)$is rd-continuous with respect to t and continuous with respect to u. Clearly, $|f(t,u)| < 1$. Hence, by Theorem 3.2, the BVP(5.3)–(5.4) has at least one solution.

Example 5.3. Consider $\mathbb{T} = \mathbb{R}$

$$D_{0+}^{1.5}u(t) = t + \sin u, \tag{5.5}$$

$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0, \tag{5.6}$$

where $f(t,u) = t + \sin u,\ \alpha = 1.5,\ t \in [0,1] = J$. Then $f(t,u)$is rd-continuous with respect to t and continuous with respect to u. Clearly, $|f(t,u)| < 2$.

Moreover,

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = 0, \quad \lim_{u\to 0} \frac{f(t,u)}{u} = \infty.$$

Hence, by Theorem 4.2, the BVP(5.5)–(5.6) has at least one positive solution.

Conclusion

This paper studies the existence and uniqueness of solution of boundary value problem for fractional differential equations on time scales. We also discuss the existence of positive solutions. We provide some sufficient conditions to get our results. When $\mathbb{T} = \mathbb{R}$, we can find the corresponding results in [33]. At the foundation of this paper, one can make further research on fractional differential equations on time scales.

Acknowledgments

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript. This research is supported by the Natural Science Foundation of China (61374074), Natural Science Outstanding Youth Foundation of Shandong Province (JQ201119) and supported by Shandong Provincial Natural Science Foundation (ZR2012AM009, ZR2013AL003).

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