ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 2, pp. 285–295

Title:

Sufficiency and duality for a nonsmooth vector optimization problem with generalized α-d_I-type-I univexity over cones

Author(s):

H. Jiao

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 2, pp. 285–295
Online ISSN: 1735-8515

SUFFICIENCY AND DUALITY FOR A NONSMOOTH VECTOR OPTIMIZATION PROBLEM WITH GENERALIZED α-d_I-TYPE-I UNIVEXITY OVER CONES

H. JIAO

(Communicated by Ali Ghaffari)

Abstract. In this paper, using Clarke's generalized directional derivative and d_I-invexity, we introduce new concepts of nonsmooth K-α-d_I-invex and generalized type I univex functions over cones for a nonsmooth vector optimization problem with cone constraints. We obtain some sufficient optimality conditions and Mond-Weir type duality results under the foresaid generalized invexity and type I cone-univexity assumptions.
Keywords: Vector optimization, generalized d_I-invexity, type I univexity, cones, optimality, duality.
MSC(2010): Primary: 26A51; Secondary: 90C29, 90C46.

1. Introduction

Convexity plays a vital role in optimality and duality of mathematical programming, see [1, 2]. During the past several decades many attempts have been made to weaken convexity hypothesis, see [3–5]. In this endeavor, Hanson and Mond [6] introduced type I function for a scalar optimization problem. Later, various generalized type I functions have been presented and a number of optimality conditions and duality results have been obtained by using these functions, see [7–10] and the references therein.

Jayswal and Kumar [11] proposed d-V-type-I univex functions for a multiobjective optimization problem in R^n and established several sufficient optimality criteria and duality results. Jayswal [12] defined generalized α-univex type-I vector-valued functions for a multiobjective programming problem in R^n and obtained some K-T type sufficient optimality conditions and Mond-Weir type duality results. Then, Suneja et al. [13] introduced various generalized

Article electronically published on April 30, 2016.
Received: 5 October 2013, Accepted: 20 December 2014.

type-I functions over cones for a nonsmooth vector minimization problem using Clarke's generalized gradients and established a few sufficient optimality conditions and duality results under cone generalized type-I assumptions.

In this paper, using Clarke's generalized directional derivative of locally Lipschitz functions we study a nonsmooth vector optimization problem with cone constraints. Utilizing an idea of [4], we introduce nonsmooth K-α-d_I-invex function over cones and various generalized type I univex functions over cones, which we call nonsmooth $(K \times Q)$-α-d_I-type-I univex, nonsmooth $(K \times Q)$-α-d_I-type-I quasi-pseudo univex and nonsmooth $(K \times Q)$-α-d_I-type-I pseudo-quasi univex and obtain some sufficient optimality conditions under the foresaid generalized invexity and type I cone-univexity assumptions. Moreover, a Mond-Weir type dual is formulated and weak and converse duality results are established. The results obtained in this paper generalize and extend the previously known results in this area.

2. **Preliminaries and definitions**

Throughout this paper, denote $intK$ the interior of $K \subseteq R^m$ in which $R^m = \{(x_1, x_2, \ldots, x_m) | x_i \in R, i = 1, 2, \ldots, m\}$. We assume that the spaces R^m and R^p are ordered by cones $K \subseteq R^m$ and $Q \subseteq R^p$ respectively, which are pointed, closed, convex and with nonempty interiors. The dual cone of K is defined as

$$K^* = \{u^* \in R^m : \langle u^*, x \rangle \geq 0,\ \forall x \in K\}.$$

The cone K induces a partial order $\leq_K$ on R^m given by
$x, y \in R^m$, $x \leqq_K y \Longleftrightarrow y - x \in K$;
$x, y \in R^m$, $x \leq_K y \Longleftrightarrow y - x \in K \backslash \{0\}$;
$x, y \in R^m$, $x <_K y \Longleftrightarrow y - x \in intK$.
Similarly, Q induces a partial order on R^p.

The following important property is from [14], which will be used in the sequel.

Lemma 2.1. *[14] Let $K \subseteq R^m$ be a convex cone with $intK \neq \varnothing$. Then,*
(a) $\forall\ u^* \in K^* \setminus \{0\},\ x \in intK \Rightarrow \langle u^*, x \rangle > 0$;
(b) $\forall\ u^* \in intK^*,\ x \in K \setminus \{0\} \Rightarrow \langle u^*, x \rangle > 0$.

A function $\omega : R^n \to R$ is said to be locally Lipschitz at $u \in R^n$, if there exists $s > 0$ such that $|\omega(x) - \omega(y)| \leq s\|x - y\|$, for all x, y in a neighbourhood of u.

A function is locally Lipschitz on R^n, if it is locally Lipschitz at each point of R^n.

Definition 2.2. [15] Let $\omega : R^n \to R$ be a locally Lipschitz function, then $\omega^\circ(u; \upsilon)$ denotes the Clarke's generalized directional derivative of ω at $u \in R^n$

in the direction v and is defined as

$$\omega^{\circ}(u;v)=\limsup_{y\to u\ t\to 0}\frac{\omega(y+tv)-\omega(y)}{t}, t>0, y\in R^n.$$

And the usual directional derivative of ω at u in the direction v is defined as

$$\omega'(u;v)=\lim_{t\to 0}\frac{\omega(u+tv)-\omega(u)}{t},$$

whenever this limit exists. Obviously, $\omega^{\circ}(u;v)\geq\omega'(u;v)$.

We say that ω is directionally differentiable at u, if for all $v\in R^n$, its directional derivative $\omega'(u;v)$ exists finite.

Let $f:R^n\to R^m$ be given by $f=(f_1,f_2,\ldots,f_m)$, where $f_i:R^n\to R, i=1,2,\ldots,m$. We say that f is locally Lipschitz on R^n if each f_i is locally Lipschitz on R^n. The generalized directional derivative of a locally Lipschitz function f at u in the direction $v=(v_1,v_2,\ldots,v_m)$ is defined as

$$f^{\circ}(u;v)=(f_1^{\circ}(u;v_1),f_2^{\circ}(u;v_2),\ldots,f_m^{\circ}(u;v_m)).$$

Note that $f^{\circ}(u;v)$ reduces to the notion of [15] if $v_1=v_2=\ldots=v_m$.

Definition 2.3. [12] A subset $E\subseteq R^n$ is said to be an α-invex set, if there exists $\tau:E\times E\to R^n, \alpha:E\times E\to R_+$ such that

$$\bar{x}+\lambda\alpha(x,\bar{x})\tau(x,\bar{x})\in E,\ \forall x,\bar{x}\in E,\ \lambda\in[0,1].$$

Note that, for $\alpha(x,\bar{x})\equiv 1$, α-invex set becomes the invex set. However, the α-invex need not be convex sets, see [16].

Definition 2.4. [12] A function $h:E\to R^m$ is said to be α-preinvex function, if there exist $\tau:E\times E\to R^n$, $\alpha:E\times E\to R_+$ such that $h(\bar{x}+\lambda\alpha(x,\bar{x})\tau(x,\bar{x}))\leq\lambda h(x)+(1-\lambda)h(\bar{x}),\ \forall x,\bar{x}\in E,\ \lambda\in[0,1]$.

Definition 2.5. [11] Let $h:E\to R^m$ be directionally differentiable at $\bar{x}\in E$. h is said to be α-d-invex at $\bar{x}$ with respect to $\tau:E\times E\to R^n$ if for any $x\in E$,

$h(x)-h(\bar{x})\geq\alpha(x,\bar{x})h'(\bar{x};\tau(x,\bar{x}))$.

It is clear that every directionally differentiable α-preinvex function is an α-d-invex function.

From now onwards, we always assume that $f:R^n\to R^m$ and $g:R^n\to R^p$ are locally Lipschitz and that $\alpha:R^n\times R^n\to R_+\setminus\{0\}$, $\eta_i,\theta_j:R^n\times R^n\to R^n, i=1,2,\ldots,m, j=1,2,\ldots,p$ are fixed mappings,where $R_+=\{x|x\geq 0\}$. Denote $\eta=(\eta_1,\eta_2,\ldots,\eta_m)$ and $\theta=(\theta_1,\theta_2,\ldots,\theta_p)$.

Now, we extend Definition 2.4 to the function over cones in the following way.

Definition 2.6. Let $f:R^n\to R^m$ be locally Lipschitz at $\bar{x}\in R^n$. f is said to be nonsmooth K-α-d_I-invex at $\bar{x}$ with respect to η if for any $x\in R^n$ and $u^*\in K^*$,

$\langle u^*,f(x)-f(\bar{x})\rangle\geq\alpha(x,\bar{x})(u^*\circ f)^{\circ}(\bar{x};\eta(x,\bar{x}))$, where $(u^*\circ f)(x)=u^*(f(x))$.

Remark 2.7. If f is directionally differentiable, $K^* = K_+^m$ and $\eta_1 = \eta_2 = \cdots = \eta_m$, then the above definition reduces to α-d-invex [11]. If $\alpha(x,\bar{x}) \equiv 1$, for all $x, \bar{x} \in R^n$ and $\eta_1 = \eta_2 = \cdots = \eta_m$, then the above definition reduces to the notion of K-nonsmooth invex [18].

In this paper, we consider the vector optimization problem with cone constraints as follows

$$(VP) \begin{cases} K - min\ f(x) \\ s.t.\ -g(x) \in Q, \\ x \in X, \end{cases}$$

where $f : X \to R^m,\ g : X \to R^p$ are locally Lipschitz functions on $X \subseteq R^n$, K and Q are closed convex cones with $intK \neq \varnothing$ and $intQ \neq \varnothing$.

Denote $F = \{x \in X | -g(x) \in Q\}$ the feasible set of problem (VP). Let $b_0, b_1 : X \times X \to R_+$ and $\phi_0, \phi_1 : R \to R$.

Next, following Jayswal and Kumar [11], Jayswal [12] and Suneja et al. [13] we introduce various nonsmooth $(K \times Q)$-α-d_I-type-I cone-univex functions.

Definition 2.8. (f, g) is said to be nonsmooth $(K \times Q)$-α-d_I-type-I univex at $\bar{x} \in X$, if for each $x \in X$, there exist $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ and θ such that for all $u^* \in K^*$ and $v^* \in Q^*$

$$b_0(x,\bar{x})\phi_0\langle u^*, f(x) - f(\bar{x})\rangle \geq \alpha(x,\bar{x})(u^* \circ f)^\circ(\bar{x}, \eta(x,\bar{x})),$$
$$- b_1(x,\bar{x})\phi_1\langle v^*,\ g(\bar{x})\rangle \geq \alpha(x,\bar{x})(v^* \circ g)^\circ(\bar{x}, \theta(x,\bar{x})).$$

Definition 2.9. (f, g) is said to be nonsmooth $(K \times Q)$-α-d_I-type-I quasi-pseudo univex at $\bar{x} \in X$, if for each $x \in X$, there exist $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ and θ such that for all $u^* \in K^*$ and $v^* \in Q^*$

$$b_0(x,\bar{x})\phi_0\langle u^*, f(x) - f(\bar{x})\rangle \leq 0 \Rightarrow \alpha(x,\bar{x})(u^* \circ f)^\circ(\bar{x}; \eta(x,\bar{x})) \leq 0,$$
$$\alpha(x,\bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(x,\bar{x})) \geq 0 \Rightarrow -b_1(x,\bar{x})\phi_1\langle v^*, g(\bar{x})\rangle \geq 0.$$

If in the second relation, we have

$$\alpha(x,\bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(x,\bar{x})) \geq 0 \Rightarrow -b_1(x,\bar{x})\phi_1\langle v^*, g(\bar{x})\rangle > 0,$$

then we say that (f, g) is nonsmooth $(K \times Q)$-α-d_I-type-I quasi-strict-pseudo univex at $x \in X$.

Definition 2.10. (f, g) is said to be nonsmooth $(K \times Q)$-α-d_I-type-I pseudo-quasi univex at $\bar{x} \in X$, if for each $x \in X$, there exist $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ and θ such that for all $u^* \in K^*$ and $v^* \in Q^*$

$$\alpha(x,\bar{x})(u^* \circ f)^\circ(\bar{x}; \eta(x,\bar{x})) \geq 0 \Rightarrow b_0(x,\bar{x})\phi_0\langle u^*, f(x) - f(\bar{x})\rangle \geq 0,$$
$$- b_1(x,\bar{x})\phi_1\langle v^*, g(\bar{x})\rangle \leq 0 \Rightarrow \alpha(x,\bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(x,\bar{x})) \leq 0.$$

Remark 2.11. The notions defined above are different from those in Slimani and Radjef [4], Yu and Liu [5], Jayswal and Kumar [11], Jayswal [12], Suneja et al. [13] and Mishra et al. [17].

Definition 2.12. We say that $\bar{x} \in F$ is a weakly efficient (or an efficient) solution of problem (VP), if there exists no $x \in F$ such that

$$f(x) <_K f(\bar{x}) \ (or \ f(x) \leq_K f(\bar{x})).$$

3. Optimality criteria

In this section, we establish a few sufficient optimality conditions for problem (VP) under the assumptions of various nonsmooth $(K \times Q)$-α-d_I-invexity and $(K \times Q)$-α-d_I-type-I univexity.

Theorem 3.1. *Let f be nonsmooth K-α-d_I-invex at $\bar{x} \in F$ with respect to η and g be nonsmooth Q-α-d_I-invex at $\bar{x} \in F$ with respect to θ. Suppose that there exist $u^* \in K^* \setminus \{0\}, v^* \in Q^*$ such that*

$$(u^* \circ f)^\circ(\bar{x};\ \eta(x, \bar{x})) + (v^* \circ g)^\circ(\bar{x};\ \theta(x, \bar{x})) \geq 0,\ \forall x \in F, \tag{3.1}$$

$$\langle v^*, g(\bar{x})\rangle = 0. \tag{3.2}$$

Then $\bar{x}$ is a weakly efficient solution of (VP).

Proof. Since f is nonsmooth K-α-d_I-invex at $\bar{x}$ with respect to η and g is nonsmooth Q-α-d_I-invex at $\bar{x}$ with respect to θ, we get

$$\langle u^*, f(x) - f(\bar{x})\rangle \geq \alpha(x, \bar{x})(u^* \circ f)^\circ(\bar{x}; \eta(x, \bar{x})), \tag{3.3}$$

$$\langle v^*, g(x) - g(\bar{x})\rangle \geq \alpha(x, \bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(x, \bar{x})). \tag{3.4}$$

Let if possible $\bar{x}$ be not a weakly efficient solution of (VP). Then there exists $x \in F$ such that $f(x) <_K f(\bar{x})$. By $u^* \in K^* \setminus \{0\}$ and Lemma 2.1, we have $\langle u^*, f(x) - f(\bar{x})\rangle < 0$.

From (3.3) and $\alpha(x, \bar{x}) > 0$, we deduce

$$(u^* \circ f)^\circ(\bar{x}, \eta(x, \bar{x})) < 0. \tag{3.5}$$

As $x \in F$, $-g(x) \in Q$ gives $\langle v^*, g(x)\rangle \leq 0$, for all $v^* \in Q^*$. Considering (3.2), we obtain $\langle v^*, g(x) - g(\bar{x})\rangle \leq 0$.

From (3.4), it follows that $\alpha(x, \bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(x, \bar{x})) \leq 0$.

Hence,

$$(v^* \circ g)^\circ(\bar{x}; \theta(x, \bar{x})) \leq 0. \tag{3.6}$$

Adding (3.5) and (3.6), we get

$$(u^* \circ f)^\circ(\bar{x};\ \eta(x, \bar{x})) + (v^* \circ g)^\circ(\bar{x};\ \theta(x, \bar{x})) < 0,$$

which contradicts (3.1).

Therefore, $\bar{x}$ is a weakly efficient solution of (VP). $\square$

Theorem 3.2. *Assume that there exist* $\bar{x} \in F$, $u^* \in K^* \setminus \{0\}$ *(or* $u^* \in intK^*$*) and* $v^* \in Q^*$ *such that* (3.1) *and* (3.2) *hold. Moreover, suppose any one of the following conditions is satisfied:*
(a) (f, g) *is nonsmooth* $(K\times Q)$-α-d_I*-type-I univex at* $\bar{x}$ *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ *and* θ*;*
(b) (f, g) *is nonsmooth* $(K \times Q)$-α-d_I*-type-I pseudo-quasi univex at* $\bar{x}$ *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ *and* θ*;*
(c) (f, g) *is nonsmooth* $(K \times Q)$-α-d_I*-type-I quasi-strict-pseudo univex at* $\bar{x}$ *with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ *and* θ*.*
Further, assume that $a < 0 \Rightarrow \phi_0(a) < 0$ *and* $a \leq 0 \Rightarrow \phi_1(a) \leq 0$ *and* $b_0(x, \bar{x}) > 0$ *and* $b_1(x, \bar{x}) > 0$*.*

Then $\bar{x}$ *is a weakly efficient (or an efficient) solution of* (VP)*.*

Proof. Suppose, on the contrary, we assume that $\bar{x}$ is not a weakly efficient (or an efficient) solution of (VP). Then there exists a feasible solution $\breve{x}$ of (VP) such that

$$f(\breve{x}) <_K f(\bar{x}) \ (or \ f(\breve{x}) \leq_K f(\bar{x})).$$

Since $u^* \in K^* \setminus \{0\}$ (or $u^* \in intK^*$), from Lemma 2.1, we have

$$\langle u^*, f(\breve{x}) - f(\bar{x})\rangle < 0. \tag{3.7}$$

From $a < 0 \Rightarrow \phi_0(a) < 0$ and $b_0(\breve{x}, \bar{x}) > 0$, it follows that

$$b_0(\breve{x}, \bar{x})\phi_0\langle u^*, f(\breve{x}) - f(\bar{x})\rangle < 0.$$

By condition (a), we deduce

$$\alpha(\breve{x}, \bar{x})(u^* \circ f)^\circ(\bar{x}; \eta(\breve{x}, \bar{x})) < 0.$$

On account of positivity of $\alpha(\breve{x}, \bar{x})$, we get

$$(u^* \circ f)^\circ(\bar{x}; \eta(\breve{x}, \bar{x})) < 0. \tag{3.8}$$

According to $a \leq 0 \Rightarrow \phi_1(a) \leq 0$ and $b_1(\breve{x}, \bar{x}) > 0$ and (3.2), we obtain

$$-b_1(\breve{x}, \bar{x})\phi_1\langle v^*, g(\bar{x})\rangle \leq 0.$$

By condition (a), we also have

$$\alpha(\breve{x}, \bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(\breve{x}, \bar{x})) \leq 0.$$

From $\alpha(\breve{x}, \bar{x}) > 0$, the above inequality gives

$$(v^* \circ g)^\circ(\bar{x}; \theta(\breve{x}, \bar{x})) \leq 0. \tag{3.9}$$

Adding the inequalities (3.8) and (3.9), we obtain

$$(u^* \circ f)^\circ(\bar{x}; \eta(\breve{x}, \bar{x})) + (v^* \circ g)^\circ(\bar{x}; \theta(\breve{x}, \bar{x})) < 0,$$

which contradicts (3.1).
By condition (b), the above inequality $-b_1(\breve{x}, \bar{x})\phi_1\langle v^*, g(\bar{x})\rangle \leq 0$ also yields

$$\alpha(\breve{x}, \bar{x})(v^* \circ g)^\circ(\bar{x}; \theta(\breve{x}, \bar{x})) \leq 0.$$

From $\alpha(\breve{x},\bar{x})>0$, we get

$$(v^*\circ g)^\circ(\bar{x};\theta(\breve{x},\bar{x}))\le 0.$$

Combining the above inequality and (3.1), we have

$$(u^*\circ f)^\circ(\bar{x};\ \eta(\breve{x},\bar{x}))\ge 0.$$

Hence, $\alpha(\breve{x},\bar{x})(u^*\circ f)^\circ(\bar{x};\ \eta(\breve{x},\bar{x}))\ge 0$.
By condition (b) again, we obtain

$$b_0(\breve{x},\bar{x})\phi_0\langle u^*, f(\breve{x})-f(\bar{x})\rangle\ge 0.$$

From $a<0\Rightarrow\phi_0(a)<0$ and $b_0(\breve{x},\bar{x})>0$, it follows that

$$\langle u^*, f(\breve{x})-f(\bar{x})\rangle\ge 0,$$

which is a contradiction to (3.7).
Using the above inequality $b_0(\breve{x},\bar{x})\phi_0\langle u^*, f(\breve{x})-f(\bar{x})\rangle<0$ and condition (c), we have

$$\alpha(\breve{x},\bar{x})(u^*\circ f)^\circ(\bar{x};\eta(\breve{x},\bar{x}))\le 0,$$

that is , $(u^*\circ f)^\circ(\bar{x};\eta(\breve{x},\bar{x}))\le 0$.
By (3.1), the above inequality implies

$$(v^*\circ g)^\circ(\bar{x};\theta(\breve{x},\bar{x}))\ge 0.$$

Therefore, $\alpha(\breve{x},\bar{x})(v^*\circ g)^\circ(\bar{x};\theta(\breve{x},\bar{x}))\ge 0$.
Applying condition (c) again, we obtain

$$(-b_1(\breve{x},\bar{x})\phi_1\langle v^*, g(\bar{x})\rangle>0.)$$

From $b_1(\breve{x},\bar{x})>0$ and $a\le 0\Rightarrow\phi_1(a)\le 0$, it follows that

$$-\langle v^*, g(\bar{x})\rangle>0,$$

which is in contradiction with (3.2).

The proof is completed. □

4. **Duality**

In relation to (VP), we formulate the following Mond-Weir type dual problem

$$(4.1)\quad (VD)\begin{cases} K-max\ f(y)\\ s.t.\ (u^*\circ f)^\circ(y;\ \eta(x,y))+(v^*\circ g)^\circ(y;\ \theta(x,y))\ge 0,\ \forall x\in F,\\ \langle v^*, g(y)\rangle\ge 0,\\ y\in X,\ u^*\in K^*,\ v^*\in Q^*.\end{cases}$$

Denote the feasible set of problem (VD) by G, i.e., $G=\{(y,u^*,v^*):\ (u^*\circ f)^\circ(y;\ \eta(x,y))+(v^*\circ g)^\circ(y;\ \theta(x,y))\ge 0,\ \langle v^*, g(y)\rangle\ge 0,\ \forall x\in F,\ y\in X,\ u^*\in K^*,\ v^*\in Q^*\}$.

Now, we establish weak and converse duality results.

Theorem 4.1. *(Weak duality) Let $x \in F,\ (y, u^*, v^*) \in G$ and $u^* \in K^* \setminus \{0\}$ (or $u^* \in intK^*$). Furthermore, suppose any one of the following conditions is satisfied:*
(a) (f, g) is nonsmooth $(K \times Q)$-α-d_I-type-I univex at $y \in F$ with respect to $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ and θ;
(b) (f, g) is nonsmooth $(K \times Q)$-α-d_I-type-I pseudo-quasi univex at $y \in F$ with respect to $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ and θ;
(c) (f, g) is nonsmooth $(K \times Q)$-α-d_I-type-I quasi-strict-pseudo univex at $y \in F$ with respect to $b_0, b_1, \phi_0, \phi_1, \alpha, \eta$ and θ.
Further assume that $a < 0 \Rightarrow \phi_0(a) < 0$ and $a \leq 0 \Rightarrow \phi_1(a) \leq 0$ and $b_0(x, y) > 0$ and $b_1(x, y) > 0$.
Then $f(x) \not<_K f(y)$ (or $f(x) \not\leq_K f(y)$).

Proof. Assume to the contrary that there exist $\breve{x} \in F,\ (y, u^*, v^*) \in G$ such that
$f(\breve{x}) <_K f(y)$ (or $f(\breve{x}) \leq_K f(y)$).
By $u^* \in K^* \setminus \{0\}$ (or $u^* \in intK^*$) and Lemma 2.1, we have

$$\langle u^*,\ f(\breve{x}) - f(y)\rangle < 0. \tag{4.2}$$

In view of the fact that $a < 0 \Rightarrow \phi_0(a) < 0$ and $b_0(\breve{x}, y) > 0$, we get

$$b_0(\breve{x}, y)\phi_0\langle u^*,\ f(\breve{x}) - f(y)\rangle < 0.$$

By condition (a), the above inequality gives

$$\alpha(\breve{x}, y)(u^* \circ f)^\circ(y;\ \eta(\breve{x}, y)) < 0. \tag{4.3}$$

From $(y, u^*, v^*) \in G$, it follows that

$$-\langle v^*, g(y)\rangle \leq 0. \tag{4.4}$$

By $a \leq 0 \Rightarrow \phi_1(a) \leq 0$ and $b_1(\breve{x}, y) > 0$, we deduce

$$-b_1(\breve{x}, y)\phi_1\langle v^*, g(y)\rangle \leq 0.$$

Using condition (a), we obtain

$$\alpha(\breve{x}, y)(v^* \circ g)^\circ(y;\ \theta(\breve{x}, y)) \leq 0. \tag{4.5}$$

Since $\alpha(\breve{x}, y) > 0$, adding (4.3) and (4.5), we have

$$(u^* \circ f)^\circ(y;\ \eta(\breve{x}, y)) + (v^* \circ g)^\circ(y;\ \theta(\breve{x}, y)) < 0,$$

which is a contradiction to (4.1).
By the above inequality $-b_1(\breve{x}, y)\phi_1\langle v^*, g(y)\rangle \leq 0$ and condition (b), we have

$$\alpha(\breve{x}, y)(v^* \circ g)^\circ(y;\ \theta(\breve{x}, y)) \leq 0.$$

From $\alpha(\breve{x}, y) > 0$, it follows that

$$(v^* \circ g)^\circ(y;\ \theta(\breve{x}, y)) \leq 0.$$

Taking (4.1) into account, we obtain

$$(u^* \circ f)^\circ(y;\ \eta(\breve{x}, y)) \geq 0.$$

Thus, $\alpha(\breve{x}, y)(u^* \circ f)^\circ(y;\ \eta(\breve{x}, y)) \geq 0$. By condition (b) again, the above inequality leads to

$$b_0(\breve{x}, y)\phi_0\langle u^*, f(\breve{x}) - f(y)\rangle \geq 0.$$

According to $b_0(\breve{x}, y) > 0$ and $a < 0 \Rightarrow \phi_0(a) < 0$, we get

$$\langle u^*, f(\breve{x}) - f(y)\rangle \geq 0,$$

which contradicts (4.2).
By condition (c) and $\alpha(\breve{x}, y) > 0$, the above relation $b_0(\breve{x}, y)\phi_0\langle u^*,\ f(\breve{x}) - f(y)\rangle < 0$ gives

$$(u^* \circ f)^\circ(y;\ \eta(\breve{x}, y)) \leq 0.$$

Considering (4.1) and positivity of $\alpha(\breve{x}, y)$, we obtain

$$\alpha(\breve{x}, y)(v^* \circ g)^\circ(y;\ \theta(\breve{x}, y)) \geq 0.$$

Using condition (c) again, we get

$$-b_1(\breve{x}, y)\phi_1\langle v^*, g(y)\rangle > 0.$$

From $a \leq 0 \Rightarrow \phi_1(a) \leq 0$ and $b_1(\breve{x}, y) > 0$, the above relation yields

$$-\langle v^*, g(y)\rangle > 0,$$

which is in contradiction with (4.4).

Therefore, the theorem is proved. □

Theorem 4.2. *(Converse duality) Let $(\bar{y}, \bar{u}^*, \bar{v}^*)$ be a weakly efficient (or an efficient) solution of problem (VD). Assume that $\bar{u}^* \in K^* \backslash \{0\}$ (or $\bar{u}^* \in intK^*$) and that all conditions of Theorem 4.1 hold at $\bar{y}$. Then $\bar{y}$ is a weakly efficient (or an efficient) solution of (VP).*

Proof. Assume to the contrary that $\bar{y}$ is not a weakly efficient (or an efficient) solution of (VP), then there exists $\breve{y} \in F$ such that

$$f(\breve{y}) <_K f(\bar{y}) \ (or\ f(\breve{y}) \leq_K f(\bar{y})).$$

From $\bar{u}^* \in K^* \setminus \{0\}$ (or $\bar{u}^* \in intK^*$) and Lemma 2.1, it follows that

$$\langle \bar{u}^*, f(\breve{y}) - f(\bar{y})\rangle < 0. \tag{4.6}$$

By $(\bar{y}, \bar{u}^*, \bar{v}^*) \in G$, we have

$$(\bar{u}^* \circ f)^\circ(\bar{y};\ \eta(\breve{y}, \bar{y})) + (\bar{v}^* \circ g)^\circ(\bar{y};\ \theta(\breve{y}, \bar{y})) \geq 0, \tag{4.7}$$

$$\langle \bar{v}^*, g(\bar{y})\rangle \geq 0. \tag{4.8}$$

By $a < 0 \Rightarrow \phi_0(a) < 0$ and $b_0(\breve{y}, \bar{y}) > 0$, (4.6) gives

$$b_0(\breve{y}, \bar{y})\phi_0\langle \bar{u}^*, f(\breve{y}) - f(\bar{y})\rangle < 0.$$

If condition (a) of Theorem 4.1 holds, then the above inequality yields

$$\alpha(\breve{y},\bar{y})(\bar{u}^*\circ f)^\circ(\bar{y};\ \eta(\breve{y},\bar{y}))<0. \tag{4.9}$$

Similarly, from $a\leq 0\Rightarrow \phi_1(a)\leq 0$ and $b_1(\breve{y},\bar{y})>0$ and condition (a), (4.8) implies

$$\alpha(\breve{y},\bar{y})(\bar{v}^*\circ g)^\circ(\bar{y};\ \theta(\breve{y},\bar{y}))\leq 0. \tag{4.10}$$

Since $\alpha(\breve{y},\bar{y})>0$, summing (4.9) and (4.10), we get

$$(\bar{u}^*\circ f)^\circ(\bar{y};\ \eta(\breve{y},\bar{y}))+(\bar{v}^*\circ g)^\circ(\bar{y};\ \theta(\breve{y},\bar{y}))<0,$$

which contradicts (4.7).

If condition (b) or (c) of Theorem 4.1 holds, by an argument similar to that of Theorem 4.1, we obtain

$$\langle \bar{u}^*, f(\breve{y})-f(\bar{y})\rangle\geq 0, \tag{4.11}$$

or

$$-\langle \bar{v}^*, g(\bar{y})\rangle>0. \tag{4.12}$$

The inequalities (4.11) and (4.12) contradict (4.6) and (4.8), respectively.

Therefore, the proof is completed. □

Acknowledgements

The author would like to thank the anonymous referees for valuable comments and suggestions for the improvements of the paper. This work is supported by the National Natural Science Foundation of China (No. 61373174), the Key Research Project of Yangtze Normal University (No. 2013XJZD006) and Research Startup Project of Introduction Talent of Yangtze Normal University (No. 2014KYQD02).

References

[1] T. Antczak, Optimality conditions and duality for nondifferentiable multiobjective programming problems involving d-r-type I functions, *J. Comput. Appl. Math.* **225** (2009), no. 1, 236–250.

[2] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, Nonlinear Programming, Theory and Algorithms, third edition, Wiley-Interscience [John Wiley & Sons], Hoboken, 2006.

[3] L. Batista dos Santos, R. Osuna-Gómez, M. A. Rojas-Medar, and A. Rufián-Lizana, Preinvex functions and weak efficient solutions for some vectorial optimization problem in Banach spaces, *Comput. Math. Appl.* **48** (2004), no. 5-6, 885–895.

[4] B. D. Craven, Control and Optimization, Chapman & Hall, London, 1995.

[5] F. H. Clarke, Optimization and Nonsmooth Analysis, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1983.

[6] M. Hachimi and B. Aghezzaf, Sufficiency and duality in nondifferentiable multiobjective programming involving generalized type I functions, *J. Math. Anal. Appl.* **319** (2006), no. 1, 110–123.

[7] M. A. Hanson and B. Mond, Necessary and sufficient conditions in constrained optimization, *Math. Programming* **37** (1987), no. 1, 51–58.

[8] M. A. Hanson, R. Pini, and C. Singh, Multiobjective programming under generalized type-I invexity. *J. Math. Anal. Appl.* **261** (2001), no. 2, 562–577.
[9] A. Jayswal, On sufficiency and duality in multiobjective programming problem under generalized α-type I univexity, *J. Global Optim.* **46** (2010), no. 2, 207–216.
[10] A. Jayswal and R. Kumar, Some nondifferentiable multiobjective programming under generalized d-V-type-I univexity, *J. Comput. Appl. Math.* **229** (2009), no. 1, 175–182.
[11] O. L. Mangasarian, Nonlinear Programming, McGraw-Hill Book Co., New York-London-Sydney, 1969.
[12] S. K. Mishra, S. Y. Wang, and K. K. Lai, Nondifferentiable multiobjective programming under generalized d-univexity, *European J. Oper. Res.* **160** (2005), no. 1, 218–226.
[13] M. A. Noor, On generalized preinvex functions and monotonicities, *JIPAM. J. Inequal. Pure Appl. Math.* **5** (2004), no. 4, Article 110, 9 pages.
[14] H. Slimani and M. S. Radjef, Nondifferentiable multiobjective programming under generalized d_I-invexity, *European J. Oper. Res.* **202** (2010), no. 1, 32–41.
[15] S. K. Suneja and M. K. Srivastava, Optimality and duality in nondifferentiable multiobjective optimization involving d-type I and related functions, *J. Math. Anal. Appl.* **206** (1997), no. 2, 465–479.
[16] S. K. Suneja, S. Khurana and M. Bhatia, Optimality and duality in vector optimization involving generalized type I functions over cones, *J. Global Optim.* **49** (2011), no. 1, 23–35.
[17] N. D. Yen and P. H. Sach, On locally Lipschitz vector-valued invex functions, *Bull. Austral. Math. Soc.* **47** (1993) 259–271.
[18] G. L. Yu and S. Y. Liu, Some vector optimization problems in Banach spaces with generalized convexity, *Comput. Math. Appl.* **54** (2007), no. 11-12, 1403–1410.

(Hehua Jiao) School of Mathematics and Statistics, Yangtze Normal University, Chongqing 408100, P. R. China.
E-mail address: jiaohh361@126.com