# ON ONE-SIDED IDEALS OF RINGS OF LINEAR TRANSFORMATIONS 

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Communicated by Heydar Radjavi
With kind regards, dedicated to Chandler Davis on the occasion of his eightieth birthday


#### Abstract

Let $D$ be a division ring, $\mathcal{V}$ a right or left vector space over $D$, and $\mathcal{L}(\mathcal{V})$ the ring of all right (resp. left) linear transformations on $\mathcal{V}$. We characterize certain one-sided ideals of the ring $\mathcal{L}(\mathcal{V})$ in terms of their rank-one idempotents. We use our result to characterize a division ring $D$ in terms of the one-sided ideals of $M_{n}(D)$. Some other consequences are presented.


## 1. One-sided ideals of rings of linear transformations

In this note, among other things, we present the counterparts of some of the results of [3] for one-sided ideals of the ring of all right (resp. left) linear transformations on a right (resp. left) vector space over a general division ring. It is worth mentioning that the proofs presented for Theorem 1.2 and Theorem 1.6 below are almost identical to those of

[^0]their counterparts presented in [3] except that in [3] one needs to make frequent use of Hahn-Banach theorem for locally convex vector spaces. However, we have included proofs of Theorem 1.2 and Theorem 1.6 for reader's convenience.

Throughout this note, unless otherwise stated, $D$ denotes a division ring, $\mathcal{V}$ and $\mathcal{W}$ right (resp. left) vector spaces over $D$, and $\mathcal{L}(\mathcal{V}, \mathcal{W})$ the set of all right (resp. left) linear transformations $A: \mathcal{V} \longrightarrow \mathcal{W}$ such that $A(x+y)=A x+A y$ and $A(x \lambda)=(A x) \lambda($ resp. $A(\lambda x)=\lambda(A x))$ for all $x, y \in \mathcal{V}$ and $\lambda \in D$. When $\mathcal{V}=\mathcal{W}$, we use the symbol $\mathcal{L}(\mathcal{V})$ to denote $\mathcal{L}(\mathcal{V}, \mathcal{W})$. It is easy to see that the set $\mathcal{L}(\mathcal{V})$ forms a ring under the addition and multiplication of linear transformations which are, respectively, defined by $(A+B)(x):=A x+B x$ and $(A B)(x):=$ $A(B x)$.

A subspace $\mathcal{M}$ is called invariant for a collection $\mathcal{F}$ in $\mathcal{L}(\mathcal{V})$ if $T \mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F}$. A collection $\mathcal{F}$ of linear transformations in $\mathcal{L}(\mathcal{V})$ is called reducible if $\mathcal{F}=\{0\}$ or it has a nontrivial invariant subspace and irreducible otherwise. A collection $\mathcal{F}$ of linear transformations in $\mathcal{L}(\mathcal{V})$ is called simultaneously triangularizable or simply triangularizable if there exists a maximal chain of the subspaces of $\mathcal{V}$ each of which is invariant under the collection $\mathcal{F}$. If the space $\mathcal{V}$ happens to be finite-dimensional, this is equivalent to saying that there exists a basis for the vector space $\mathcal{V}$ relative to which all matrices in the family are upper triangular.

We define the dual space of $\mathcal{V}$ to be $\mathcal{L}(\mathcal{V}, D)$, where $D$ is regarded as a one-dimensional vector space over itself with the same chirality as that of $\mathcal{V}$. (Here, by the chirality of a vector space $\mathcal{V}$ over a division ring, we mean whether it is a left or a right vector space according to which we define $\mathcal{V}$ to have left or right chirality, respectively.) As is usual, we use the symbol $\mathcal{V}^{\prime}$ for $\mathcal{L}(\mathcal{V}, D)$. The members of $\mathcal{V}^{\prime}$ are called linear functionals on $\mathcal{V}$. Also, when $\mathcal{V}$ is a right (resp. left) vector space, $\mathcal{V}^{\prime}$ is a left (resp. right) vector space over $D$ endowed with the addition and the scalar multiplication defined by $(f+g)(x):=f(x)+g(x)$ and $(\lambda f)(x):=\lambda f(x)($ resp. $(f \lambda)(x):=f(x) \lambda)$ for all $x \in \mathcal{V}$ and $\lambda \in D$. The second dual of $\mathcal{V}$, denoted by $\mathcal{V}^{\prime \prime}$, is the dual of $\mathcal{V}^{\prime}$. The space $\mathcal{V}^{\prime \prime}$ has the same chirality as that of $\mathcal{V}$ over $D$. It is easily seen that $\mathcal{V}$ naturally imbeds into $\mathcal{V}^{\prime \prime}$ via the natural mapping ${ }^{\wedge}: \mathcal{V} \rightarrow \mathcal{V}^{\prime \prime}(x \mapsto \widehat{x})$ defined by $\widehat{x}(f)=f(x)$ for all $f \in \mathcal{V}^{\prime}$, and that the natural mapping is an isomorphism of the vector spaces $\mathcal{V}$ and $\mathcal{V}^{\prime \prime}$ if and only if the space $\mathcal{V}$ is finite-dimensional. Let $\left\{x_{i}\right\}_{i \in I}$ be an independent subset of $\mathcal{V}$. It is easy to see that there is an independent subset $\left\{f^{i}\right\}_{i \in I}$ of linear functionals
on $\mathcal{V}$ satisfying $f^{i}\left(x_{j}\right)=\delta_{i j}$, where $i, j \in I$ and $\delta$ denotes the Kronecker delta. Every such independent subset of $\mathcal{V}^{\prime}$ is called a dual independent subset with respect to $\left\{x_{i}\right\}_{i \in I}$. If $\left\{x_{i}\right\}_{1 \leq i \leq n}$ happens to be a basis for $\mathcal{V}$, then the corresponding set $\left\{f^{i}\right\}_{1 \leq i \leq n}$ of linear functionals is unique and forms a basis for $\mathcal{V}^{\prime}$ and is called the dual basis of $\left\{x_{i}\right\}_{1 \leq i \leq n}$. For a collection $\mathcal{C}$ of vectors in a right (resp. left) vector space $\mathcal{V}$ over $D,\langle\mathcal{C}\rangle$ is used to denote the right (resp. left) linear subspace spanned by $\mathcal{C}$. For a subset $S$ of $\mathcal{V}$, we define $S^{\perp}:=\left\{f \in \mathcal{V}^{\prime}: f(S)=0\right\}$. It is plain that $S^{\perp}$ is a subspace of $\mathcal{V}^{\prime}$. For $T \in \mathcal{L}(\mathcal{V}, \mathcal{W}), T^{\prime} \in \mathcal{L}\left(\mathcal{W}^{\prime}, \mathcal{V}^{\prime}\right)$ denotes the adjoint of $T$ which is defined by $\left(T^{\prime} f\right)(v):=f(T v)$, where $f \in \mathcal{W}^{\prime}$ and $v \in \mathcal{V}$.

We start off with a known lemma, which can be a good exercise for the beginners with division rings. Throughout the remainder of the paper, we assume that $\mathcal{V}$ and $\mathcal{W}$ have a fixed common chirality.

Lemma 1.1. Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces over $D$ and $\mathcal{C} \subseteq$ $\mathcal{L}(\mathcal{V}, \mathcal{W})$. Then the following equalities hold.

$$
\left(\bigcap_{T \in \mathcal{C}} \operatorname{ker} T\right)^{\perp}=\left\langle\bigcup_{T \in \mathcal{C}} T^{\prime} \mathcal{W}^{\prime}\right\rangle, \quad\left(\bigcap_{T \in \mathcal{C}} \operatorname{ker} T^{\prime}\right)=\left\langle\bigcup_{T \in \mathcal{C}} T \mathcal{V}\right\rangle^{\perp}
$$

An important subset of $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is the class of rank-one linear transformations. It can be shown that every rank-one linear transformation is of the form $x \otimes f$ for some $x \in \mathcal{W}$ and $f \in \mathcal{V}^{\prime}$, where $(x \otimes f)(y):=x f(y)$ or $(x \otimes f)(y):=f(y) x$ for all $y \in \mathcal{V}$ depending on whether the space $\mathcal{W}$ is a right or a left vector space over $D$.

For a family $\mathcal{F} \subset \mathcal{L}(\mathcal{V})$, we use $\operatorname{ri}(\mathcal{F})$ (resp. $\operatorname{li}(\mathcal{F})$ ) to denote the right (resp. left) ideal generated by $\mathcal{F}$. If $\mathcal{F}=\{A\}$, it is obvious that $\operatorname{ri}(A)=A \mathcal{L}(\mathcal{V})($ resp. $\operatorname{li}(A)=\mathcal{L}(\mathcal{V}) A)$. Also, by the image and the kernel of the family $\mathcal{F}$, denoted by $\operatorname{im}(\mathcal{F})$ and $\operatorname{ker}(\mathcal{F})$, respectively, we mean $\langle\{A x: A \in \mathcal{F}, x \in \mathcal{V}\}\rangle$ and $\bigcap_{A \in \mathcal{F}}$ ker $A$. The coimage and cokernel of the family $\mathcal{F}$, denoted by $\operatorname{coim}(\mathcal{F})$ and $\operatorname{coker}(\mathcal{F})$, respectively, are defined as $\mathcal{V} / \operatorname{ker} \mathcal{F}$ and $\mathcal{V} / \operatorname{im}(\mathcal{F})$. The following theorem characterizes all right (resp. left) ideals in $\mathcal{L}(\mathcal{V})$ whose image (resp. coimage) is finite-dimensional. The theorem shows that rank-one idempotents play an important role in characterizing such one-sided ideals of $\mathcal{L}(\mathcal{V})$.

Theorem 1.2. Let $D$ be a division ring, $\mathcal{V}$ a right or left vector space over $D$, and $\mathcal{I}$ a nonzero right (resp. left) ideal in $\mathcal{L}(\mathcal{V})$.
(i) If the image (resp. coimage) of $\mathcal{I}$ is finite-dimensional, then there are $x_{i} \in \mathcal{V}$ and $f_{i} \in \mathcal{V}^{\prime}(1 \leq i \leq r)$ which are dual to each other, where $r=\operatorname{dimim}(\mathcal{I}) \quad$ (resp. $r=\operatorname{dim} \operatorname{coim}(\mathcal{I})$ ), and such that

$$
\mathcal{I}=x_{1} \otimes f_{1} \mathcal{L}(\mathcal{V})+\cdots+x_{r} \otimes f_{r} \mathcal{L}(\mathcal{V})=A \mathcal{L}(\mathcal{V})
$$

(resp.

$$
\left.\mathcal{I}=\mathcal{L}(\mathcal{V}) x_{1} \otimes f_{1}+\cdots+\mathcal{L}(\mathcal{V}) x_{r} \otimes f_{r}=\mathcal{L}(\mathcal{V}) A\right)
$$

where $A=x_{1} \otimes f_{1}+\cdots+x_{r} \otimes f_{r}$ is an idempotent in $\mathcal{I}$. Therefore, every right (resp. left) ideal of $\mathcal{L}(\mathcal{V})$ whose image (resp. coimage) is finitedimensional is principal; in fact the right (resp. left) ideal is generated by a finite-rank idempotent whose rank is equal to the dimension of the image (resp. coimage) of the right (resp. left) ideal.
(ii) Let $A \in \mathcal{L}(\mathcal{V})$. Then the following are equivalent.
(a) $\operatorname{rank}(A)=r$.
(b) $\operatorname{ri}(A)=A \mathcal{L}(\mathcal{V})=x_{1} \otimes f_{1} \mathcal{L}(\mathcal{V})+\cdots+x_{r} \otimes f_{r} \mathcal{L}(\mathcal{V})$, where $\left\{x_{i}\right\}_{1 \leq i \leq r}$ is a basis for $\operatorname{im}(A)$ and $f_{i}$ 's are dual to $x_{i}$ 's $(1 \leq i \leq r)$.
(c) $\operatorname{li}(A)=\mathcal{L}(\mathcal{V}) A=\mathcal{L}(\mathcal{V}) x_{1} \otimes f_{1}+\cdots+\mathcal{L}(\mathcal{V}) x_{r} \otimes f_{r}$, where $\left\{x_{i}+\right.$ ker $A\}_{1 \leq i \leq r}$ is a basis for $\operatorname{coim}(A)$ and $f_{i}$ 's are dual to $x_{i}$ 's $(1 \leq i \leq r)$.

Proof. (i) With a fixed chirality for $\mathcal{V}$, we first assume that $\mathcal{I}$ is a right ideal in $\mathcal{L}(\mathcal{V})$. Choose $A_{i} \in \mathcal{I}$ and $y_{i} \in \mathcal{V}(1 \leq i \leq r)$ such that $\left\{A_{i} y_{i}\right\}_{1 \leq i \leq r}$ is a basis for $\operatorname{im}(\mathcal{I})$. Set $x_{i}:=A_{i} y_{i}$ and enlarge $\left\{x_{i}\right\}_{1 \leq i \leq r}$ to a basis $\mathcal{B} \cup\left\{x_{i}\right\}_{1 \leq i \leq r}$ for $\mathcal{V}$, where the set $\mathcal{B}$ is linearly independent. Now, let $\left\{f_{i}\right\}_{1 \leq i \leq r}$ be a dual subset with respect to $\mathcal{B} \cup\left\{x_{i}\right\}_{1 \leq i \leq r}$ so that $\langle\mathcal{B}\rangle \subseteq \operatorname{ker} f_{i}$ and $f_{i}\left(x_{j}\right)=\delta_{i j}$ for each $i, j=1, \ldots, r$. We show that $\mathcal{I}=x_{1} \otimes f_{1} \mathcal{L}(\mathcal{V})+\cdots+x_{r} \otimes f_{r} \mathcal{L}(\mathcal{V})$. Define $E_{i}:=x_{i} \otimes f_{i}$. We have $E_{i}=x_{i} \otimes f_{i}=A_{i}\left(y_{i} \otimes f_{i}\right)$. Since $A_{i} \in \mathcal{I}$ and $\mathcal{I}$ is a right ideal in $\mathcal{L}(\mathcal{V})$, it follows that $E_{i}=x_{i} \otimes f_{i} \in \mathcal{I}$ for each $i=1, \ldots, r$, whence $x_{1} \otimes f_{1} \mathcal{L}(\mathcal{V})+\cdots+x_{r} \otimes f_{r} \mathcal{L}(\mathcal{V}) \subseteq \mathcal{I}$. On the other hand, since $\left\{f_{i}\right\}_{1 \leq i \leq r}$ is dual to $\left\{x_{i}\right\}_{1 \leq i \leq r}$ and $\left\{x_{i}\right\}_{1 \leq i \leq r}$ is a basis for $\operatorname{im}(\mathcal{I})$, it is easily seen that $B=x_{1} \otimes f_{1} B+\cdots+x_{r} \otimes f_{r} B=\left(x_{1} \otimes f_{1}+\cdots+x_{r} \otimes f_{r}\right) B$ for all $B \in \mathcal{I}$. So we have shown that $\mathcal{I}=x_{1} \otimes f_{1} \mathcal{L}(\mathcal{V})+\cdots+x_{r} \otimes f_{r} \mathcal{L}(\mathcal{V})=A \mathcal{L}(\mathcal{V})$, where $A \in \mathcal{I}$ is the idempotent $x_{1} \otimes f_{1}+\cdots+x_{r} \otimes f_{r} \in \mathcal{L}(\mathcal{V})$. This finishes the proof in this case.

Next, let $\mathcal{I}$ be a left ideal in $\mathcal{L}(\mathcal{V})$ whose coimage is finite-dimensional. Let $r=\operatorname{dim}(\mathcal{V} / \operatorname{ker} \mathcal{I})$. Choose $x_{1} \notin \operatorname{ker} \mathcal{I}$ so that $A_{1} x_{1} \neq 0$ for some $A_{1} \in \mathcal{I}$. Let $f$ be a linear functional such that $f\left(A_{1} x_{1}\right)=1$. Then $E_{1}=x_{1} \otimes f A_{1}=\left(x_{1} \otimes f\right) A_{1}$ is a rank-one idempotent in $\mathcal{I}$ sending $x_{1}$ to $x_{1}$, annihilating $\operatorname{ker} \mathcal{I}$. Let $1<m \leq r$ be an integer. Assume that
we have found linearly independent vectors $x_{1}+\operatorname{ker} \mathcal{I}, \ldots, x_{m-1}+\operatorname{ker} \mathcal{I}$ in $\mathcal{V} / \operatorname{ker} \mathcal{I}$ and a family of rank-one idempotents $E_{1}, \ldots, E_{m-1}$ in $\mathcal{I}$ such that $E_{i}(\operatorname{ker} \mathcal{I})=0$ and that $E_{i} x_{j}=\delta_{i j} x_{j}$ for $i, j=1, \ldots, m-1$. Therefore, $F_{m}:=E_{1}+\cdots+E_{m-1}$ is an idempotent in $\mathcal{I}$ of rank $m-1$ having $\operatorname{ker} \mathcal{I}$ in its kernel and the vectors $\left\{x_{1}, \ldots, x_{m-1}\right\}$ in its range. Now choose $x_{m}$ in the kernel of $F_{m}$ such that $x_{1}+\operatorname{ker} \mathcal{I}, \ldots, x_{m-1}+$ $\operatorname{ker} \mathcal{I}, x_{m}+\operatorname{ker} \mathcal{I}$ are linearly independent in $\mathcal{V} / \operatorname{ker} \mathcal{I}$. Again, there exists a rank-one idempotent $C_{m} \in \mathcal{I}$ sending $x_{m}$ to itself, annihilating ker $\mathcal{I}$. Obviously, $E_{m}:=C_{m}\left(I-F_{m}\right)=C_{m}-C_{m} F_{m}$ is a rank-one idempotent in $\mathcal{I}$ sending $x_{m}$ to itself and including the range of $F_{m}$ in its kernel. Since $\operatorname{dim}(\mathcal{V} / \operatorname{ker} \mathcal{I})<\infty$, finite induction implies that there exist a basis $\left\{x_{1}+\operatorname{ker} \mathcal{I}, \ldots, x_{r}+\operatorname{ker} \mathcal{I}\right\}$ of $\mathcal{V} / \operatorname{ker} \mathcal{I}$ and a family of rank-one idempotents $\left\{E_{1}, \ldots, E_{r}\right\} \subset \mathcal{I}$ such that $E_{i}(\operatorname{ker} \mathcal{I})=0$ and $E_{i} x_{j}=\delta_{i j} x_{j}$ for all $i, j=1, \ldots, r$. It is plain that we can write $E_{i}=x_{i} \otimes f_{i}$ for some linear functional $f_{i}(i=1, \ldots, r)$. Therefore, we have $f_{i}(\operatorname{ker} \mathcal{I})=0$ and $f_{i}\left(x_{j}\right)=\delta_{i j}$ for all $i, j=1, \ldots, r$. Now, since $x_{i} \otimes f_{i} \in \mathcal{I}$ for all $i=1, \ldots, r$, it follows that $\mathcal{L}(\mathcal{V}) x_{1} \otimes f_{1}+\cdots+\mathcal{L}(\mathcal{V}) x_{r} \otimes f_{r} \subseteq \mathcal{I}$. On the other hand, if $B \in \mathcal{I}$ is arbitrary, as the set $\left\{x_{1}+\operatorname{ker} \mathcal{I}, \ldots, x_{r}+\operatorname{ker} \mathcal{I}\right\}$ is a basis for $\mathcal{V} / \operatorname{ker} \mathcal{I}$, we easily see that $B=B x_{1} \otimes f_{1}+\cdots+B x_{r} \otimes f_{r}$, proving that $\mathcal{I} \subseteq \mathcal{L}(\mathcal{V}) x_{1} \otimes f_{1}+\cdots+\mathcal{L}(\mathcal{V}) x_{r} \otimes f_{r}$. Hence,

$$
\mathcal{I}=\mathcal{L}(\mathcal{V}) x_{1} \otimes f_{1}+\cdots+\mathcal{L}(\mathcal{V}) x_{r} \otimes f_{r}=\mathcal{L}(\mathcal{V}) A
$$

where $A=x_{1} \otimes f_{1}+\cdots+x_{r} \otimes f_{r} \in \mathcal{I}$ is an idempotent. This is what we want, finishing the proof.
(ii) Just note that, if $\operatorname{rank}(A)=r$, then, by the First Isomorphism Theorem for modules (see [1, Theorem IV.1.7]), we have $r=\operatorname{dimim}(A)=$ $\operatorname{dim} \operatorname{coim}(A)$. So (i) above applies, establishing the theorem.

Remark 1.3. If, in the theorem, the space $\mathcal{V}$ were finite-dimensional, then it would be enough to present the proof of the assertion for right ideals, which is short and simple. The proof for left ideals would then follow by taking adjoints.

Theorem 1.4. Let $D$ be a division ring, $\mathcal{V}$ a right or left vector space over $D$, and $\mathcal{I}$ a nonzero right (resp. left) ideal in $\mathcal{L}(\mathcal{V})$ containing a linear transformation $A$ whose rank $r \in \mathbb{N}$ is maximal among all elements of $\mathcal{I}$. Then, there are $x_{i} \in \mathcal{V}$ and $f_{i} \in \mathcal{V}^{\prime}(1 \leq i \leq r)$ which are dual to each other and such that

$$
\mathcal{I}=\operatorname{ri}(A)=A \mathcal{L}(\mathcal{V})=x_{1} \otimes f_{1} \mathcal{L}(\mathcal{V})+\cdots+x_{r} \otimes f_{r} \mathcal{L}(\mathcal{V})
$$

(resp.

$$
\left.\mathcal{I}=\operatorname{li}(A)=\mathcal{L}(\mathcal{V}) A=\mathcal{L}(\mathcal{V}) x_{1} \otimes f_{1}+\cdots+\mathcal{L}(\mathcal{V}) x_{r} \otimes f_{r}\right) .
$$

Moreover, the $x_{i}$ 's can be chosen to be in the range (resp. the complement of the kernel) of $A$. In particular, if $\mathcal{V}$ is finite-dimensional, then the above holds for all right (resp. left) ideals of $\mathcal{L}(\mathcal{V})$.

Proof. The proof is just an imitation of the proof of part (i) of the preceding theorem, which is omitted for the sake of brevity.

On the other hand, if the image (resp. coimage) of the right (resp. left) ideal $\mathcal{I}$ is the whole space, then we can say the following, which characterizes all one-sided ideals of $\mathcal{L}(\mathcal{V})$ that are irreducible.

Theorem 1.5. Let $\mathcal{V}$ be a vector space over a division ring $D$ and $\mathcal{I}$ a right (resp. left) ideal in the ring $\mathcal{L}(\mathcal{V})$. Then the following are equivalent.
(i) $\mathcal{I}$ includes all finite-rank transformations in $\mathcal{L}(\mathcal{V})$.
(ii) $\mathcal{I}$ is irreducible.
(iii) $\operatorname{im}(\mathcal{I})=\mathcal{V}($ resp. $\operatorname{coim}(\mathcal{I})=\mathcal{V}$, which is equivalent to $\operatorname{ker}(\mathcal{I})=$ $0)$.

Therefore, when $\mathcal{V}$ is finite-dimensional, then $\mathcal{I}=\mathcal{L}(\mathcal{V})$ if and only if the right (resp. left) ideal $\mathcal{I}$ is irreducible if and only if $\operatorname{im}(\mathcal{I})=\mathcal{V}$ (resp. $\operatorname{coim}(\mathcal{I})=\mathcal{V})$.

Proof. The implications "(i) $\Longrightarrow$ (ii)" and "(ii) $\Longrightarrow$ (iii)" are obvious. So it suffices to prove (iii) $\Longrightarrow$ (i). To this end, first, let $\mathcal{I}$ be a right ideal in $\mathcal{L}(\mathcal{V})$ whose image is $\mathcal{V}$. To prove the assertion, in view of Theorem 1.2(ii), it suffices to show that $\mathcal{I}$ contains all rank-one linear transformations. Let $x \otimes f$ be an arbitrary rank-one linear transformation, where $x \in \mathcal{V}$ and $f \in \mathcal{V}^{\prime}$. It follows from the hypothesis that there are $A_{j} \in \mathcal{I}$ and $y_{j} \in \mathcal{V}(1 \leq j \leq m)$ such that $x=\sum_{1 \leq j \leq m} A_{j} y_{j}$. We can write

$$
x \otimes f=\left(\sum_{1 \leq j \leq m} A_{j} y_{j}\right) \otimes f=\sum_{1 \leq j \leq m} A_{j}\left(y_{j} \otimes f\right),
$$

implying that $x \otimes f \in \mathcal{I}$ because $A_{j}$ 's belong to $\mathcal{I}$ and $\mathcal{I}$ is a right ideal in $\mathcal{L}(\mathcal{V})$. This is what we wanted.

As for left ideals, let $\mathcal{I}$ be a left ideal in $\mathcal{L}(\mathcal{V})$ whose kernel is zero. Again, in view of Theorem 1.2(ii), it suffices to show that $x \otimes f \in \mathcal{I}$ for all $x \in \mathcal{V}$ and $f \in \mathcal{V}^{\prime}$. To this end, let $x \otimes f \in \mathcal{L}(\mathcal{V})$ be given. It follows
from Lemma 1.1 that $\mathcal{V}^{\prime}=0^{\perp}=\left(\bigcap_{A \in \mathcal{I}} \text { ker } A\right)^{\perp}=\left\langle\cup_{A \in \mathcal{I}} A^{\prime} \mathcal{V}^{\prime}\right\rangle$. This implies that for $f \in \mathcal{V}^{\prime}$, there are $A_{i} \in \mathcal{I}$ and $f_{i} \in \mathcal{V}^{\prime}(1 \leq i \leq m, m \in \mathbb{N})$ such that $f=\sum_{1 \leq i \leq m} A_{i}^{\prime} f_{i}=\sum_{1 \leq i \leq m} f_{i} A_{i}$. We can write

$$
x \otimes f=x \otimes\left(\sum_{1 \leq i \leq m} f_{i} A_{i}\right)=\sum_{1 \leq i \leq m}\left(x \otimes f_{i}\right) A_{i}
$$

implying that $x \otimes f \in \mathcal{I}$, for $A_{i}$ 's belong to $\mathcal{I}$ and $\mathcal{I}$ is a left ideal in $\mathcal{L}(\mathcal{V})$. This settles the proof.

The following result characterizes all one-sided ideals of $\mathcal{L}(\mathcal{V})$ that are triangularizable.

Theorem 1.6. Let $\mathcal{V}$ be a vector space over a division ring $D$ and $\mathcal{I}$ a nonzero right (resp. left) ideal in the ring $\mathcal{L}(\mathcal{V})$. Then the following are equivalent.
(i) $\mathcal{I}$ is triangularizable.
(ii) $\mathcal{I}$ is generated by a rank-one idempotent.
(iii) $\mathcal{I}$ consists of all linear transformations of rank at most one.
(iv) The rank of $A B-B A$ is at most one for all $A, B \in \mathcal{I}$.

In particular, a linear transformation $A \in \mathcal{L}(\mathcal{V})$ has rank one if and only if one of the one-sided ideals generated by $A$ is triangularizable.

Proof. The implications "(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv)" are obvious. We establish the theorem by proving that "(i) $\Longrightarrow$ (ii)" and "(iv) $\Longrightarrow$ (i)".
(i) $\Longrightarrow$ (ii): First, let $\mathcal{I}$ be a triangularizable right ideal. In view of the proof of Theorem 1.2(i), it suffices to show that $\operatorname{im}(\mathcal{I})=\langle A y\rangle$ for some $A \in \mathcal{I}$ and $y \in \mathcal{V}$. To this end, let $\mathcal{C}$ be a maximal chain of subspaces of $\mathcal{V}$ each of which is invariant under the right ideal $\mathcal{I}$. Use $0_{+}$to denote $\bigcap_{0 \neq \mathcal{M} \in \mathcal{C}} \mathcal{M}$. We claim that $0_{+}=\langle A y\rangle=\operatorname{im}(\mathcal{I})$ for any choices of $A \in \mathcal{I}$ and $y \in \mathcal{V}$ for which $A y \neq 0$. Since $\mathcal{C}$ is a maximal chain of subspaces of $\mathcal{V}$, we see that $\operatorname{dim}\left(0_{+}\right) \leq 1$. To prove our claim, it suffices to show that for all $0 \neq \mathcal{M} \in \mathcal{C}, y \in \mathcal{V}$, and $A \in \mathcal{I}$, we have $A y \in \mathcal{M}$. For such an $\mathcal{M}$, choose a nonzero $z \in \mathcal{M}$ and a linear functional $g$ such that $g(z)=1$. We note that the linear transformation $E:=A y \otimes g=A(y \otimes g)$ belongs to the ideal $\mathcal{I}$. We have $A y=E z \in \mathcal{M}$ because $\mathcal{M}$ is invariant under $\mathcal{I}$ and $z \in \mathcal{M}$. This is what we want, finishing the proof for right ideals.

Next, let $\mathcal{I}$ be a triangularizable left ideal. Again, in view of the proof of Theorem 1.2(i), it suffices to show that $\operatorname{coim}(\mathcal{I})=\mathcal{V} / \operatorname{ker} \mathcal{I}=$ $\langle x+\operatorname{ker} \mathcal{I}\rangle$ for some $x \notin \operatorname{ker} \mathcal{I}$. To this end, let $\mathcal{C}$ be a maximal chain of subspaces of $\mathcal{V}$ each of which is invariant under the left ideal $\mathcal{I}$. Use
$\mathcal{V}_{-}$to denote $\bigcup_{\mathcal{V} \neq \mathcal{M} \in \mathcal{C}} \mathcal{M}$. We claim that $\mathcal{I}\left(\mathcal{V}_{-}\right)=0$. To prove this by contradiction, assume that there are $x \in \mathcal{V}_{-}$and $A \in \mathcal{I}$ such that $A x \neq 0$. Since $x \in \mathcal{V}_{-}$, it follows that there is an $\mathcal{M}_{x} \in \mathcal{C}$ such that $x \in \mathcal{M}_{x} \neq \mathcal{V}$. Now, since $\mathcal{M}_{x}$ is a proper subspace, there is a vector $y \in \mathcal{V} \backslash \mathcal{M}_{x}$. Let $f$ be a linear functional such that $f(A x)=1$. Then the rank-one linear transformation $E:=y \otimes f A=(y \otimes f) A$ belongs to $\mathcal{I}$ because $\mathcal{I}$ is a left ideal and $A \in \mathcal{I}$. On the other hand, as the space $\mathcal{M}_{x}$ is invariant under $\mathcal{I}, E \in \mathcal{I}$, and $x \in \mathcal{M}_{x}$, we must have $y=E x \in \mathcal{M}_{x}$, which is impossible. Therefore, $\mathcal{I}\left(\mathcal{V}_{-}\right)=0$, yielding $\mathcal{V}_{-} \subset \operatorname{ker}(\mathcal{I}) \subset \mathcal{V}$. From this, we see that $\operatorname{ker}(\mathcal{I})=\mathcal{V}_{-}$and $\operatorname{dim} \mathcal{V} / \mathcal{V}_{-}=1$ because the left ideal $\mathcal{I}$ is nonzero and that the chain $\mathcal{C}$ is a maximal chain of subspaces of $\mathcal{V}$. That is, we have shown that $\operatorname{dim}(\operatorname{coim}(\mathcal{I}))=\operatorname{dim}(\mathcal{V} / \operatorname{ker} \mathcal{I})=1$ which is what we wanted, finishing the proof.
(iv) $\Longrightarrow$ (i): First, let $\mathcal{I}$ be a right ideal in $\mathcal{L}(\mathcal{V})$ with the property that the rank of $A B-B A$ is at most one for all $A, B \in \mathcal{I}$. We claim that $\operatorname{im}(\mathcal{I})$ is one-dimensional. If not, choose two independent vectors $x_{i}:=$ $A_{i} y_{i} \in \operatorname{im}(\mathcal{I})(i=1,2)$. From the proof of Theorem 1.2(i), we see that we can obtain linear functionals $f_{i}$ 's so that they are dual to $x_{i}$ 's $(i=1,2)$ such that $x_{1} \otimes f_{2}, x_{2} \otimes f_{1} \in \mathcal{I}$ whence $x_{1} \otimes f_{2} \mathcal{L}(\mathcal{V})+x_{2} \otimes f_{1} \mathcal{L}(\mathcal{V}) \subseteq \mathcal{I}$. Then, we can write

$$
\left(x_{1} \otimes f_{2}\right)\left(x_{2} \otimes f_{1}\right)-\left(x_{2} \otimes f_{1}\right)\left(x_{1} \otimes f_{2}\right)=x_{1} \otimes f_{1}-x_{2} \otimes f_{2}
$$

implying that the linear transformation $\left(x_{1} \otimes f_{2}\right)\left(x_{2} \otimes f_{1}\right)-\left(x_{2} \otimes f_{1}\right)\left(x_{1} \otimes\right.$ $\left.f_{2}\right) \in \mathcal{I}$ has rank two, contradicting the hypothesis. Therefore, $\operatorname{im}(\mathcal{I})$ is one-dimensional. Now, mimicking the proof of Theorem 1.2(i), we see that $\mathcal{I}=x \otimes f \mathcal{L}(\mathcal{V})$ for some $x=A y \in \operatorname{im}(\mathcal{I})$ spanning $\operatorname{im}(\mathcal{I})$ and any linear functional $f$ for which $f(x)=1$. It is now plain that $\mathcal{I}$ is triangularizable which is what we want.

Next, let $\mathcal{I}$ be a left ideal in $\mathcal{L}(\mathcal{V})$ satisfying the hypothesis. We claim that $\mathcal{V} / \operatorname{ker} \mathcal{I}$ is one-dimensional. If not, mimicking the proof of Theorem 1.2(i) for left ideals, we can obtain linearly independent vectors $x_{1}+\operatorname{ker} A, x_{2}+\operatorname{ker} A \in \mathcal{V} / \operatorname{ker} \mathcal{I}$ and linear functionals $f_{1}, f_{2} \in \mathcal{V}^{\prime}$ such that $f_{i}(\operatorname{ker} \mathcal{I})=0$, that $f_{i}\left(x_{j}\right)=\delta_{i j}$ for all $i, j=1,2$ and that $x_{1} \otimes$ $f_{1}, x_{2} \otimes f_{2} \in \mathcal{I}$. Now, choose rank-one linear transformations $A, B$ taking $x_{1}$ to $x_{2}$ and $x_{2}$ to $x_{1}$, respectively. From this, it follows that $x_{2} \otimes f_{1}=$ $A x_{1} \otimes f_{1} \in \mathcal{I}$ and $x_{1} \otimes f_{2}=B x_{2} \otimes f_{2} \in \mathcal{I}$ whence $\mathcal{L}(\mathcal{V}) x_{1} \otimes f_{2}+\mathcal{L}(\mathcal{V}) x_{2} \otimes$ $f_{1} \subseteq \mathcal{I}$. Therefore, the linear transformation $\left(x_{1} \otimes f_{2}\right)\left(x_{2} \otimes f_{1}\right)-\left(x_{2} \otimes\right.$ $\left.f_{1}\right)\left(x_{1} \otimes f_{2}\right) \in \mathcal{I}$ has rank two which contradicts the hypothesis. So we conclude that $\mathcal{V} / \operatorname{ker} \mathcal{I}$ is one-dimensional. Again, mimicking the proof
of Theorem $1.2(\mathrm{i})$, we will see that $\mathcal{I}=\mathcal{L}(\mathcal{V}) x \otimes f A$, where $x \notin \operatorname{ker} \mathcal{I}$, $A \in \mathcal{I}, A x \neq 0$, and $f$ is any linear functional for which $f(A x)=1$. It is now plain that $\mathcal{I}$ is triangularizable which is what we want.

Let $D^{n}$ (resp. $\left.D_{n}=\left(D^{n}\right)^{\prime}\right)$ denote the right (resp. left) vector space of $n \times 1$ column (resp. $1 \times n$ row) vectors with entries in $D$; that is, the addition $x+y$ is defined componentwise and the multiplication of the scalar $\lambda \in D$ into the vector $x=\left(x_{i}\right)_{i=1}^{n} \in D^{n}$ (resp. $D_{n}$ ) is defined by $x \lambda:=\left(x_{i} \lambda\right)_{i=1}^{n}\left(\right.$ resp. $\left.\lambda x:=\left(\lambda x_{i}\right)_{i=1}^{n}\right)$. The members of $M_{n}(D)$ can be viewed as linear transformations acting on the left of $D^{n}$ (resp. right of $D_{n}=\left(D^{n}\right)^{\prime}$ ) via the usual matrix multiplication; that is, we can write $M_{n}(D)=\mathcal{L}\left(D^{n}\right)\left(\right.$ resp. $\left.M_{n}(D)=\mathcal{L}\left(D_{n}\right)\right)$, where $\mathcal{L}\left(D^{n}\right)\left(\right.$ resp. $\left.\mathcal{L}\left(D_{n}\right)\right)$ is the ring of right (resp. left) linear transformations acting on the left of $D^{n}$ (resp. right of $D_{n}$ ).

Here is a quick consequence of the preceding results.
Corollary 1.7. Let $D$ be a division ring, and $n \in \mathbb{N}$. Then the following hold.
(i) The ideal $M_{n}(D)$ is the only irreducible left (resp. right) ideal in $M_{n}(D)$.
(ii) The only triangularizable one-sided ideals of $M_{n}(D)$ are those of the form $A M_{n}(D)$ or $M_{n}(D) A$ for some rank-one idempotent $A \in$ $M_{n}(D)$.

Proof. Just note that, as explained above, $M_{n}(D)=\mathcal{L}\left(D^{n}\right)$ and that $D^{n}$ is $n$-dimensional. Thus, Theorem 1.5 and Theorem 1.6 apply, proving (i) and (ii) above.

Let $<R,+, .>$ be a ring. We use the symbol $R^{o p}$ to denote the opposite ring of $R$ which is defined as the ring $<R,+, \circ>$ where $a \circ$ $b:=b . a$. Here are some easy-to-check facts about opposite rings. (i) A ring $R$ is unital if and only if its opposite ring is unital; (ii) A ring $R$ is a division ring if and only if $R^{o p}$ is; (iii) $\left(R^{o p}\right)^{o p}=R$. If $A, B \in$ $M_{n}\left(R^{o p}\right)$, then the (opposite) product of $A$ and $B$, denoted by $A \circ B$, is said to be the $n \times n$ matrix whose $i j$ entry is defined by $(A \circ B)_{i j}=$ $\sum_{i=1}^{n} a_{i k} \circ b_{k j}=\sum_{i=1}^{n} b_{k j} a_{i k}$. If $A^{t}$ and $B^{t}$ denote the transposes of $A, B \in M_{n}(R)$, it is then easy to check that $(A B)^{t}=B^{t} \circ A^{t}$. From this, we see that if $S \in M_{n}(R)$ is invertible, i.e., there is $S^{-1} \in M_{n}(R)$ such that $S^{-1} S=S S^{-1}=I_{n}$, then $S^{t} \circ\left(S^{-1}\right)^{t}=\left(S^{-1}\right)^{t} \circ S^{t}=I_{n}$,
implying that $\left(S^{t}\right)^{-1}=\left(S^{-1}\right)^{t}$ in $M_{n}\left(R^{o p}\right)$. It is worth mentioning that if $\mathcal{V}$ is any $n$-dimensional right (resp. left) vector space over a division ring $D$, then, by fixing a basis for $\mathcal{V}$, we see that the correspondence of a linear transformation to its matrix representation with respect to the fixed basis defines a ring isomorphism from $\mathcal{L}(\mathcal{V})$ onto $M_{n}(D)$ (resp. $M_{n}\left(D^{o p}\right)$ [1, Theorem VII.1.4]). Let $R$ be a ring. For $A \in M_{n}(R)$, we use $\operatorname{col}_{i}(A)$ (resp. $\left.\operatorname{row}_{i}(A)\right)$ to denote the $i$-th column (resp. row) of the matrix $A$. For $\mathcal{F} \subset M_{n}(D)$, the $i$ th $\operatorname{column}^{\operatorname{col}}(\mathcal{F})$ (resp. the $i$ th row $\operatorname{row}_{i}(\mathcal{F})$ ) of $\mathcal{F}$ is defined to be the collection of all $i$ th columns (resp. rows) of the members of $\mathcal{F}$. Let $\mathcal{I}$ be a left (resp. right) ideal in $M_{n}(D)$. It is not difficult to see that the columns (resp. rows) of the ideal $\mathcal{I}$ are either $D^{n}$ or $0^{n}$ (resp. $D_{n}$ or $0_{n}$ ) (see Lemma 1.8 below). In view of this, the $i$ th and $j$ th columns (resp. rows) of $\mathcal{I}(1 \leq i, j \leq n, i \neq j)$ are said to be linked if there exists a nonzero $a \in D$ such that $\operatorname{col}_{j}(A)=\operatorname{col}_{i}(A) \cdot a$ $\left(\operatorname{resp} \operatorname{row}_{j}(A)=a \operatorname{row}_{i}(A)\right)$ for all $A \in \mathcal{I}$.

Also, we say that the $i$ th and $j$ th columns of $\mathcal{I}(1 \leq i, j \leq n, i \neq$ $j$ ) are independent if $\left\{\left(\operatorname{col}_{i}(A), \operatorname{col}_{j}(A)\right): A \in \mathcal{I}\right\}=D^{n} \times D^{n}$ (resp. $\left.\left\{\left(\operatorname{row}_{i}(A), \operatorname{row}_{j}(A)\right): A \in \mathcal{I}\right\}=D_{n} \times D_{n}\right)$.

In order to present our next result, which characterizes a division ring $D$ in terms of the one-sided ideals of $M_{n}(D)$, we need the following lemma. It is worth noting that the equivalence of the parts (i)-(iv) of the lemma is well-known. However, for reader's convenience, we include a proof of the well-known parts as hinted in [1, Exercise III.2.7].

Lemma 1.8. Let $D$ be a ring such that $D^{2} \neq 0$. Then the following are equivalent.
(i) The ring $D$ is a division ring.
(ii) Zero is the only proper left ideal in $D$.
(iii) Zero is the only proper right ideal in $D$.
(iv) Zero is the only proper two-sided ideal in $D$ and that $D$ has the property that for all $x, y \in D$ there is $z \in D$ such that $x y=z x$.
(v) The only left ideals in $M_{n}(D)$ are those collections whose columns are $0^{n}$ or $D^{n}$.
(vi) The only right ideals in $M_{n}(D)$ are those collections whose rows are $0_{n}$ or $D_{n}$.

Moreover, if $D$ is a division ring and $\mathcal{I}$ a left (resp. right) ideal in $M_{n}(D)$, then every two nonzero columns (resp. rows) of the ideal $\mathcal{I}$ are either linked or they are independent.

Proof. The implications "(i) $\Longrightarrow$ (ii)" and "(i) $\Longrightarrow$ (iii)" are obvious.
(ii) $\Longrightarrow$ (i): From $D^{2} \neq 0$, it follows that the ideal $\{x \in D: D x=0\}$ is $\{0\}$ because, otherwise $\{x \in D: D x=0\}=D$, implying $D^{2}=0$, a contradiction. Let $d \in D$ be nonzero. Then the set $\{x \in D: x d=0\}$ is a proper left ideal of $D$, implying that $\{x \in D: x d=0\}=\{0\}$. Thus, from $x_{1} d=x_{2} d$, we see that $x_{1}=x_{2}$. On the other hand, if $d x_{1}=d x_{2}$ for some $x_{1}, x_{2} \in D$, then $x_{1}=x_{2}$, for, otherwise the left ideal $\{x \in D$ : $\left.x\left(x_{1}-x_{2}\right)=0\right\}$ is nonzero, and hence $\left\{x \in D: x\left(x_{1}-x_{2}\right)=0\right\}=D$ a contradiction. So the cancelation law holds in $D$. Now, since $D d$ is a nonzero left ideal in $D$, it follows that $D d=D$. Therefore, there is an $e \in D$ such that $e d=d$. Letting $x \in D$ be arbitrary, we can write $x e d=x d$. Canceling $d$, we get $x e=x$ for all $x \in D$. In particular, $d e=d$. Hence, $d e x=d x$ for all $x \in D$. Canceling $d$, we get $e x=x$ for all $x \in D$. Therefore, $e$ is the identity element of $D$. Now, let $x$ be a nonzero element of $D$. We have $D x=D$, for $D x$ is a nonzero left ideal in $D$. It follows that, there is $x^{\prime} \in D$ such that $x^{\prime} x=e$. We can write $x^{\prime} x x^{\prime}=e x^{\prime}=x^{\prime}=x^{\prime} e$, yielding $x x^{\prime}=e$, in view of the cancelation property. That is, every nonzero member of $D$ has an inverse. Therefore, $D$ is a division ring, as desired (see [1, Exercise III.2.7]).
(iii) $\Longrightarrow$ (i): This is similar to "(ii) $\Longrightarrow$ (i)".
(i) $\Longrightarrow$ (iv): This is obvious.
(iv) $\Longrightarrow$ (i): Since for all $x, y \in D$ there is $z \in D$ such that $x y=z x$, it follows that every left ideal in $D$ is a two-sided ideal. Therefore, zero is the only proper left ideal in $D$. So $D$ is a division ring by the equivalence of (i) and (ii).
(i) $\Longrightarrow(\mathrm{v})$ : Let $\mathcal{I}$ be a left ideal of $M_{n}(D)$. For $1 \leq k, l \leq n$ and $A \in M_{n}(D)$, let $(A)_{k l}$ denote the $k l$ entry of the matrix $A$. With this in mind, define $J_{k l}:=\left\{a \in D: \exists A \in \mathcal{I}\right.$ such that $\left.(A)_{k l}=a\right\}$. Multiplying the members of $\mathcal{I}$ from the left by the elementary matrix interchanging rows 1 and $k$ of the identity matrix, we will see that $J_{k l}=J_{1 l}:=J_{l}$ for all $1 \leq k, l \leq n$. Since $\mathcal{I}$ is a left ideal in $M_{n}(D)$, it follows that $J_{l}$ is a left ideal of $D$ for each $l=1, \ldots, n$. Therefore, $J_{l}=0$ or $D$, for $D$ is a division ring. Now, let $l \in\{1, \ldots, n\}$ and $d_{1}, d_{2} \in D$ be given. If there are matrices $A, B \in M_{n}(D)$ and $1 \leq i_{1}, i_{2} \leq n$ with $i_{1} \neq i_{2}$ such that $(A)_{i_{1} l}=d_{1}$ and $(B)_{i_{2} l}=d_{2}$, then we will have $(C)_{i_{1} l}=d_{1}$ and $(C)_{i_{2} l}=d_{2}$, where $C=A_{1} A+B_{1} B, A_{1}=\operatorname{diag}\left(\delta_{i_{1} 1}, \ldots, \delta_{i_{1} n}\right)$, $B_{1}=\operatorname{diag}\left(\delta_{i_{2} 1}, \ldots, \delta_{i_{2} n}\right)$, and where $\delta$ denotes the Kronecker delta. This shows that the columns of the left ideal $\mathcal{I}$ are $0^{n}$ or $D^{n}$, which is what we want.
$(\mathrm{v}) \Longrightarrow(\mathrm{i})$ : Let $I$ be a nonzero left ideal of $D$. In view of the implication "(ii) $\Longrightarrow(\mathrm{i})$ ", it suffices to show that $I=D$. To this end, just note that the set $\left(I^{n}, 0^{n}, \ldots, 0^{n}\right)$, where $I^{n}$ and $0^{n}$ denote the $n \times 1$ column vectors with entries in $I$ and 0 respectively, is a nonzero left ideal in $M_{n}(D)$. So it follows from the hypothesis that $I^{n}=D^{n}$, implying that $I=D$ which is what we want.
(i) $\Longrightarrow$ (vi) and (vi) $\Longrightarrow$ (i): These can be done by an easy modification of the proofs of the preceding two implications. We omit the details.

For the rest, we prove the assertion for left ideals. The proof for right ideals can be done similarly. Let $D$ be a division ring, $\mathcal{I}$ be a left ideal in $M_{n}(D)$, and $\operatorname{col}_{l_{1}}(\mathcal{I})$ and $\operatorname{col}_{l_{2}}(\mathcal{I})$ be nonzero and distinct. For the columns $l_{1}$ and $l_{2}$ of $\mathcal{I}\left(1 \leq l_{1}<l_{2} \leq n\right)$ there are two cases to consider: (a) there is an $A \in \mathcal{I}$ such that $(A)_{1 l_{1}}=0$ but $(A)_{1 l_{2}} \neq 0$, and (b) otherwise. In case (a), since $\operatorname{col}_{l_{j}}(\mathcal{I})=D^{n}$, it follows that there is a $B \in$ $M_{n}(D)$ such that $(B)_{1 l_{1}} \neq 0$. The matrix $B \in \mathcal{I}$ can be chosen so that $(B)_{1 l_{2}}=0$ because for the matrix $C:=B-b a^{-1} A \in \mathcal{I}$ with $b=(B)_{1 l_{2}}$, $a=(A)_{1 l_{2}}$, we have $(C)_{1 l_{1}} \neq 0$ and $(C)_{1 l_{2}}=0$. Now, since $\mathcal{I}$ is a left ideal in $M_{n}(D)$, since $D$ is a division ring, and since such matrices $A$ and $B$ in the ideal $\mathcal{I}$ exist, it easily follows that $\left\{\left((X)_{l_{1} 1},(X)_{l_{2} 1}\right): X \in \mathcal{I}\right\}=$ $D \times D$. As the rows of the left ideal $\mathcal{I}$ can arbitrarily be interchanged, we conclude that $\left\{\left(\operatorname{col}_{l_{1}}(X), \operatorname{col}_{l_{2}}(X)\right): X \in \mathcal{I}\right\}=D^{n} \times D^{n}$, which is what we want. In case (b), we see that the mapping $f: D \rightarrow D$ defined by $f\left((C)_{l_{1} 1}\right)=(C)_{l_{2} 1}$ for all $C \in \mathcal{I}$ is well-defined and, in fact, $f$ is a nonzero left linear transformation from $D$ into $D$. Thus, we have $f(x)=x f(1)$ for all $x \in D$, where $0 \neq f(1) \in D$. Hence, the mapping $f$ is one-to-one and onto. Again, as the rows of the left ideal $\mathcal{I}$ can arbitrarily be interchanged, we conclude that $\operatorname{col}_{l_{2}}(A)=\operatorname{col}_{l_{1}}(A) f(1)$ for all $A \in \mathcal{I}$. This is what we want, completing the proof.

Question. Let $R$ be a ring. Is there a description of one-sided ideals of the matrix ring $M_{n}(R)$ in terms of those of the ring $R$ ?

The following result characterizes a division ring $D$ in terms of the one-sided ideals of $M_{n}(D)$. It is worth mentioning that the implications "(i) $\Longrightarrow$ (ii)" and "(i) $\Longrightarrow$ (iii)" in the theorem below are likely known to the experts.

Theorem 1.9. Let $D$ be a unital ring. Then the following are equivalent.
(i) The ring $D$ is a division ring.
(ii) Up to similarity, the only left ideals in $M_{n}(D)$ are those that consist of all matrices in $M_{n}(D)$ whose first $r$ columns are completely arbitrary and whose last $n-r$ columns are zero, where $r \leq n$ is a nonnegative integer depending on the ideal.
(iii) Up to similarity, the only right ideals in $M_{n}(D)$ are those that consist of all matrices in $M_{n}(D)$ whose first r rows are completely arbitrary and whose last $n-r$ rows are zero, where $r \leq n$ is a nonnegative integer depending on the ideal.

If $D$ is a division ring, then the integer $r$ in (i) (resp. in (ii)) above is the dimension of the coimage (resp. the image) of the left (resp. the right) ideal.

Proof. The implications "(i) $\Longrightarrow$ (ii)" and "(i) $\Longrightarrow$ (iii)" are quick consequences of Theorem 1.2. Let $\mathcal{I}$ be a one-sided ideal in $M_{n}(D)$. View the members of $M_{n}(D)$ as right linear transformations acting on the left of the right vector space $D^{n}$. With that in mind, apply Theorem 1.2 (i) to obtain the independent set $\left\{x_{i}\right\}_{1 \leq i \leq r}$ and extend it to a basis $\left\{x_{i}\right\}_{1 \leq i \leq r} \cup \mathcal{B}$ for $D^{n}$. Now, let $\left\{f_{i}\right\}_{1 \leq i \leq r}$ be a dual subset with respect to $\left\{x_{i}\right\}_{1 \leq i \leq r}$ such that $f_{i}\left(x_{j}\right)=\delta_{i j}$ and $\langle\mathcal{B}\rangle \subseteq \operatorname{ker} f_{i}$ for each $i, j=1, \ldots, r$ so that the conclusion of Theorem $1.2(\mathrm{i})$ is true for the one-sided ideal $\mathcal{I}$. It is now easily seen that the matrix representations of the members of $\mathcal{I}$ with respect to the basis $\left\{x_{i}\right\}_{1 \leq i \leq r} \cup \mathcal{B}$ for $D^{n}$ have the desired form described in (ii) and (iii) above, which is what we want.
(ii) $\Longrightarrow$ (i): First we show that the ring $D$ has the property that for all $a, b \in D, a b=1$ if and only if $b a=1$. And this, in view of [5, Theorem 3.2.37], follows as soon as we show that the ring $D$ is left Noetherian. To see this, it suffices to prove that the left ideals of $D$ are finitely generated. To this end, let $I$ be a left ideal of $D$. We consider the left ideal $\left(I^{n}, 0^{n}, \ldots, 0^{n}\right)$ in $M_{n}(D)$, where $I^{n}$ and $0^{n}$ denote the $n \times 1$ column vectors with entries in $I$ and 0 , respectively. Then, by the hypothesis, the left ideal $\left(I^{n}, 0^{n}, \ldots, 0^{n}\right)$ is similar to the left ideal $\left(D^{n}, \ldots, D^{n}, 0^{n}, \ldots, 0^{n}\right)$ in which $D^{n}$ 's and $0^{n}$ 's occur $r$ and $n-r$ times, respectively, for some nonnegative integer $r \leq n$. Thus, there is an $S \in G L_{n}(D)$ such that $S\left(I^{n}, 0^{n}, \ldots, 0^{n}\right) S^{-1}=$ $\left(D^{n}, \ldots, D^{n}, 0^{n}, \ldots, 0^{n}\right)$. From this, since $S$ is invertible, it easily follows that $\left(I^{n}, 0^{n}, \ldots, 0^{n}\right)=\left(D^{n}, \ldots, D^{n}, 0^{n}, \ldots 0^{n}\right) S$, whence the ideal $I$ is finitely generated. In fact, we have $I=D s_{11}+\cdots+D s_{r 1}$, where $r$ is the number of times that $D^{n}$ appears in $\left(I^{n}, 0^{n}, \ldots, 0^{n}\right)=$ $\left(D^{n}, \ldots, D^{n}, 0^{n}, \ldots 0^{n}\right) S$ and that $s_{i j}$ denotes the $i j$ entry of the matrix
$S$. This shows that every left ideal $I$ of the ring $D$ is finitely generated, implying that the ring $D$ is left Noetherian.

Now, let $I$ be an arbitrary left ideal of the unital ring $D$ that is not zero. In view of Lemma 1.8 , it suffices to show that $I=D$. Following the argument presented in the preceding paragraph, if we let $T=S^{-1}$, we will obtain $\left(I^{n}, 0^{n}, \ldots, 0^{n}\right) T=\left(D^{n}, \ldots, D^{n}, 0^{n}, \ldots 0^{n}\right)$. This implies that $I t_{11}=D$, where $t_{11}$ denotes the 11 entry of the matrix $T$. This, in particular, shows that there is an $i \in I$ such that $i t_{11}=1$. Since $D$ is left Noetherian, it follows that $1=t_{11} i \in I$, implying that $I=D$. This is what we want, finishing the proof.
(iii) $\Longrightarrow$ (i): Let $\mathcal{I}$ be an arbitrary left ideal in $M_{n}\left(D^{o p}\right)$. Set $\mathcal{I}^{t}=$ $\left\{A^{t} \in M_{n}(D): A \in \mathcal{I}\right\}$. In view of the identity $(A \circ B)^{t}=B^{t} A^{t}$, we see that $\mathcal{I}^{t}$ is a right ideal in $M_{n}(D)$. So it follows from the hypothesis that there is an $S \in G L_{n}(D)$ such that $S^{-1} \mathcal{I}^{t} S=\left(D_{n}, \ldots, D_{n}, 0_{n}, \ldots, 0_{n}\right)$ in which $D_{n}$ 's and $0_{n}$ 's occur $r$ and $n-r$ times, respectively, for some nonnegative integer $r \leq n$. Taking transpose of both sides of the above equality and again noting that $(A \circ B)^{t}=B^{t} A^{t}$ for all $A^{t}, B^{t} \in$ $M_{n}(D)$ and that $\left(S^{t}\right)^{-1}=\left(S^{-1}\right)^{t}$, we conclude that $S^{t} \circ \mathcal{I} \circ\left(S^{t}\right)^{-1}=$ $\left(D^{n}, \ldots, D^{n}, 0^{n}, \ldots, 0^{n}\right)$. That is, every left ideal in $M_{n}\left(D^{o p}\right)$ is of the form described in (ii). So the preceding implication applies. Therefore, the ring $D^{o p}$ is a division ring which, in turn, implies that so is $D$, finishing the proof.

Corollary 1.10. Let $D$ be a division ring, $n$ a positive integer, and $A \in M_{n}(D)$. Then the following are equivalent.
(i) $\operatorname{rank}(A)=r$.
(ii) Up to similarity, the left ideal generated by $A$ in $M_{n}(D)$ is the left ideal that consists of all matrices in $M_{n}(D)$ whose first $r$ columns are completely arbitrary and whose last $n-r$ columns are zero.
(iii) Up to similarity, the right ideal generated by $A$ in $M_{n}(D)$ is the right ideal that consists of all matrices in $M_{n}(D)$ whose first rows are completely arbitrary and whose last $n-r$ rows are zero.

Proof. This is a quick consequence of Theorem 1.2(ii) in the same way as the implications"(i) $\Longrightarrow$ (ii)" and "(i) $\Longrightarrow$ (iii)" of the preceding theorem are consequences of Theorem 1.2(i). We omit the details.

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[^0]:    MSC(2000): Primary: 15A33, 15A30; Secondary:16S50
    Keywords: Matrix ring, Division ring, One-sided ideal, Linear transformation, (Algebraic) (Ir)reducibility, (Algebraic) Triangularizability, (Algebraic) Dual, (Algebraic) Adjoint, Finite-rank linear transformation
    The second author was supported by a grant from IPM (No. 83470318), the Mahani Mathematical Center of the University of Kerman, and NSERC
    Received: 18 August 2007, Accepted: 7 October 2007
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