ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 2, pp. 315–326

Title:

An analytic solution for a non-local initial-boundary value problem including a partial differential equation with variable coefficients

Author(s):

M. Jahanshahi and M. Darabadi

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 2, pp. 315–326
Online ISSN: 1735-8515

AN ANALYTIC SOLUTION FOR A NON-LOCAL INITIAL-BOUNDARY VALUE PROBLEM INCLUDING A PARTIAL DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

M. JAHANSHAHI AND M. DARABADI*

(Communicated by Mohammad Asadzadeh)

ABSTRACT. This paper considers a non-local initial-boundary value problem containing a first order partial differential equation with variable coefficients. At first, the non-self-adjoint spectral problem is derived. Then its adjoint problem is calculated. After that, for the adjoint problem the associated eigenvalues and the subsequent eigenfunctions are determined. Finally the convergence of series solution and the uniqueness of this solution will be proved.
Keywords: Partial differential equation, boundary value problem, self adjoint problem, non-self adjoint operators, non-local-boundary conditions
MSC(2010): Primary: 35801; Secondary: 35808.

1. **Introduction**

Initial-boundary value problems are usually discussed in classic text books, when the problems are self-adjoint and the variables of related partial differential equations are real and complex constants [7, 17]. These problems are generally the mathematical models for physics and engineering problems. When these problems contain partial differential equations with variable coefficients in a non-self-adjoint case, they are often unsolved problems. Some of these problems, when including variable coefficients as well as Navier-Stokes system [1, 12] and Benjamin-Ono equation [13] have been solved by authors. On the other hand, partial differential equations with variable coefficients appear in the other fields of mathematical physics such as inverse Sturm-Liouville problems when the potential functions and the other unknowns involve time

Article electronically published on April 30, 2016.
Received: 9 September 2014, Accepted: 11 January 2015.
*Corresponding author.

and space variables

$$
\begin{cases}
\dfrac{\partial^2 A(x,t)}{\partial x^2} - 2p(x)\dfrac{\partial B(x,t)}{\partial t} - q(x)A(x,t) = \dfrac{\partial^2 A(x,t)}{\partial t^2} \\[2ex]
\dfrac{\partial^2 B(x,t)}{\partial x^2} + 2p(x)\dfrac{\partial A(x,t)}{\partial t} - q(x)B(x,t) = \dfrac{\partial^2 B(x,t)}{\partial t^2}.
\end{cases} \tag{1.1}
$$

As is easily observed, theses equations have variable coefficients [8, 9]. The general form of these equations can be written in the following form when the unknown functions include two spatial variables and one time variable

$$
\frac{\partial^2 u(x_1,x_2,t)}{\partial t^2} = F(x_1,x_2)\frac{\partial^2 u(x_1,x_2,t)}{\partial x_2^2} + G(x_1,x_2,t)\frac{\partial^2 u(x_1,x_2,t)}{\partial x_1^2}, \tag{1.2}
$$

We will consider the equation (1.2) for the case of first order derivative with respect to time variable and spatial variables. For the second order case, the problem can be considered as an open problem.

Remark 1.1. It is intended to say that, one can consider and solve the above mentioned second order case by using the same process applied for the first case in this paper. we will consider equation (1.2) with separable case of variables. That is

$$
\frac{\partial^2 u(x_1,x_2,t)}{\partial t^2} = f_2(x_2)\frac{\partial^2 u(x_1,x_2,t)}{\partial x_2^2} + f_1(x_1)\frac{\partial^2 u(x_1,x_2,t)}{\partial x_1^2}, \tag{1.3}
$$

and for the first order:

$$
\frac{\partial u(x_1,x_2,t)}{\partial t} = f_2(x_2)\frac{\partial u(x_1,x_2,t)}{\partial x_2} + f_1(x_1)\frac{\partial u(x_1,x_2,t)}{\partial x_1}, \tag{1.4}
$$

Remark 1.2. Equation (1.4) with some non-local-boundary conditions and initial condition has been considered by authors with real and complex constants [14]. These problems have solved by the Fourier method the cases of self adjoint and non-self-adjoint problems.

It is also worth mentioning that the selfadjoint and non-self-adjoint boundary value problems have appeared in some of physics and engineering problems. For example A.Burchard et al in [4] reached a non-self-adjoint differential operators that appear as linearization of coating and rimming flows, where a thin layer of fluid coats a horizontal rotating cylinder. In [5] R.Carlson also applied the adjoint and self-adjoint differential operators on graphs. He has shown a directed graph with weighted edges can be characterized as a system of ordinary differential operators.

On the other hand, over the last decades, boundary value problems with non-local-boundary conditions have an important role in many area of researches. In these problems the values of the unknown functions on the boundary are connected to each other some of the values of the given domain, such as the boundary conditions are called non-local-boundary conditions [2, 3, 6, 10].

Authors in [11, 15, 16] have considered some complex constants in partial differential equations in non-classic cases as well as non-self-adjoint problems with non-local and non-periodic conditions.

2. Mathematical statement of problem

Consider the initial-boundary value problem

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = f_2(x_2)\frac{\partial u(x,t)}{\partial x_2} + f_1(x_1)\frac{\partial u(x,t)}{\partial x_1}, \\ &x=(x_1,x_2)\in D=\{x|x_j\in(0,1), j=1,2\}\subset R^2, \quad t>0, \end{aligned} \tag{2.1}$$

with the non local boundary conditions

$$\begin{cases} u(0,x_2,t)=a_1u(1,x_2,t), & x_2\in[0,1], \quad t\geq 0, \\ u(x_1,0,t)=a_2u(x_1,1,t), & x_1\in[0,1], \quad t\geq 0, \end{cases} \tag{2.2}$$

and initial condition

$$u(x,0)=\varphi(x), \quad x\in\overline{D}, \tag{2.3}$$

where a_1 and a_2 are arbitrary complex constants and $\varphi(x)$ is a known continuous function with a complex variable. For this problem, we consider the related spectral problem such that it is not self-adjoint. At first we calculate its spectral problem. Hence its eigenvalues are not real and the eigenfunctions do not form an orthogonal basis system.

To construct the solution as infinite series, we will use the eigenfunctions of the main spectral problem and the related adjoint problem.

3. Spectral problem of the main problem

We assume that the solution of (2.1) is in the form $u(x,t)=X(x)T(t)$ where X and T are functions of x and t respectively, and they are twice continuously differentiable functions (see [6]). Therefore

$$f_2(x_2)\frac{\partial X(x)}{\partial x_2} + f_1(x_1)\frac{\partial X(x)}{\partial x_1} = \lambda X(x), \quad x\in D, \quad \lambda\in\mathbb{C}, \tag{3.1}$$

And

$$T^{'}(t)=\lambda T(t), \quad t>0. \tag{3.2}$$

Let L be the operator of equation (3.1), i.e.

$$L = f_2(x_2)\frac{\partial}{\partial x_2} + f_1(x_1)\frac{\partial}{\partial x_1} - \lambda.$$

Separating the boundary conditions (2.2) yields

$$\begin{cases} X(0,x_2)=a_1X(1,x_2), & x_2\in[0,1], \\ X(x_1,0)=a_2X(x_1,1), & x_1\in[0,1]. \end{cases} \tag{3.3}$$

By using twice "the method of separation of variables" $X(x) = X_1(x_1)X_2(x_2)$ we have

$$f_2(x_2)\frac{X_2'(x_2)}{X_2(x_2)} + f_1(x_1)\frac{X_1'(x_1)}{X_1(x_1)} = \lambda, \tag{3.4}$$

and the boundary conditions (3.3) will be

$$X_1(0) = a_1X_1(1), \quad X_2(0) = a_2X_2(1).$$

If we let

$$f_1(x_1)\frac{X_1'(x_1)}{X_1(x_1)} = \rho, \qquad f_2(x_2)\frac{X_2'(x_2)}{X_2(x_2)} = \mu,$$

where $\lambda = \mu + \rho$. Then two boundary value problems

$$\begin{cases} f_1(x_1)X_1'(x_1) = \rho X_1(x_1), \quad x_1 \in (0,1), \\ X_1(0) = a_1X_1(1), \end{cases} \tag{3.5}$$

and

$$\begin{cases} f_2(x_2)X_2'(x_2) = \mu X_2(x_2), \quad x_2 \in (0,1), \\ X_2(0) = a_2X_2(1), \end{cases} \tag{3.6}$$

are obtained to determine $X_1(x_1)$ and $X_2(x_2)$. The general solutions of (3.5) and (3.6) are

$$X_2(x_2) = c_2\mathrm{e}^{\int_0^{x_2}\frac{\mu dx}{f_2(x)}}, \quad X_1(x_1) = c_1\mathrm{e}^{\int_0^{x_1}\frac{\rho dx}{f_1(x)}}.$$

In this part we suppose

$$F_1(x_1) = \int_0^{x_1}\frac{dx}{f_1(x)}, \quad F_2(x_2) = \int_0^{x_2}\frac{dx}{f_2(x)}, \quad F_1(1), F_2(1) \neq 0. \tag{3.7}$$

To satisfy the related boundary condition we must have $\mathrm{e}^{\rho F_1(1)} = a_1^{-1}$ and $\mathrm{e}^{\mu F_2(1)} = a_2^{-1}$ .Hence the eigenvalues and eigenfunctions are

$$\begin{cases} \rho_m = \frac{2m\pi i - \log a_1}{F_1(1)}, \\ X_{1m}(x_1) = c_{1m}\frac{\mathrm{e}^{2m\pi i G_1(x_1)}}{a_1^{G_1(x_1)}}, \end{cases} \quad m \in \mathbb{Z}, \tag{3.8}$$

and

$$\begin{cases} \mu_n = \frac{2n\pi i - \log a_2}{F_2(1)}, \\ X_{2n}(x_2) = c_{2n}\frac{\mathrm{e}^{2n\pi i G_2(x_2)}}{a_2^{G_2(x_2)}}, \end{cases} \quad n \in \mathbb{Z}, \tag{3.9}$$

such that $G_1(x_1) = \frac{F_1(x_1)}{F_1(1)}$ and $G_2(x_2) = \frac{F_2(x_2)}{F_2(1)}$. From equations (3.8),(3.9) we get

$$\text{(3.10)} \quad \begin{cases} \lambda_{mn} = \rho_m + \mu_n = \frac{2m\pi i - \log a_1}{F_1(1)} + \frac{2n\pi i - \log a_2}{F_2(1)}, \\ \\ X_{mn}(x) = c_{mn} a_1^{-G_1(x_1)} a_2^{-G_2(x_2)} \mathrm{e}^{2\pi i[mG_1(x_1)+nG_2(x_2)]}, \end{cases} \quad m, n \in \mathbb{Z}$$

where λ_{mn} and X_{mn} are the eigenvalues and the eigenfunctions of the spectral problem (3.1) and (3.3) respectively. By considering the general solution of (3.2) in the form $T_{mn}(t) = \mathrm{e}^{\lambda_{mn} t}$ the general solution of equation (2.1) will be (3.11)

$$u(x,t) = \sum_{m,n=-\infty}^{\infty} c_{mn} a_1^{-G_1 - \frac{t}{F_1(1)}} a_2^{-G_2 - \frac{t}{F_2(1)}} \mathrm{e}^{2\pi i(\frac{m}{F_1(1)} + \frac{n}{F_1(2)})t} \mathrm{e}^{2\pi i(mG_1 + nG_2)}.$$

Since each term of the series satisfies the boundary conditions, these conditions hold.Therefore, it simply remains to show that the initial condition of (2.3) is satisfied. Namely,

$$\text{(3.12)} \qquad \varphi(x) = \sum_{m,n=-\infty}^{\infty} c_{mn} a_1^{-G_1(x_1)} a_2^{-G_2(x_2)} \mathrm{e}^{2\pi i[mG_1(x_1)+nG_2(x_2)]}.$$

The unknown coefficients c_{mn} are calculated in the sequel.

4. **The adjoint of operator L**

Since the eigenfunctions (3.10) are not orthogonal, then to determine the constants C_{mn} we should find the adjoint equation (3.1). For this, we multiply both sides of equation (3.1) in arbitrary function$\overline{Z}(x)$ and integrate on the interval [0,1] we get

$$\int_0^1 dx_1 \int_0^1 f_2(x_2) \frac{\partial X(x)}{\partial x_2} \overline{Z}(x) dx_2 + \int_0^1 dx_2 \int_0^1 f_1(x_1) \frac{\partial X(x)}{\partial x_1} \overline{Z}(x) dx_1 = \lambda \int_D X(x) \overline{Z}(x) dx.$$

Finally we have

$$\int_0^1 X(x) f_2(x_2) \overline{Z}(x) \Big|_{x_2=0}^1 dx_1 + \int_0^1 X(x) f_1(x_1) \overline{Z}(x) \Big|_{x_1=0}^1 dx_2$$
$$- \int_D X(x) \left[\frac{\partial [f_2(x_2) \overline{Z}(x)]}{\partial x_2} + \frac{\partial [f_1(x_1) \overline{Z}(x)]}{\partial x_1} + \overline{\lambda} Z(x) \right] dx = 0.$$

Not that $\overline{Z}(x)$ denotes the complex conjugate function of $Z(x)$. Therefore

$$\text{(4.1)} \quad B(X,Z) = \int_0^1 X(x) f_2(x_2) \overline{Z}(x) \Big|_{x_2=0}^1 dx_1 + \int_0^1 X(x) f_1(x_1) \overline{Z}(x) \Big|_{x_1=0}^1 dx_2.$$

Let L* be the adjoint operator of L, i.e

$$\text{(4.2)} \qquad L^*[Z] = \frac{\partial [\overline{f}_2(x_2) Z(x)]}{\partial x_2} + \frac{\partial [\overline{f}_1(x_1) Z(x)]}{\partial x_1} + \overline{\lambda} Z(x) = 0,$$

and therefore [16]

$$(LX, Z) = B(X, \overline{Z}) + (X, L^*Z).$$

Substituting (3.3) in (4.1), we obtain boundary conditions of the adjoint problem. Namely

$$\begin{aligned} B(X,Z) = &\int_0^1 X(x_1,1)[f_2(1)\overline{Z}(x_1,1) - a_2 f_2(0)\overline{Z}(x_1,0)]dx_1 \\ &+ \int_0^1 X(1,x_2)[f_1(1)\overline{Z}(1,x_2) - a_1 f_1(0)\overline{Z}(0,x_2)]dx_2 = 0. \end{aligned}$$

Therefore, the boundary conditions for the adjoint problem are

$$\left\{ \begin{array}{l} \overline{f}_2(1)Z(x_1,1) = \overline{a}_2\overline{f}_2(0)Z(x_1,0), \\ \\ \overline{f}_1(1)Z(1,x_2) = \overline{a}_1\overline{f}_1(0)Z(0,x_2). \end{array} \right. \tag{4.3}$$

Let $Z(x) = Z_1(x_1)Z_2(x_2)$ then (4.2) will be

$$L^*[Z] = \left(\frac{\partial \overline{f}_2(x_2)}{\partial x_2} + \overline{f}_2(x_2)\frac{Z_2^{'}(x_2)}{Z_2(x_2)}\right) + \left(\frac{\partial \overline{f}_1(x_1)}{\partial x_1} + \overline{f}_1(x_1)\frac{Z_1^{'}(x_1)}{Z_1(x_1)}\right) + \overline{\lambda} = 0.$$

Suppose

$$\frac{\partial \overline{f}_2(x_2)}{\partial x_2} + \overline{f}_2(x_2)\frac{Z_2^{'}(x_2)}{Z_2(x_2)} = \alpha, \quad \frac{\partial \overline{f}_1(x_1)}{\partial x_1} + \overline{f}_1(x_1)\frac{Z_1^{'}(x_1)}{Z_1(x_1)} = \beta, \tag{4.4}$$

where $\alpha + \beta = -\overline{\lambda}$. ($\overline{\lambda}$ and the functions $\overline{f}_1, \overline{f}_2$ are the complex conjugate λ and f_1,f_2 respectively). So boundary conditions will be

$$\left\{ \begin{array}{l} \overline{f}_2(1)Z_2(1) = \overline{a}_2\overline{f}_2(0)Z_2(0), \\ \\ \overline{f}_1(1)Z_1(1) = \overline{a}_1\overline{f}_1(0)Z_1(0). \end{array} \right. \tag{4.5}$$

The general solutions of (4.4)-(4.5) are

$$\overline{f}_2(x_2)Z_2(x_2) = c_2 \mathrm{e}^{\alpha \overline{F}_2(x_2)}, \quad \overline{f}_1(x_1)Z_1(x_1) = c_1 \mathrm{e}^{\beta \overline{F}_1(x_1)}. \tag{4.6}$$

To satisfy the related boundary condition (4.5), we have $\mathrm{e}^{\alpha \overline{F}_2(1)} = \overline{a}_2$ and $\mathrm{e}^{\beta \overline{F}_2(1)} = \overline{a}_1$. Hence the eigenvalues and eigenfunctions are

$$\left\{ \begin{array}{l} \alpha_n = \frac{2n\pi i + \log \overline{a}_2}{\overline{F}_2(1)}, \\ \\ \overline{f}_2(x_2)Z_{2n}(x_2) = c_{2n}\mathrm{e}^{2n\pi i \overline{G}_2(x_2)}\overline{a}_2^{\overline{G}_2(x_2)}, \end{array} \right. \quad n \in \mathbb{Z}, \tag{4.7}$$

and

$$\left\{ \begin{array}{l} \beta_m = \frac{2m\pi i + \log \overline{a}_1}{\overline{F}_1(1)}, \\ \\ \overline{f}_1(x_1)Z_{1m}(x_1) = c_{1m}\mathrm{e}^{2m\pi i \overline{G}_1(x_1)}\overline{a}_1^{\overline{G}_1(x_1)}. \end{array} \right. \quad m \in \mathbb{Z}, \tag{4.8}$$

From equations (4.7)-(4.8) we get

$$
(4.9) \qquad \begin{cases} -\overline{\lambda}_{mn} = \beta_m + \alpha_n = \frac{2m\pi i + \log \overline{a}_1}{\overline{F}_1(1)} + \frac{2n\pi i + \log \overline{a}_2}{\overline{F}_2(1)}, \\ \\ Z_{mn}(x) = \dfrac{1}{\overline{f}_1 \overline{f}_2} \mathrm{e}^{2\pi i(m\overline{G}_1 + n\overline{G}_2)} \overline{a}_1^{\overline{G}_1} \overline{a}_2^{\overline{G}_2}. \end{cases} \qquad m, n \in \mathbb{Z},
$$

Now we prove that functions X_{mn} and Z_{mn} are orthogonal . Then we will have

$$
\begin{aligned}
(X_{mn}, Z_{rs}) &= \int_D X_{mn}(x) \overline{Z}_{rs}(x) dx \\
&= \int_D a_1^{-G_1} a_2^{-G_2} \mathrm{e}^{2\pi i(mG_1 + nG_2)} \frac{1}{f_1 f_2} \mathrm{e}^{-2\pi i(rG_1 + sG_2)} a_1^{G_1} a_2^{G_2} dx_1 dx_2.
\end{aligned}
$$

Using (3.7) if $m = r, n = s$ then $(X_{mn}, Z_{rs}) = F_1(1)F_2(1)$ and if $m \neq r, n \neq s$ then

$$
\begin{aligned}
(X_{mn}, Z_{rs}) &= \left(\int_0^1 F_1^{'}(x_1) \mathrm{e}^{2\pi i(m-r)G_1(x_1)} dx_1 \right) \left(\int_0^1 F_2^{'}(x_2) \mathrm{e}^{2\pi i(n-s)G_2(x_2)} dx_2 \right) \\
&= \frac{F_1(1)}{2\pi i(m-r)} \left(\mathrm{e}^{2\pi i(m-r)} - 1 \right) \frac{F_2(1)}{2\pi i(n-s)} \left(\mathrm{e}^{2\pi i(n-s)} - 1 \right) \\
&= 0.
\end{aligned}
$$

Therefore

$$
(X_{mn}, Z_{rs}) = \begin{cases} F_1(1)F_2(1) & \text{for} \quad m = r, \quad n = s, \\ \\ 0 & \text{otherwise.} \end{cases}
$$

Now, for determining coefficients c_{mn}, the function $\overline{Z}_{rs}(x)$ is multiplied to the both sides of (3.12) and integrating on D yields

(4.10)

$$
c_{rs} = \frac{1}{F_1(1)F_2(1)} \int_D \varphi(x) \overline{Z}_{rs}(x) dx = \frac{1}{F_1(1)F_2(1)} \sum_{m,n=-\infty}^{\infty} c_{mn}(X_{mn}, Z_{rs}).
$$

The solution of the problem (2.1)-(2.2)-(2.3) is therefore given by the series (3.11) where the coefficients C_{rs} are determined by the formula (4.10). In the next section we will provide the convergence conditions for the solution (3.11).

5. Convergence of solution

For the convergence of solution (3.11) we consider the asymptotic behavior of the coefficients c_{mn}. Using (3.12) and (4.10) imply

$$
\begin{aligned}
c_{mn} &= \frac{1}{F_1(1)F_2(1)} \int_D \varphi(x)\overline{Z}_{rs}(x)dx \\
&= \int_0^1 G_1' a_1^{G_1} \mathrm{e}^{-2m\pi i G_1} \left(\int_0^1 \varphi(x) a_2^{G_2} G_2' \mathrm{e}^{-2n\pi i G_2} dx_2 \right) dx_1.
\end{aligned}
$$

By considering (3.7) the internal integral can be developed by part method as follows

$$
\begin{aligned}
\int_0^1 \varphi(x) a_2^{G_2} G_2' \mathrm{e}^{-2n\pi i G_2} dx_2 &= - \left. \frac{a_2^{G_2(x_2)} \varphi(x_1, x_2) \mathrm{e}^{-2n\pi i G_2(x_2)}}{2n\pi i} \right|_{x_2=0}^{1} \\
&\quad + \frac{1}{2n\pi i} \int_0^1 \frac{\partial}{\partial x_2} \left(\varphi(x) a_2^{G_2} \right) \mathrm{e}^{-2n\pi i G_2} dx_2 \\
&= - \frac{a_2 \varphi(x_1, 1) - \varphi(x_1, 0)}{2n\pi i} \\
&\quad + \frac{1}{2n\pi i} \int_0^1 \frac{\partial}{\partial x_2} \left(\varphi(x) a_2^{G_2} \right) \mathrm{e}^{-2n\pi i G_2} dx_2,
\end{aligned}
$$

such that $n \neq 0$. If the function $\varphi(x)$ satisfies in boundary conditions (2.2) and its second derivative with respect to x_2 exists, then for coefficients c_{mn} we have the following asymptotic behavior

$$c_{mn} = O(n^{-2}).$$

This caused the formal series in (3.11) and (3.12) to uniformly converge with respect to x_2. Similarly if this process applies for $\varphi(x)$ with respect to x_1, we will have the same asymptotic behavior for c_{mn}, that is

$$c_{mn} = O(m^{-2}).$$

6. Uniqueness of solution

To establish the uniqueness of solution, at first we get the Laplace transform of equation (2.1), i.e

$$\int_0^\infty \mathrm{e}^{-st} \frac{\partial u}{\partial t} dt = \int_0^\infty \mathrm{e}^{-st} f_2(x_2) \frac{\partial u}{\partial x_2} dt + \int_0^\infty \mathrm{e}^{-st} f_1(x_1) \frac{\partial u}{\partial x_1} dt,$$

where s is complex value with positive real part. If $\tilde{u}$ is the Laplace transform of u then

$$\left. u(x,t)\mathrm{e}^{-st} \right|_{t=0}^{\infty} + \int_0^\infty s\mathrm{e}^{-st} u(x,t) dt = f_2(x_2) \frac{\partial \tilde{u}}{\partial x_2} + f_1(x_1) \frac{\partial \tilde{u}}{\partial x_1},$$

and

$$-\varphi(x)+s\tilde{u}(x,s)=f_2(x_2)\frac{\partial\tilde{u}(x,s)}{\partial x_2}+f_1(x_1)\frac{\partial\tilde{u}(x,s)}{\partial x_1}, \tag{6.1}$$

and the boundary conditions will be

$$\begin{cases} \tilde{u}(0,x_2,s)=a_1\tilde{u}(1,x_2,s), \\ \tilde{u}(x_1,0,s)=a_2\tilde{u}(x_1,1,s). \end{cases} \tag{6.2}$$

To prove the uniqueness of solution of problem (2.1) we show that the above associated homogenous problem has only a trivial solution, that is

$$s\tilde{u}(x,s)=f_2(x_2)\frac{\partial\tilde{u}(x,s)}{\partial x_2}+f_1(x_1)\frac{\partial\tilde{u}(x,s)}{\partial x_1}, \quad s\in\mathbb{C}, \tag{6.3}$$

with the boundary conditions of (6.2). If $\tilde{u}(x,s)\neq 0$ be a solution then by division both side of the relation (6.3) by $f_1(x_1)f_2(x_2)\tilde{u}(x,s)$ and integrating on D, yield

$$\int_0^1\frac{1}{f_1}\log\tilde{u}\Big|_{x_2=0}^1dx_1+\int_0^1\frac{1}{f_2}\log\tilde{u}\Big|_{x_1=0}^1dx_2=s\int_0^1\int_0^1\frac{1}{f_1}\frac{1}{f_2}dx_1dx_2,$$

and

$$\int_0^1\frac{1}{f_1(x_1)}\log\frac{\tilde{u}(x_1,1,s)}{\tilde{u}(x_1,0,s)}dx_1+\int_0^1\frac{1}{f_2(x_2)}\log\frac{\tilde{u}(1,x_2,s)}{\tilde{u}(0,x_2,s)}dx_2=sF_1(1)F_2(1).$$

By using conditions (6.2) and (3.7) we have

$$-F_1(1)\log a_2-F_2(1)\log a_1=sF_1(1)F_2(1).$$

It means that s is a constant value and it is a contradiction. therefore $\tilde{u}(x,s)=0$ is only a trivial solution of a homogeneous problem (6.3), with the boundary conditions (6.2). Consequently this completes the uniqueness of the solution. So we conclude the following theorem.

Theorem 6.1. *Assume that the function $\varphi(x)$ has second derivatives with respect to x_1 , x_2 and $\varphi(x)$ satisfies the boundary conditions (3.3) and $\frac{1}{f_1(x_1)}$, $\frac{1}{f_2(x_2)}$ are integrable on $[0,1]$. Then the problem (2.1)-(2.2)-(2.3) has a unique solution in the form of (3.11).*

7. Examples

In the following examples suppose

$$x=(x_1,x_2)\in D=\{x|x_j\in(0,1),j=1,2\}\subset R^2,\ i=\sqrt{-1},\ t>0.$$

Example 7.1. Consider the initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x_2} + \left(\frac{1}{6x_1^2 - 6x_1 + 2}\right)\frac{\partial u(x,t)}{\partial x_1}, \tag{7.1}$$

with the non-local-boundary conditions

$$\begin{cases} u(0,x_2,t) = u(1,x_2,t), & x_2 \in [0,1], \quad t \geq 0, \\ u(x_1,0,t) = u(x_1,1,t), & x_1 \in [0,1]], \quad t \geq 0, \end{cases}$$

and initial condition

$$u(x,0) = \varphi(x_1,x_2) = \frac{4x_2^2 - 4x_2 + 1}{1 + \frac{1}{2}\mathrm{e}^{2\pi i(2x_1^3 - 3x_1^2 + 2x_1)}}, \quad x \in \overline{D}.$$

Solution: Using (3.11) and (3.12) the solution is

$$u(x,t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_{mn}\mathrm{e}^{2\pi i(m+n)t}\mathrm{e}^{2\pi i[m(2x_1^3 - 3x_1^2 + 2x_1) + nx_2]}, \tag{7.2}$$

and

$$u(x,0) = \varphi(x) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_{mn}\mathrm{e}^{2\pi i[m(2x_1^3 - 3x_1^2 + 2x_1) + nx_2]}.$$

If we consider the double Fourier series of $\varphi(x_1,x_2)$ then (See [17])

$$\begin{aligned} \varphi(x_1,x_2) = &\sum_{m=0}^{\infty}(\frac{1}{3})\frac{(-1)^m}{2^m}\mathrm{e}^{2\pi mi(2x_1^3 - 3x_1^2 + 2x_1)} \\ &+ \sum_{m=0}^{\infty}\sum_{\substack{n=-\infty,\\ n\neq 0}}^{\infty}\frac{(-1)^m}{2^{m-1}\pi^2 n^2}\mathrm{e}^{2\pi i[m(2x_1^3 - 3x_1^2 + 2x_1) + nx_2]}. \end{aligned}$$

Therefore

$$c_{mn} = \begin{cases} (\frac{1}{3})\frac{(-1)^m}{2^m}, & \text{for } n = 0, \\ \frac{(-1)^m}{2^{m-1}\pi^2 n^2}, & \text{for } n \neq 0, \end{cases} \tag{7.3}$$

such that $m,n \in \mathbb{Z}$ and $m \geqslant 0$. The solution of the problem (7.1) is given by the series (7.2), where the coefficients c_{mn} are determined by the formulae (7.3).

Remark 7.2. Without using the double Fourier series of $\varphi(x_1,x_2)$, we can calculate c_{mn} by relation (4.10).

Example 7.3. Consider the initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = i(1+x_2)\frac{\partial u(x,t)}{\partial x_2} + (1+x_1^2)\frac{\partial u(x,t)}{\partial x_1}, \tag{7.4}$$

with the non-local-boundary conditions

$$\begin{cases} u(0,x_2,t) = u(1,x_2,t), & x_2 \in [0,1], \quad t \geq 0, \\ u(x_1,0,t) = u(x_1,1,t), & x_1 \in [0,1]], \quad t \geq 0, \end{cases}$$

and initial condition

$$u(x,0) = \varphi(x_1,x_2) = \cos^2[\pi(x_1+x_2)]. \tag{7.5}$$

Solution: Using (3.11) the solution is

$$u(x,t) = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} c_{mn} \mathrm{e}^{2\pi i(\frac{4m}{\pi}+\frac{in}{\log 2})t} \mathrm{e}^{2\pi i[m\frac{4\tan^{-1}x_1}{\pi}+n\frac{\log(1+x_2)}{\log 2}]}. \tag{7.6}$$

For determining coefficients c_{mn}, if we use the relations (4.9) , (4.10) and initial condition of (7.5) then

$$c_{mn} = \int_D \frac{4}{\pi\log 2}\frac{\cos^2[\pi(x_1+x_2)]}{(1+x_2)(1+x_1^2)} \mathrm{e}^{-2\pi i[m\frac{4\tan^{-1}x_1}{\pi}+n\frac{\log(1+x_2)}{\log 2}]}dx, \tag{7.7}$$

such that $m,n \in \mathbb{Z}$ and $n \geq 0$. Numerical approximation of the coefficients c_{mn} for some values of m and n are given in table 1. Clearly, by means of Theorem 6.1, we can provide more examples.

TABLE 1. The numerical values of coefficients c_{mn} obtained by the dblquad built-in Matlab function.

m \ n	0	1	2	3	4
-2	-0.0020-0.0026i	0.0002+0.0025i	-0.0006-0.0001i	-0.0002-0.0001i	-0.0001-0.0001i
-1	0.0142-0.0223i	0.0041+0.0005i	0.0004-0.0026i	0.0004-0.0016i	0.0003-0.0010i
0	0.4936	0.0067+0.0284i	-0.0034+0.0011i	-0.0011+0.0001i	-0.0005+0.0000i
1	0.0142+0.0223i	0.1810-0.1621i	0.0192+0.0184i	0.0035+0.0062i	0.0013+0.0028i
2	-0.0020+0.0026i	0.0255+0.0116i	-0.0011+0.0029i	-0.0005+0.0007i	-0.0002+0.0003i

REFERENCES

[1] N. Aliev, M. Jahanshahi and S. M. Hosseini, An analytical-numerical method for investigation and solving three dimensional steady state Navier-Stokes equations, II, *Int. J. Differ. Equ. Appl.* **10** (2005), no. 2, 135–145.

[2] G. Avalishvili and M. Avalishvili, On a nonlocal problem with integral boundary conditions for a multidimensional elliptic equation, *Appl. Math. Lett* **24** (2011), no. 4, 566–571.

[3] G. Berikelashvili and N. Khomeriki, On the convergence of difference schemes for one nonlocal boundary-value problem, *Lith. Math. J.* **52** (2012), no. 4, 353–362.

[4] A. Burchard and M. Chugunova, On computing the instability index of a non-self-adjoint differential operator associated with coating and rimming flows, *SIAM J. Math. Anal.* **43** (2011), no. 1, 367–388.

[5] R. Carlson, Adjoint and self-adjoint differential operators on graphs, *Electron. J. Differential Equations* (1998), no. 6, 10 pages.

[6] R. Ciupaila, M. Sapagovas and O. Stikonien, Numerical solution of nonlinear elliptic equation with nonlocal condition, *Nonlinear Anal. Model. Control* **18** (2013), no. 4, 412–426.

[7] G. Evans, J. Blackledge and P. Yardley, Analytic Methods for Partial Differential Equations, Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2000.

[8] G. Freiling and A. Yarco, Inverse problem for Sturm-Liouville equation with boundary conditions, *J. Inverse Problem* **20** (2010).

[9] M. G. Gasymov and G. S. Guseĭnov, Determination of a diffusion operator from spectral data, *Akad. Nauk Azerbaĭdzhan. SSR Dokl.* **37** (1981), no. 2, 19–23.

[10] D. G. Gordeziani and G. A. Avalishvili, Time-nonlocal problems for Schrodinger type equations, II, Results for specific problems, *Differ. Equ.* **41** (2005), no. 6, 852–859.

[11] M. Jahanshahi and A. Ahmadkhanloo, The wave equation in non-classic cases: non-self adjoint with non-local and non-periodic boundary conditions, *Iran. J. Math. Sci. Inform.* **9** (2014), no. 1, 1–12, 100.

[12] M. Jahanshahi, N. Aliev and S. M. Hosseini, An analytic method for investigation and solving two-dimensional steady state navier-stokes equations I, *Southeast Asian Bull. Math.* **33** (2009), no. 6, 1075–1089.

[13] M. Jahanshahi and N. Aliev, Reduction of linearized Benjamin-Ono equation to the Schrdinger equation, *Int. Math. Forum* **2** (2007), no. 11, 543–549,

[14] M. Jahanshahi and M. Darabadi, Existence and uniqueness of solution of a non-self-adjoint initial-boundary value problem for partial differential equation with non-local boundary conditions, *Vietnam J. Math.*, Accepted.

[15] M. Jahanshahi and M. Fatehi, Analytic solution for the cauchyriemann equation with nonlocal boundary conditions in the first quarter, *Int. J. Pure Appl. Math.* **46** (2008), no. 2, 245–249,

[16] M. Jahanshahi and M. Sajjadmanesh, A new method for investigation and recognizing of self-adjoint boundary value problems, *Journal of Science (Kharazmi University)* **12** (2012), no. 1, 295–304.

[17] T. Myint-U and L. Debnath, Linear partial Differential Equations for Scientists and Engineers, Birkhäuser, Boston, 2007.

(Mohammad Jahanshahi) Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

E-mail address: `Jahanshahi@azaruniv.edu`

(Mojtaba Darabadi) Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

E-mail address: `mojtaba.darabadi@yahoo.com`