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PARTIAL PROOF OF GRAHAM HIGMAN'S CONJECTURE RELATED TO COSET DIAGRAMS

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(Communicated by Jamshid Moori)

Abstract. Higman has defined coset diagrams for $PSL(2,\mathbb{Z})$. These diagrams are composed of fragments, and the fragments are further composed of two or more circuits. In 1983, Q. Mushtaq has proved that existence of a certain fragment γ of a coset diagram in a coset diagram is a polynomial f in $\mathbb{Z}[z]$. Higman has conjectured that, the polynomials related to the fragments are monic and for a fixed degree, there are finite number of such polynomials. In this paper, we consider a family $\mathcal{F}$ of fragments such that each fragment in $\mathcal{F}$ contains one vertex v fixed by

$$F_v\left[\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3},(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}\right]$$

where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$, and prove this conjecture for the polynomials obtained from the fragments in $\mathcal{F}$.

Keywords: Modular group, coset diagrams, projective line over finite field.
MSC(2010): Primary: 20G40; Secondary: 05C25.

1. Introduction

It is well known that the modular group $PSL(2,\mathbb{Z})$ [2] has the finite presentation $< x, y : x^2 = y^3 = 1 >$ where x and y are the linear fractional transformations defined by $z \to \frac{-1}{z}$ and $z \to \frac{z-1}{z}$ respectively. By adjoining a new element $t : z \to \frac{1}{z}$ to x and y, we obtain a presentation

$$< x, y, t : x^2 = y^3 = t^2 = (xt)^2 = (yt)^2 = 1 >$$

of the extended modular group $PGL(2,\mathbb{Z})$.

Let q be a power of a prime p. Then by the projective line over the finite field F_q, denoted by $PL\left(F_q\right)$, we mean $F_q \cup \{\infty\}$.

The group $PGL(2, q)$ has its customary meaning, as the group of all linear fractional transformations $z \to \frac{az+b}{cz+d}$ such that a, b, c, d are in F_q and $ad - bc$ is

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non-zero, while $PSL(2,q)$ is its subgroup consisting of all those where $ad-bc$ is a quadratic residue in F_q.

In 1978, Higman introduced a new type of graph called coset diagrams for the modular group $PSL(2,\mathbb{Z})$, and in 1983 Mushtaq [4] laid its foundation. Since there are only two generators, namely x and y, it is possible to avoid using colours as well as the orientation of edges associated with the involution x. For y, which has order 3, there is a need to distinguish y from y^2. The three cycles of y are therefore represented by small triangles, with the convention that y permutes its vertices counter-clockwise, while the fixed points of x and y, if any, are denoted by heavy dots. Thus the geometry of the figure makes the distinction between x-edges and y-edges obvious. For more on coset diagrams, we suggest reading [1, 2, 6, 7] and [9].

Two homomorphisms α and β from $PGL(2,\mathbb{Z})$ to $PGL(2,q)$ are called conjugate if $\beta=\alpha\rho$ for some inner automorphism ρ on $PGL(2,q)$. We call α to be non-degenerate if neither of x,y lies in the kernel of α. In [5] it has been shown that there is a one to one correspondence between the conjugacy classes of non-degenerate homomorphisms from $PGL(2,\mathbb{Z})$ to $PGL(2,q)$ and the elements $\theta\neq 0,3$ of F_q under the correspondence which maps each class to its parameter θ. As in [5], the coset diagram corresponding to the action of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ via a homomorphism α with parameter θ is denoted by $D(\theta,q)$.

2. Occurrence of fragments in $D(\theta,q)$

By a circuit in a coset diagram for $PGL(2,\mathbb{Z})$, we shall mean a closed path of triangles and edges. Let $k\geq 1$ and $n_1,n_2,...,n_{2k}$ be a sequence of positive integers. The circuit which contains a vertex, xed by $w=(xy)^{n_1}(xy^{-1})^{n_2}.....(xy^{-1})^{n_{2k}}$ $\in PSL(2,\mathbb{Z})$, we mean the circuit in which n_1 triangles have one vertex inside the circuit and n_2 triangles have one vertex outside the circuit and so on.

For a given sequence of positive integers $n_1,n_2,...,n_{2k}$ the circuit of the type $(n_1,n_2,...,n_{2k'},n_1,n_2,...,n_{2k'},...,n_1,n_2,...,n_{2k'})$ where k' divides k, is said to have a period of length $2k'$. A circuit which is not of this type is called non-periodic circuit. A circuit is called a simple circuit, if each vertex of the circuit is fixed by a unique word w or its inverse w^{-1}. Two circuits $(n_1,n_2,...,n_{2k})$ and $(m_1,m_2,...,m_{2k})$ are connected, if they have at least one vertex in common.

Consider two non-periodic and simple circuits $(n_1,n_2,...,n_{2k})$ and $(m_1,m_2,...,m_{2k})$. Let v_i be any vertex of $(n_1,n_2,...,n_{2k})$ fixed by a word w_i and v_j be any vertex of $(m_1,m_2,...,m_{2k})$ fixed by a word w_j. In order to connect these two circuits at v_i and v_j, we choose, without loss of generality $(n_1,n_2,...,n_{2k})$ and apply w_j on v_i in such a way that w_j ends at v_i. Consequently, we get a fragment, denoted by γ. As in [8], a pair of words that fixes a vertex $v=v_i=v_j$ in γ is denoted by $F_v[w_i,w_j]$.

The coset diagram $D(\theta, q)$ is made of fragments. It is therefore necessary to ask, when a fragment exists in $D(\theta, q)$. In [3] this question is answered in the following way.

Theorem 2.1. *Given a fragment, there is a polynomial f in $\mathbb{Z}[z]$ such that*
(i) if the fragment occurs in $D(\theta, q)$, then $f(\theta) = 0$,
(ii) if $f(\theta) = 0$ then the fragment, or a homomorphic image of it occurs in $D(\theta, q)$ or in $\overline{PL(F_q)}$.

In [3], the method of calculating a polynomial from a fragment is given. Here we describe this method briefly. Since a fragment is composed of two non-periodic and connected circuits $(n_1, n_2, ..., n_{2k})$ and $(m_1, m_2, ..., m_{2k})$ with a common fixed vertex say v, then there is a pair of words $w_i = (xy)^{l_1} \left(xy^{-1}\right)^{l_2} ... \left(xy^{-1}\right)^{l_{2k_1}}$, $w_j = (xy)^{m_1} \left(xy^{-1}\right)^{m_2} ... \left(xy^{-1}\right)^{m_{2k_2}}$ such that $(v)\, w_i = v$ and $(v)\, w_j = v$. Let X and Y be the matrices corresponding to x and y of $PGL(2, q)$. Then w_i and w_j can be expressed as

$$W_i = (XY)^{l_1} \left(XY^{-1}\right)^{l_2} ... \left(XY^{-1}\right)^{l_{2k_1}}$$

$$W_j = (XY)^{m_1} \left(XY^{-1}\right)^{m_2} ... \left(XY^{-1}\right)^{m_{2k_2}}$$

where $k_1, k_2 > 0$. Since X and Y are the matrices with entries from F_q and satisfy

$$X^2 = Y^3 = \lambda I. \tag{2.1}$$

We can take X, Y to be represented by

$$X = \begin{pmatrix} a & kc \\ c & -a \end{pmatrix}, \qquad Y = \begin{pmatrix} d & kf \\ f & -d-1 \end{pmatrix}$$

where a, c, d, f, k are elements of F_q. We shall write

$$a^2 + kc^2 = -\Delta \neq 0 \tag{2.2}$$

and require that

$$d^2 + d + kf^2 + 1 = 0 \tag{2.3}$$

This certainly gives elements satisfying the relations (2.1).

We note that the matrix M, representing xy, has the trace $r = a(2d+1) + 2kcf$ and the determinant $\Delta = -(a^2 + kc^2)$, because det $(Y) = 1$. This means that det $(X) = \Delta$ and trace $(X) = 0$; and so the characteristic equation of X will be

$$X^2 + \Delta I = 0. \tag{2.4}$$

Similarly, since det $(Y) = q$ and trace $(Y) = -q$, the characteristic equation of Y will be

$$Y^2 + Y + I = 0. \tag{2.5}$$

Furthermore, det $(XY) = \Delta$ and trace $(XY) = r$ imply that the characteristic equation of the matrix XY will be

$$(XY)^2 - r(XY) + \Delta I = 0. \tag{2.6}$$

On recursion, Equation (2.6) yields

$$(XY)^n = \left\{ \binom{n-1}{0} r^{n-1} - \binom{n-2}{1} r^{n-3}\Delta + ... \right\} XY - \left\{ \binom{n-2}{0} r^{n-2}\Delta - \binom{n-3}{1} r^{n-4}\Delta^2 + ... \right\} I. \tag{2.7}$$

After suitable manipulation, Equations (2.4), (2.5) and (2.6) give the following equations

$$XYX = rX + \Delta I + \Delta Y. \tag{2.8}$$

$$XYY = -X - XY \tag{2.9}$$

$$YXY = rY + X. \tag{2.10}$$

$$YX = rI - X - XY. \tag{2.11}$$

Thus, by making use of Equations (2.4) to (2.11) the matrices W_i and W_j can be expressed linearly as

$$W_i = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$
$$W_j = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

where λ_i and μ_i, for $i = 0, 1, 2, 3$ are polynomials in r and Δ. Since $(v)\, w_i = v$ and $(v)\, w_j = v$ the 2×2 matrices W_i and W_j have an eigenvector in common. This by Lemma 3.1 of [3] means that the algebra generated by W_i and W_j has dimension 3. The algebra contains I, W_i, W_j, W_iW_j and so these must be linearly dependent. Using Equations (2.4) to (2.11) the matrix W_iW_j can be expressed as

$$W_iW_j = \nu_0 I + \nu_1 X + \nu_2 Y + \nu_3 XY$$

where v_i, for $i = 0, 1, 2, 3$ can be calculated in terms of the λ_i and μ_i, using (2.4) to (2.11). The condition that I, W_i, W_j and W_iW_j are linearly dependent, can be expressed as

$$(2.12) \qquad \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} = 0.$$

If we carry out the calculation of v_1, v_2, v_3 in terms of λ_i and μ_i and substitute, in (2.12), we find that this is equivalent to

$$(2.13) \quad \begin{aligned} &(\lambda_2\mu_3 - \mu_2\lambda_3)^2 + \Delta(\lambda_3\mu_1 - \mu_3\lambda_1)^2 + (\lambda_1\mu_2 - \mu_1\lambda_2)^2 \\ &+ r(\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_3\mu_1 - \mu_3\lambda_1) + (\lambda_2\mu_3 - \mu_2\lambda_3)(\lambda_1\mu_2 - \mu_1\lambda_2) = 0. \end{aligned}$$

This gives a homogeneous equation in Δ and r. In [5], θ is defined as $\frac{r^2}{\Delta}$, so we can substitute $\Delta\theta$ for r^2 to get a polynomial in θ.

Higman has conjectured that, the polynomials related to the fragments are monic and for a fixed degree, there are finite number of such polynomials. In this paper, we consider a family $\mathcal{F}$ of fragments such that each fragment in $\mathcal{F}$ contains one vertex v fixed by

$$F_v\left[\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3}, (xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}\right]$$

where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$, and prove the Higman's conjecture for the polynomials obtained from $\mathcal{F}$.

3. Main results

The following three theorems have been proved in [8]. Since we use them in this paper frequently, we therefore reproduce their statements here.

Theorem 3.1. *Let the fragment γ be constructed by joining a vertex v_i of $(n_1, n_2, ..., n_{2k})$ with the vertex v_j of $(m_1, m_2, ..., m_{2k})$. Then γ is obtainable also, if the vertex $(v_i)\,w$ of $(n_1, n_2, ..., n_{2k})$ is joined with the vertex $(v_j)\,w$ of $(m_1, m_2, ..., m_{2k})$.*

If $w = xy^{\eta_1}xy^{\eta_2}...xy^{\eta_n}$ ($\eta = 1$ or -1) is a word, then let $w^* = xy^{-\eta_1}xy^{-\eta_2}...xy^{-\eta_n}$.

Theorem 3.2. *If the fragment γ has one vertex v fixed by $F_v[w_i, w_j]$, then its mirror image γ^* has one vertex fixed by $F_{v^*}\left[w_i^*, w_j^*\right]$.*

Theorem 3.3. *The polynomials obtained from the fragment γ and its mirror image γ^* are the same.*

Consider two circuits (n_1, n_2) and (m_1, m_2).

$\mathcal{F}$ is constructed by joining

$$e_{3i_1} \text{ with } u_{3j_1+1} \text{ and } v_{3j_2+1},$$
$$f_{3i_2} \text{ with } u_{3j_1+1} \text{ and } v_{3j_2+1}$$
$$u_{3j_1} \text{ with } e_{3i_1+1} \text{ and } f_{3i_2+1},$$

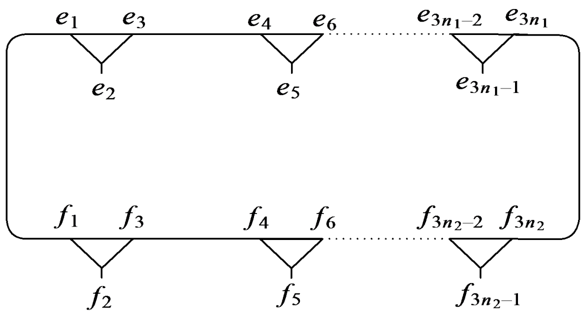

FIGURE 1.

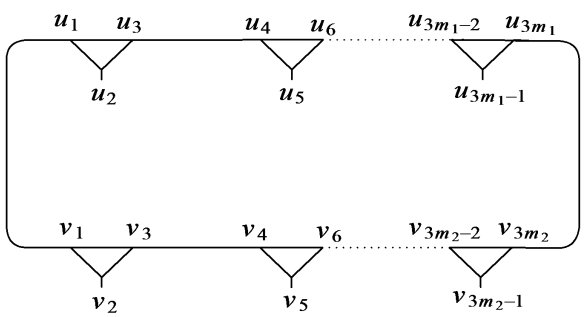

FIGURE 2.

$$v_{3j_1} \text{ with } e_{3i_1+1} \text{ and } f_{3i_2+1}$$

where

$$i_1 = 1, 2, ..., n_1 - 1,\ \ i_2 = 1, 2, ..., n_2 - 1$$

and

$$j_1 = 1, 2, ..., m_1 - 1,\ \ j_2 = 1, 2, ..., m_2 - 1.$$

Theorem 3.4. *Number of triangles in any fragment $\gamma \in F$ is*

$$s_1 + s_2 + s_3 + q_1 + q_2 + q_3 - 2.$$

Proof. Let γ be any fragment in F. Then its one vertex say v, is a fixed point of the circuits $\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3}$ and $(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}$, where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$. Diagrammatically, it means:

From the diagram it is clear that, $\gamma \in F$ has $s_1 + s_2 + s_3 + q_1 + q_2 + q_3 - 2$ triangles. $\square$

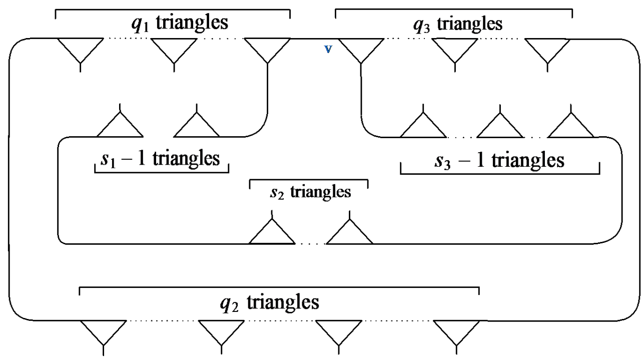

FIGURE 3.

Proposition 3.5. *If $w = (x\ y^{-1})^{q_2}$ where $q_2 \in \mathbb{Z}^+$. Then the corresponding matrix can be expressed linearly as*

$$\begin{aligned}(XY^{-1})^{q_2} &= (-1)^{q_2+1}\left\{\binom{q_2-2}{0} r^{q_2-2}\Delta - \binom{q_2-3}{1} r^{q_2-4}\Delta^2 + \ldots\right\} I + \\ &\quad (-1)^{q_2}\left\{\binom{q_2-1}{0} r^{q_2-1} - \binom{q_2-2}{1} r^{q_2-3}\Delta + \ldots\right\} X + \\ &\quad (-1)^{q_2}\left\{\binom{q_2-1}{0} r^{q_2-1} - \binom{q_2-2}{1} r^{q_2-3}\Delta + \ldots\right\} XY.\end{aligned}$$

The proof is obtained by using mathematical induction.

Total number of triangles in the circuit $(x\ y)^{q_1}(xy^{-1})^{q_2}(x\ y)^{q_3}$ are $q_1 + q_2 + q_3$, let $q_1 + q_2 + q_3 = \tau_1$ and

$$\epsilon_1 = \begin{cases} 1 & \text{if } q_1 < q_3 \\ 0 & \text{if } q_1 \geq q_3 \end{cases}, \quad \epsilon_2 = \begin{cases} 3 & \text{if } q_1 < q_3 \\ 1 & \text{if } q_1 \geq q_3 \end{cases},$$

$$\alpha_1 = \begin{cases} 0 & \text{if } q_3 - q_1 = 1 \\ 1 & \text{otherwise} \end{cases}, \quad \alpha_2 = \begin{cases} 0 & \text{if } q_1 = q_3 \\ 1 & \text{otherwise} \end{cases}.$$

Since $(x\ y)^{q_1}(xy^{-1})^{q_2}(x\ y)^{q_3}$ can be expressed linearly as

$$(XY)^{q_1}(XY^{-1})^{q_2}(XY)^{q_3} = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

where μ_i, for $i = 0, 1, 2, 3$ is polynomial in r and Δ, we use $max\,(\mu_i)$ for the term containing the highest power of r, in μ_i.

Theorem 3.6. *If $w = (x\ y)^{q_1}(xy^{-1})^{q_2}(x\ y)^{q_3}$, where $q_1, q_2, q_3 \in \mathbb{Z}^+$, then the corresponding matrix can be expressed linearly as $W = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$, such that*

$$max\,(\mu_0) = (-1)^{q_2+1}\, r^{\tau_1-2}\Delta,$$

$$max\,(\mu_1) = (-1)^{q_2+\epsilon_1+2}\,\alpha_1 r^{\tau_1-2\min(q_1,q_3)-\epsilon_2}\Delta^{\min(q_1,q_3)+\epsilon_1},$$
$$max\,(\mu_2) = (-1)^{q_2+\epsilon_1+2}\,\alpha_2 r^{\tau_1-2\min(q_1,q_3)-2}\Delta^{\min(q_1,q_3)+1},$$
$$max\,(\mu_3) = (-1)^{q_2+2}\,r^{\tau_1-1}.$$

Proof. By Proposition 3.5

$$\begin{aligned}\left(XY^{-1}\right)^{q_2} \;=\; & (-1)^{q_2+1}\left\{\binom{q_2-2}{0}r^{q_2-2}\Delta-\binom{q_2-3}{1}r^{q_2-4}\Delta^2+...\right\}I+\\ & (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2-1}-\binom{q_2-2}{1}r^{q_2-3}\Delta+...\right\}X+\\ & (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2-1}-\binom{q_2-2}{1}r^{q_2-3}\Delta+...\right\}XY.\end{aligned}$$

Now

$$XY\left(XY^{-1}\right)^{q_2}XY$$

$$\begin{aligned}= \;& (-1)^{q_2+1}\left\{\binom{q_2-2}{0}r^{q_2-2}\Delta-\binom{q_2-3}{1}r^{q_2-4}\Delta^2+...\right\}(XY)^2+\\ & (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2-1}-\binom{q_2-2}{1}r^{q_2-3}\Delta+...\right\}XYXXY+\\ & (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2-1}-\binom{q_2-2}{1}r^{q_2-3}\Delta+...\right\}(XY)^3.\end{aligned}$$

By making use of Equations (2.4) to (2.11), we get

$$XY\left(XY^{-1}\right)^{q_2}XY$$

$$\begin{aligned}= \;& (-1)^{q_2+1}\left\{\begin{array}{c}\binom{q_2-2}{0}r^{q_2-2}\Delta-\\ \binom{q_2-3}{1}r^{q_2-4}\Delta^2+...\end{array}\right\}(-\Delta I+rXY)+\\ & (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2-1}-\binom{q_2-2}{1}r^{q_2-3}\Delta+...\right\}(\Delta X+\Delta XY)+\\ & (-1)^{q_2}\left\{\begin{array}{c}\binom{q_2-1}{0}r^{q_2-1}-\\ \binom{q_2-2}{1}r^{q_2-3}\Delta+...\end{array}\right\}\left(-r\Delta I+(r^2-\Delta)XY\right)\\ = \;& (-1)^{q_2+1}\left\{\begin{array}{c}\binom{q_2-1}{0}r^{q_2}\Delta-\\ ...\end{array}\right\}I+(-1)^{q_2}\left\{\begin{array}{c}\binom{q_2-1}{0}r^{q_2-1}\Delta-\\ ...\end{array}\right\}X+\\ & (-1)^{q_2}\left\{\binom{q_2-1}{0}r^{q_2+1}-...\right\}XY.\end{aligned}$$

Hence the result is true for $XY\left(XY^{-1}\right)^{q_2}XY$.
Let it be true for $(XY)^k\left(XY^{-1}\right)^{q_2}(XY)^k$, that is

$$(XY)^k\left(XY^{-1}\right)^{q_2}(XY)^k =$$

$(-1)^{q_2+1}\left\{r^{k+q_2+k-2}\Delta-...\right\}I+(-1)^{q_2+2}\left\{r^{k+q_2+k-2k-1}\Delta^k-...\right\}X+$
$0Y+(-1)^{q_2+2}\left\{r^{k+q_2+k-1}-...\right\}XY.$

Now

$$\begin{aligned}&(XY)^{k+1}\left(XY^{-1}\right)^{q_2}(XY)^{k+1}\\ =\;&(-1)^{q_2+1}\left\{r^{k+q_2+k-2}\Delta-...\right\}(XY)^2+\\ &(-1)^{q_2+2}\left\{r^{k+q_2+k-2k-1}\Delta^k-...\right\}XYXXY+\\ &0Y+(-1)^{q_2+2}\left\{r^{k+q_2+k-1}-...\right\}(XY)^3.\end{aligned}$$

By making use of Equations (2.4) to (2.11), we obtain

$$\begin{aligned}&(XY)^{k+1}\left(XY^{-1}\right)^{q_2}(XY)^{k+1}\\ =\;&(-1)^{q_2+1}\left\{r^{k+q_2+k-2}\Delta-...\right\}(-\Delta I+rXY)+\\ &(-1)^{q_2+2}\left\{r^{q_2-1}\Delta^k-...\right\}(\Delta X+\Delta XY)+\\ &(-1)^{q_2+2}\left\{r^{k+q_2+k-1}-...\right\}\left(-r\Delta I+(r^2-\Delta)XY\right)\\ =\;&(-1)^{q_2+1}\left\{r^{(k+1)+q_2+(k+1)-2}\Delta-...\right\}I+\\ &(-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k+1}-...\right\}X+\\ &(-1)^{q_2+2}\left\{r^{(k+1)+q_2+(k+1)-1}-...\right\}XY.\end{aligned}$$

This shows that the result is true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$, where $q_1=q_3,q_2\in\mathbb{Z}^+$. So, for $k_1=k_3$

$$\begin{aligned}&(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}\\ =\;&(-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-2}\Delta-...\right\}I+(-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_3}-...\right\}X+\\ &(-1)^{q_2+2}\left\{r^{k_1+q_2+k_3-1}-...\right\}XY.\end{aligned}$$

Therefore

$$\begin{aligned}&(XY)^{k_1+1}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}\\ =\;&(-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-2}\Delta-...\right\}XYI+\\ &(-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_3}-...\right\}XYX+\\ &(-1)^{q_2+2}\left\{r^{k_1+q_2+k_3-1}-...\right\}XYXY.\end{aligned}$$

By making use of Equations (2.4) to (2.11), we have

$$(XY)^{k_1+1}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}$$

$$\begin{aligned}
&= (-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-2}\Delta - ...\right\} XY + \\
&\quad (-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_3} - ...\right\}(rX + \Delta Y) + \\
&\quad (-1)^{q_2+2}\left\{r^{k_1+q_2+k_3-1} - ...\right\}(-\Delta I + rXY) \\
&= (-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-1}\Delta - ...\right\} I + (-1)^{q_2+2}\left\{r^{q_2}\Delta^{k_3} - ...\right\} X + \\
&\quad (-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_3+1} - ...\right\} Y + (-1)^{q_2+2}\left\{r^{k_1+q_2+k_3} - ...\right\} XY.
\end{aligned}$$

Hence the result is true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ such that $q_1 - q_3 = 1$. Let it be true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ such that $q_1 - q_3 = n$, that is

$$(XY)^{k_3+n}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}$$

$$\begin{aligned}
&= (-1)^{q_2+1}\left\{r^{k_3+n+q_2+k_3-2}\Delta - ...\right\} I + \\
&\quad (-1)^{q_2+2}\left\{r^{n+q_2-1}\Delta^{k_3} - ...\right\} X + \\
&\quad (-1)^{q_2+2}\left\{r^{n+q_2-2}\Delta^{k_3+1} - ...\right\} Y + \\
&\quad (-1)^{q_2+2}\left\{r^{k_3+n+q_2+k_3-1} - ...\right\} XY.
\end{aligned}$$

Now

$$(XY)^{k_3+(n+1)}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}$$

$$\begin{aligned}
&= (-1)^{q_2+1}\left\{r^{k_3+n+q_2+k_3-2}\Delta - ...\right\} XYI + \\
&\quad (-1)^{q_2+2}\left\{r^{n+q_2-1}\Delta^{k_3} - ...\right\} XYX + \\
&\quad (-1)^{q_2+2}\left\{r^{n+q_2-2}\Delta^{k_3+1} - ...\right\} XYY + \\
&\quad (-1)^{q_2+2}\left\{r^{k_3+n+q_2+k_3-1} - ...\right\} XYXY.
\end{aligned}$$

By making use of Equations (2.4) to (2.11), we have

$$(XY)^{k_3+(n+1)}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}$$

$$\begin{aligned}
&= (-1)^{q_2+1}\left\{r^{k_3+n+q_2+k_3-2}\Delta - ...\right\} XY + \\
&\quad (-1)^{q_2+2}\left\{r^{n+q_2-1}\Delta^{k_3} - ...\right\}(rX + \Delta Y) + \\
&\quad (-1)^{q_2+2}\left\{r^{n+q_2-2}\Delta^{k_3+1} - ...\right\}(-X - XY) + \\
&\quad (-1)^{q_2+2}\left\{r^{k_3+n+q_2+k_3-1} - ...\right\}(-\Delta I + rXY) \\
&= (-1)^{q_2+1}\left\{r^{k_3+n+1+q_2+k_3-2}\Delta - ...\right\} I + \\
&\quad (-1)^{q_2+2}\left\{r^{n+1+q_2-1}\Delta^{k_3} - ...\right\} X + \\
&\quad (-1)^{q_2+2}\left\{r^{n+1+q_2-2}\Delta^{k_3+1} - ...\right\} Y + \\
&\quad (-1)^{q_2+2}\left\{r^{k_3+n+1+q_2+k_3-1} - ...\right\} XY.
\end{aligned}$$

Hence the result is true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ such that $q_1 > q_3$.
Again for $k_1 = k_3$, we have

$$(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_3}$$

$$\begin{aligned} = \quad & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-2}\Delta - ...\right\}I + (-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_1} - ...\right\}X + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_3-1} - ...\right\}XY. \end{aligned}$$

Therefore

$$(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_3+1}$$

$$\begin{aligned} = \quad & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-2}\Delta - ...\right\}IXY + \\ & (-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_1} - ...\right\}XXY + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_3-1} - ...\right\}XYXY \end{aligned}$$

By making use of Equations (2.4) to (2.11), we have

$$(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_3+1}$$

$$\begin{aligned} = \quad & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-2}\Delta - ...\right\}XY + \\ & (-1)^{q_2+2}\left\{r^{q_2-1}\Delta^{k_1} - ...\right\}(-\Delta Y) + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_3-1} - ...\right\}(-\Delta I + rXY) \\ = \quad & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_3-1}\Delta - ...\right\}I + 0X + \\ & (-1)^{q_2+1}\left\{r^{q_2-1}\Delta^{k_1+1} - ...\right\}Y + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_3} - ...\right\}XY. \end{aligned}$$

Hence the result is true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ such that $q_3 - q_1 = 1$.
Let it be true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ such that $q_3 - q_1 = n$, that is

$$(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_1+n}$$

$$\begin{aligned} = \quad & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_1+n-2}\Delta - ...\right\}I + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-3}\Delta^{k_1+1} - ...\right\}X + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-2}\Delta^{k_1+1} - ...\right\}Y + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_1+n-1} - ...\right\}XY. \end{aligned}$$

Now

$$(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_1+(n+1)}$$

$$\begin{aligned} = \; & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_1+n-2}\Delta - ...\right\} IXY + \\ & (-1)^{q_2+1+2}\left\{r^{k_1+q_2+k_3+n-2k_1-3}\Delta^{k_1+1} - ...\right\} XXY + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-2}\Delta^{k_1+1} - ...\right\} YXY + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_1+n-1} - ...\right\} XYXY. \end{aligned}$$

By making use of Equations (2.4) to (2.11), we have

$$(XY)^{k_1}\left(XY^{-1}\right)^{q_2}(XY)^{k_1+(n+1)}$$

$$\begin{aligned} = \; & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_1+n-2}\Delta - ...\right\} XY + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-3}\Delta^{k_1+1} - ...\right\} (-\Delta Y) + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-2}\Delta^{k_1+1} - ...\right\} (X + rY) + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_1+n-1} - ...\right\} (-\Delta I + rXY) \end{aligned}$$

$$\begin{aligned} = \; & (-1)^{q_2+1}\left\{r^{k_1+q_2+k_1+n-1}\Delta - ...\right\} I + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-2}\Delta^{k_1+1} - ...\right\} X + \\ & (-1)^{q_2+3}\left\{r^{q_2+n-1}\Delta^{k_1+1} - ...\right\} Y + \\ & (-1)^{q_2+2}\left\{r^{k_1+q_2+k_1+n} - ...\right\} XY. \end{aligned}$$

Hence the result is true for $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ such that $q_1 < q_3$. □

Total number of triangles in the circuit $(x\ y)^{s_1}(xy^{-1})^{s_2}(x\ y)^{s_3}$ are $s_1 + s_2 + s_3$, let $s_1 + s_2 + s_3 = \tau_2$ and $\beta_1 = \begin{cases} 0 & \text{if } s_1 = s_3 \\ 1 & \text{otherwise} \end{cases}$.

Since $(xy^{-1})^{s_1}(xy)^{s_2}(xy^{-1})^{s_3}$ can be expressed linearly as

$$(XY^{-1})^{s_1}(XY)^{s_2}(XY^{-1})^{s_3} = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$

where λ_i, for $i = 0, 1, 2, 3$ is polynomial in r and Δ, we use $max\,(\lambda_i)$ for the term containing the highest power of r, in λ_i.

By using mathematical induction, we have the following Theorem.

Theorem 3.7. *If $w = (xy^{-1})^{s_1}(xy)^{s_2}(xy^{-1})^{s_3}$ where $s_1, s_2, s_3 \in \mathbb{Z}^+$ and $s_1 \geq s_3$, then the corresponding matrix can be expressed linearly as $W = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$, such that*

$$max\,(\lambda_0) = (-1)^{s_1+s_3+1}\, r^{\tau_2-2}\Delta,$$

$$max\,(\lambda_1) = (-1)^{s_1+s_3}\, r^{\tau_2-1},$$

$$max\,(\lambda_2) = (-1)^{s_1+s_3+1}\, \beta_1 r^{\tau_2-2s_3-2}\Delta^{s_3+1},$$

$$max\,(\lambda_3) = (-1)^{s_1+s_3}\, r^{\tau_2-1}.$$

Theorem 3.8. *Let $\gamma \in F$ such that $s_1 \geq s_3$. Then degree of the polynomial $f(\theta)$ obtained from γ is $s_1 + s_2 + s_3 + q_1 + q_2 + q_3 - 2$. Moreover $f(\theta)$ is monic.*

Proof. Since $\gamma \in F$ and $s_1 \geq s_3$, therefore its one vertex v, is a fixed point of the circuits $\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3}$ and $(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}$. The matrices corresponding to $(xy^{-1})^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3}$ and $(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}$ are $(XY^{-1})^{s_1}(XY)^{s_2}\left(XY^{-1}\right)^{s_3}$ and $(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3}$ respectively, and these can be written as a linear combination of I, X, Y and XY, that is

$$(XY^{-1})^{s_1}(XY)^{s_2}\left(XY^{-1}\right)^{s_3} = \lambda_0 I + \lambda_1 X + \lambda_2 Y + \lambda_3 XY$$

and

$$(XY)^{q_1}\left(XY^{-1}\right)^{q_2}(XY)^{q_3} = \mu_0 I + \mu_1 X + \mu_2 Y + \mu_3 XY$$

where λ_i and μ_i for $i = 0, 1, 2, 3$ are polynomials in r and Δ. By Theorems 3.6 and 3.7, we have

$$max\,(\mu_1) = (-1)^{q_2+\epsilon_1+2}\,\alpha_1 r^{\tau_1 - 2\min(q_1,q_3)-\epsilon_2}\Delta^{\min(q_1,q_3)+\epsilon_1}.$$

$$max\,(\mu_2) = (-1)^{q_2+\epsilon_1+2}\,\alpha_2 r^{\tau_1 - 2\min(q_1,q_3)-2}\Delta^{\min(q_1,q_3)+1}.$$

$$max\,(\mu_3) = (-1)^{q_2+2}\,r^{\tau_1-1}.$$

$$max\,(\lambda_1) = (-1)^{s_1+s_3}\,r^{\tau_2-1}.$$

$$max\,(\lambda_2) = (-1)^{s_1+s_3+1}\,\beta_1 r^{\tau_2-2s_3-2}\Delta^{s_3+1}.$$

$$max\,(\lambda_3) = (-1)^{s_1+s_3}\,r^{\tau_2-1}.$$

Now

$$max\,(\lambda_2\mu_3) = (-1)^{s_1+s_3+q_2+3}\,\beta_1 r^{\tau_1+\tau_2-2s_3-3}\Delta^{s_3+1}$$

and

$$max\,(\lambda_3\mu_2) = (-1)^{s_1+s_3+q_2+\epsilon_1+2}\,\alpha_2 r^{\tau_1+\tau_2-2\min(q_1,q_3)-3}\Delta^{\min(q_1,q_3)+1}.$$

Let $p = \begin{cases} s_1+s_3+q_2+3 & \text{if } s_3 = \min(q_1,q_3,s_3) \\ s_1+s_3+q_2+\epsilon_1+1 & \text{if } \min(q_1,q_3) = \min(q_1,q_3,s_3) \end{cases}$, and $g = \min(q_1,q_3,s_3)$. Then

$$max\,(\lambda_2\mu_3 - \lambda_3\mu_2) = (-1)^p\,\beta_1 r^{\tau_1+\tau_2-2g-3}\Delta^{g+1} \tag{3.1}$$

shows that

$$max\,(\lambda_2\mu_3 - \lambda_3\mu_2)^2 = \beta_1 r^{2(\tau_1+\tau_2-2g-3)}\Delta^{2(g+1)}. \tag{3.2}$$

Now

(3.3)

$$max\,(\lambda_3\mu_1) = (-1)^{q_2+s_1+s_3+\epsilon_1+2}\,\alpha_1 r^{\tau_1+\tau_2-2\min(q_1,q_3)-\epsilon_2-1}\Delta^{\min(q_1,q_3)+\epsilon_1}$$

and

$$max\,(\lambda_1\mu_3) = (-1)^{s_1+s_3+q_2+2}\,r^{\tau_1+\tau_2-2} \tag{3.4}$$

together imply that

$$max\,(\lambda_3\mu_1 - \lambda_1\mu_3) = (-1)^{s_1+s_3+q_2+1}\,r^{\tau_1+\tau_2-2} \tag{3.5}$$

or

$$max\left(\Delta\left(\lambda_3\mu_1-\lambda_1\mu_3\right)^2\right)=r^{2(\tau_1+\tau_2-2)}\Delta. \tag{3.6}$$

Now

$$max\left(\lambda_1\mu_2\right)=(-1)^{s_1+s_3+q_2+\epsilon_1+2}\,\alpha_2 r^{\tau_1+\tau_2-2\min(q_1,q_3)-3}\Delta^{\min(q_1,q_3)+1} \tag{3.7}$$

and

$$\begin{aligned} max\left(\lambda_2\mu_1\right) &= (-1)^{s_1+s_3+q_2+\epsilon_1+1}\,\alpha_1\beta_1 \\ & r^{\tau_1+\tau_2-2\min(q_1,q_3)-2s_3-2-\epsilon_2}\Delta^{s_3+\min(q_1,q_3)+\epsilon_1+1}. \end{aligned} \tag{3.8}$$

So

$$max\left(\lambda_1\mu_2-\lambda_2\mu_1\right)=(-1)^{s_1+s_3+q_2+\epsilon_1+2}\,\alpha_2 r^{\tau_1+\tau_2-2\min(q_1,q_3)-3}\Delta^{\min(q_1,q_3)+1}. \tag{3.9}$$

and

$$max\left(\lambda_1\mu_2-\lambda_2\mu_1\right)^2=\alpha_2 r^{2(\tau_1+\tau_2-2\min(q_1,q_3)-3)}\Delta^{2(\min(q_1,q_3)+1)}. \tag{3.10}$$

By using Equations (3.1) and (3.5), we obtain

$$max\left(r\left(\lambda_2\mu_3-\lambda_3\mu_2\right)\left(\lambda_3\mu_1-\lambda_1\mu_3\right)\right)=(-1)^{\epsilon_1}\,r^{2(\tau_1+\tau_2-2-g)}\Delta^{g+1}. \tag{3.11}$$

Also by using Equations (3.1) and (3.9), we get

$$\begin{aligned} max\left(\left(\lambda_2\mu_3-\lambda_3\mu_2\right)\left(\lambda_1\mu_2-\lambda_2\mu_1\right)\right) &= (-1)^{s_1+s_3+q_2+\epsilon_1+2+p}\,\alpha_2\beta_1 \\ & r^{2(\tau_1+\tau_2-\min(q_1,q_3)-g-3)} \\ & \Delta^{\min(q_1,q_3)+g+2}. \end{aligned} \tag{3.12}$$

The term containing the highest power of θ, in the polynomial equation 2.13 yields degree and leading coefficient of the polynomial obtained from γ. By using Equations 3.2 to 3.12, we have

$$max\left(\begin{array}{c}\left(\lambda_2\mu_3-\mu_2\lambda_3\right)^2+\Delta\left(\lambda_3\mu_1-\mu_3\lambda_1\right)^2+\left(\lambda_1\mu_2-\mu_1\lambda_2\right)^2+\\ r\left(\lambda_2\mu_3-\mu_2\lambda_3\right)\left(\lambda_3\mu_1-\mu_3\lambda_1\right)+\left(\lambda_2\mu_3-\mu_2\lambda_3\right)\left(\lambda_1\mu_2-\mu_1\lambda_2\right)\end{array}\right)=$$

$$r^{2(\tau_1+\tau_2-2)}\Delta.$$

Since $r^2=\Delta\theta$, therefore

$$max\left(\begin{array}{c}\left(\lambda_2\mu_3-\mu_2\lambda_3\right)^2+\Delta\left(\lambda_3\mu_1-\mu_3\lambda_1\right)^2+\left(\lambda_1\mu_2-\mu_1\lambda_2\right)^2+\\ r\left(\lambda_2\mu_3-\mu_2\lambda_3\right)\left(\lambda_3\mu_1-\mu_3\lambda_1\right)+\left(\lambda_2\mu_3-\mu_2\lambda_3\right)\left(\lambda_1\mu_2-\mu_1\lambda_2\right)\end{array}\right)=$$

$$\theta^{\tau_1+\tau_2-2}\Delta^{\tau_1+\tau_2-1}.$$

We can omit $\Delta^{s_1+s_2+s_3+q_1+q_2+q_3-1}$ as it is square in F_q. Hence degree of the polynomial obtained from γ is $s_1+s_2+s_3+q_1+q_2+q_3-2$. Also this polynomial is monic. □

Lemma 3.9. *Corresponding to each fragment δ containing a vertex fixed by*

$$F_v\left[\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3},(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}\right]$$

where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$ and $s_1 < s_3$, there is a fragment δ^* containing a vertex fixed by

$$F_{u^*}\left[\left(xy^{-1}\right)^{s_3}(xy)^{s_2}\left(xy^{-1}\right)^{s_1},(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}\right]$$

such that δ and δ^* have the same polynomial.

Proof. Let the fragment δ contains a vertex fixed by

$$F_v\left[\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3},(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}\right]$$

where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$ and $s_1 < s_3$. Clearly δ is created by joining a vertex v fixed by $\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3}$ of (n_1, n_2) with the vertex v' fixed by $(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}$ of (m_1, m_2). By Theorem 3.1, δ is obtainable also, if we join the vertex $(v)\,x$ with the vertex $\left(v'\right)x$. This implies that, δ has also a vertex $u = (v)\,x = \left(v'\right)x$ fixed by

$$F_u\left[(xy)^{s_3}\left(xy^{-1}\right)^{s_2}(xy)^{s_1},\left(xy^{-1}\right)^{q_3}(xy)^{q_2}\left(xy^{-1}\right)^{q_1}\right].$$

By Theorem 3.2, mirror image of δ has a vertex fixed by

$$F_{u^*}\left[\left(xy^{-1}\right)^{s_3}(xy)^{s_2}\left(xy^{-1}\right)^{s_1},(xy)^{q_3}\left(xy^{-1}\right)^{q_2}(xy)^{q_1}\right].$$

By Theorem 3.3, the polynomials obtained from δ and its mirror image δ^* are the same. □

Theorem 3.10. *Let $\delta \in \digamma$ such that $s_1 < s_3$. Then degree of the polynomial $g(\theta)$ obtained from δ is $s_1 + s_2 + s_3 + q_1 + q_2 + q_3 - 2$. Moreover $g(\theta)$ is monic.*

Proof. Since δ is a fragment in $\digamma$ such that $s_1 < s_3$. Therefore its one vertex is fixed by

$$F_v\left[\left(xy^{-1}\right)^{s_1}(xy)^{s_2}\left(xy^{-1}\right)^{s_3},(xy)^{q_1}\left(xy^{-1}\right)^{q_2}(xy)^{q_3}\right]$$

where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$ and $s_1 < s_3$. Consider a fragment η containing a vertex fixed by

$$F_{u^*}\left[\left(xy^{-1}\right)^{s_3}(xy)^{s_2}\left(xy^{-1}\right)^{s_1},(xy)^{q_3}\left(xy^{-1}\right)^{q_2}(xy)^{q_1}\right].$$

Let $g(\theta)$ and $h(\theta)$ be the polynomials obtained from δ and η. Since $s_3 > s_1$, therefore by Theorem 3.8, degree of the polynomial $h(\theta)$ is $s_1 + s_2 + s_3 + q_1 + q_2 + q_3 - 2$, and $h(\theta)$ is monic. By Lemma 3.9, the polynomials $h(\theta)$ and $g(\theta)$ are the same. □

Theorem 3.11. *Degree of the polynomial* $f(\theta)$ *obtained from any fragment* $\gamma \in \mathcal{F}$ *is* $s_1 + s_2 + s_3 + q_1 + q_2 + q_3 - 2$, *and* $f(\theta)$ *is monic.*

The proof is an immediate consequence of Theorems 3.8 and 3.10.

Theorem 3.12. *Let* $\gamma \in \mathcal{F}$ *and* $T(\gamma)$ *and* $Deg(f)$ *denote the number of triangles in* γ *and the degree of the polynomial obtained from* γ *respectively. Then* $Deg(f) = T(\gamma)$.

The proof is an immediate consequence of Theorems 3.4 and 3.11.

Theorem 3.13. *No polynomial of degree* n *such that* $n \leq 3$, *is obtained from the fragments in* $\mathcal{F}$.

Proof. Let $f(\theta)$ be any polynomial of degree 3, obtained from the fragment of $\mathcal{F}$. Since the degree of all the polynomials obtained from $\mathcal{F}$ is $q_1 + q_2 + q_3 + s_1 + s_2 + s_3 - 2$, where $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$, therefore $q_1 + q_2 + q_3 + s_1 + s_2 + s_3 - 2 = 3$ for some $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$. As there is no possibility for $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$ such that $q_1 + q_2 + q_3 + s_1 + s_2 + s_3 = 5$, therefore, there is no polynomial of degree 3, obtained from the fragments in $\mathcal{F}$.

Similarly, the same result is obtained for $n = 2$ and $n = 1$. □

Theorem 3.14. *There are finite number of polynomials of a fixed degree* n, *obtained from the fragments in* $\mathcal{F}$.

Proof. By Theorem 3.11, degree of all the polynomials obtained from $\mathcal{F}$ is $q_1 + q_2 + q_3 + s_1 + s_2 + s_3 - 2$. Since there are a finite number of possibilities for $s_1, s_2, s_3, q_1, q_2, q_3 \in \mathbb{Z}^+$ such that $q_1 + q_2 + q_3 + s_1 + s_2 + s_3 - 2 = n$, there are only a finite number of polynomials of a fixed degree n, obtained from the fragments in $\mathcal{F}$. □

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References

[1] B. Everitt, Alternating quotients of the $(3, q, r)$ triangle groups, *Comm. Algebra* **26** (1997), no. 6, 1817–1832.

[2] G. Higman and Q. Mushtaq, Coset diagrams and relations for PSL(2, **Z**), *Arab Gulf J. Sci. Res.* **1** (1983), no. 1, 159–164 .

[3] Q. Mushtaq, A condition for the existence of a fragment of a coset diagram, *Quart. J. Math. Oxford Ser. (2)* **39** (1988), no. 153, 81–95.

[4] Q. Mushtaq, Coset diagrams for the modular group, D. Phil. Thesis, University of Oxford, Oxford, 1983.

[5] Q. Mushtaq, Parametrization of all homomorphisms from PGL$(2, \mathbf{Z})$ into PGL$(2, q)$, *Comm. Algebra* **20** (1992), no. 4, 1023–1040.

[6] Q. Mushtaq and G. C. Rota, Alternating groups as quotients of two generator groups, *Adv. Math.* **96** (1993), no. 1, 113–121.

[7] Q. Mushtaq and H. Servatius, Permutation representation of the symmetry groups of regular hyperbolic tessellations, *J. Lond. Math. Soc. (2)* **48** (1993), no. 1, 77–86.

[8] Q. Mushtaq and A. Razaq, Equivalent pairs of words and points of connection, *Sci. World J.* **2014** (2014), Article ID 505496, 8 pages.

[9] A. Torstensson, Coset diagrams in the study of finitely presented groups with an application to quotients of the modular group, *J. Commut. Algebra* **2** (2010), no. 4, 501–514.

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