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TOROIDALIZATION OF LOCALLY TOROIDAL MORPHISMS OF 3-FOLDS

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Abstract. A toroidalization of a dominant morphism $\varphi : X \to Y$ of algebraic varieties over a field of characteristic zero is a toroidal lifting of φ obtained by performing sequences of blow ups of nonsingular subvarieties above X and Y. We give a proof of toroidalization of locally toroidal morphisms of 3-folds.
Keywords: Toroidalization, resolution of morphisms, principalization.
MSC(2010): Primary: 14M99; Secondary: 14B25, 14B05.

1. **Introduction**

The problem of toroidalization is to obtain, for a dominant morphism $\varphi : X \to Y$ of varieties over an algebraically closed field $\mathfrak{k}$ of characteristic zero, a morphism $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ such that there exists a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\varphi}} & \widetilde{Y} \\ \lambda \downarrow & & \downarrow \pi \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

where $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ are sequences of monoidal transforms, i.e., blow ups with nonsingular centers, $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular, and there exist simple normal crossing (SNC) divisors $D_{\widetilde{Y}}$ and $D_{\widetilde{X}} = \tilde{\varphi}^*(D_{\widetilde{Y}})_{\text{red}}$ on $\widetilde{Y}$ and $\widetilde{X}$ respectively, such that $\tilde{\varphi}$ is toroidal with respect to $D_{\widetilde{X}}$ and $D_{\widetilde{Y}}$, i.e., $\tilde{\varphi}$ is locally given by monomials in appropriate *é*tale local parameters on $\widetilde{X}$ with respect to $D_{\widetilde{X}}$ and $D_{\widetilde{Y}}$. The toroidal morphism $\tilde{\varphi}$ is called a *toroidalization* of φ.

The precise definitions of toroidal varieties and their morphisms are in [17], and more recently, in [8, Definition 4.3]. In the case of a nonsingular variety

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X, the choice of a SNC divisor on X makes it into a toroidal variety – see [7], or Section 2 of this paper for definition of SNC divisor.

The idea of toroidalization, which is fundamental in studying the structure of birational morphisms, is first proposed in [1, Problem 6.2.1]. This problem does not have a positive answer in positive characteristic $p > 0$, even for maps of curves, for instance, $y = x^p + x^{p+1}$ [7].

The existence of toroidalization has been proven completely when Y is a curve, or when X and Y are of dimension $\leqslant 3$. When Y is a curve, toroidalization follows from embedded resolution of hypersurface singularities [16], or from any of the simplified proofs including [3, 4, 14].

In the case when X and Y are surfaces, several proofs of toroidalization have been constructed – see, for instance, Corollary 6.2.3 [1], or [13], which includes the case when only tame ramification occurs in positive characteristic.

In [6], and with a simpler proof in [9], toroidalization has been solved for morphisms from 3-folds to surfaces by S. D. Cutkosky, where he introduced the concept of strongly prepared morphism. Toroidalization of a strongly prepared morphism from an n-fold to a surface has also been proven by Cutkosky and Kascheyeva in [11]. In [8], Cutkosky proves toroidalization for dominant morphisms of 3-folds.

Toroidalization, locally along a fixed valuation, has been proven in all dimensions by Cutkosky in [5]. This led up to the notion of locally toroidal morphism.

Suppose that $\varphi : X \to Y$ is a dominant morphism of nonsingular varieties over an algebraically closed field of characteristic zero. Let J be a finite set. The morphism φ is *locally toroidal* if there exist open covers $\{U_j\}_{j\in J}$ of X and $\{V_j\}_{j\in J}$ of Y, and SNC divisors D_j on U_j and E_j on V_j such that for all $j \in J$, $\varphi_j := \varphi|_{U_j} : U_j \to V_j$, $\varphi_j^*(E_j)_{\mathrm{red}} = D_j$, $\varphi_j : U_j \setminus D_j \to V_j \setminus E_j$ is smooth, and $\varphi_j : U_j \to V_j$ is toroidal with respect to E_j and D_j. We will say that φ is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$ if these conditions hold – also, see Definition 1.3 [15].

Patching problems for locally toroidal morphisms which are local resolutions of singularities along a valuation appear in [19] and [18].

The following question of existence of toroidalization for locally toroidal morphisms, proposed by S. D. Cutkosky, has been considered by Hanumanthu in [15] where he provided a positive answer to the question in the case of a locally toroidal morphism from an n-fold to a surface.

Question 1.1. [Cutkosky, Question 1.4 [15]] Suppose that $\varphi : X \to Y$ is a morphism of nonsingular varieties which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Does there exist a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\varphi}} & \widetilde{Y} \\ \lambda\downarrow & & \downarrow\pi \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

such that $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ are sequences of monoidal transforms, $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular, and there exist SNC divisors $\widetilde{E}$ and $\widetilde{D} = \tilde{\varphi}^*(\widetilde{E})_{\text{red}}$ on $\widetilde{Y}$ and $\widetilde{X}$ respectively, such that $\tilde{\varphi}$ is toroidal with respect to $\widetilde{E}$ and $\widetilde{D}$?

In this paper, we prove toroidalization for locally toroidal morphisms of 3-folds.

Theorem 1.2 (Main Theorem)**.** *Suppose $\varphi : X \to Y$ is a morphism of nonsingular 3-folds which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\varphi}} & \widetilde{Y} \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

such that $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ are sequences of monoidal transforms, $\widetilde{X}$ and $\widetilde{Y}$ are nonsingular, and there exist SNC divisors $\widetilde{E}$ and $\widetilde{D} = \tilde{\varphi}^*(\widetilde{E})_{\text{red}}$ on $\widetilde{Y}$ and $\widetilde{X}$ respectively, such that $\tilde{\varphi}$ is toroidal with respect to $\widetilde{E}$ and $\widetilde{D}$.

This theorem is proven in subsection 4.2. It is expected that the methods of this paper can be extended to give a positive answer to Question 1.1.

2. Notations, Definitions, and Main Ideas

Throughout this paper, $\mathfrak{k}$ is an algebraically closed field of characteristic zero. A variety is a quasi projective variety over $\mathfrak{k}$. A curve, surface or 3-fold is a variety of respective dimension 1, 2 or 3. If $D = \sum d_i D_i$ is an effective divisor with $d_i \in \mathbb{Z}_{>0}$ and D_i prime divisors, then $(\sum d_i D_i)_{\text{red}} := \sum D_i$.

An effective divisor D on a nonsingular variety X is *simple normal crossing (SNC)* if at each $p \in X$ there exist regular parameters $(x_1, \ldots, x_n)$ in $\mathcal{O}_{X,p}$ and natural numbers $a_1, \ldots, a_n$ such that $\mathcal{I}_{D,p} = x_1^{a_1} \cdots x_n^{a_n} \mathcal{O}_{X,p}$ where $\mathcal{I}_D \subset \mathcal{O}_X$ is the ideal sheaf of D.

2.1. Locally Toroidal Morphisms of 3-folds.

In this subsection, we provide a necessary and sufficient condition for a morphism of 3-folds to be locally toroidal using the list of toroidal forms for a dominant morphism of 3-folds ([8] pages 21-22).

Definition 2.1 ([8] page 19)**.** Suppose that V is a nonsingular three dimensional variety over an algebraically closed field of characteristic zero and F is a reduced SNC divisor on V. Suppose that $q \in V$ is a closed point. q is called an n-point for F if q lies in exactly n irreducible components of F. We have that $0 \leq n \leq 3$. We say that u, v, w are (formal) permissible parameters at q (for F) if u, v, w are regular parameters in $\hat{\mathcal{O}}_{V,q}$ and

1) $u = 0$ is a (formal) local equation of F if q is a 1-point,

2) $uv = 0$ is a (formal) local equation of F at q if q is a 2-point and
3) $uvw = 0$ is a (formal) local equation of F at q if q is a 3-point.

We say that permissible parameters u, v, w are algebraic permissible parameters if $u, v, w \in \mathcal{O}_{X,q}$.

Proposition 2.2 ([8] pages 21-22)**.** *Suppose that $\varphi : X \to Y$ is a morphism of nonsingular 3-folds and $\{U_j\}_J, \{V_j\}_J$ are open covers of X and Y respectively and D_j is a SNC divisor on U_j, E_j is a SNC divisor on V_j such that for all $j \in J$, $\varphi_j : U_j \to V_j$, $\varphi_j^*(E_j)_{\text{red}} = D_j$ and $U_j \setminus D_j \to V_j \setminus E_j$ is smooth.*

Then $\varphi : X \to Y$ is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$ if and only if the following condition holds for all $j \in J$ and $p \in D_j$.

Let $q = \varphi(p) = \varphi_j(p) \in E_j$. Then there exist (algebraic) permissible parameters u, v, w at q for E_j and (formal) permissible parameters x, y, z at p for D_j such that one of the following forms holds:

(T1) *p is a 3-point of D_j and q is a 3-point of E_j,*

$$u = x^a y^b z^c,\ v = x^d y^e z^f,\ w = x^g y^h z^i,$$

where $a, b, c, d, e, f, g, h, i \in \mathbb{N}$ and

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.$$

(T2) *p is a 2-point of D_j and q is a 3-point of E_j,*

$$u = x^a y^b,\ v = x^d y^e,\ w = x^g y^h (z + \alpha),$$

with $0 \neq \alpha \in \mathfrak{k}$ and $a, b, d, e, f, g, h \in \mathbb{N}$ satisfy $ae - bd \neq 0$ and $(g, h) \neq (0, 0)$.

(T3) *p is a 1-point of D_j and q is a 3-point of E_j,*

$$u = x^a,\ v = x^d (y + \alpha),\ w = x^g (z + \beta),$$

with $0 \neq \alpha,\ 0 \neq \beta \in \mathfrak{k}$ and $a, d, g > 0$.

(T4) *p is a 2-point of D_j and q is a 2-point of E_j,*

$$u = x^a y^b,\ v = x^d y^e,\ w = z,$$

with $ae - bd \neq 0$.

(T5) *p is a 1-point of D_j and q is a 2-point of E_j,*

$$u = x^a,\ v = x^d (y + \alpha),\ w = z,$$

with $0 \neq \alpha \in \mathfrak{k}$ and $a, d > 0$.

(T6) *p is a 1-point of D_j and q is a 1-point of E_j,*

$$u = x^a,\ v = y,\ w = z,$$

with $a > 0$. $\square$

2.2. **Embedded Resolution of Surface Singularities and Its Properties.** In order to have a SNC divisor (a toroidal structure) on Y containing all E_j, we construct the reduced divisor $\tilde{E}_0 = (\sum_{j\in J} \overline{E}_j)_{\mathrm{red}}$ on Y where $\overline{E}_j$ is the Zariski closure of E_j in Y. However $\tilde{E}_0$ is not necessarily SNC and we need to apply the algorithm of embedded resolution of surface singularities. Theorem 2.4 below, follows from this algorithm and its proof, starting with the resolution datum $\mathcal{R} = (\emptyset, \emptyset, \tilde{E}_0, Y)$ (Theorem 5.19 [7]).

Due to the fact that all centers in the algorithm are permissible (Definition 5.21 [7]), we establish further properties of the centers (conclusions 1), 2) of Theorem 2.4) which are basic to the procedure for proving our main result.

Definition 2.3. Suppose that D is a SNC divisor on a nonsingular variety X, Z is a nonsingular subvariety of X, and $p \in X$. We say that Z **makes SNCs with** D at p if there exist regular parameters $x_1, \dots, x_d$ at p, $r \geq 0$ and $m_1, \dots, m_d \geq 0$ such that $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} = 0$ is a local equation of D at p and $x_1 = \cdots = x_r = 0$ are local equations of Z at p. (If $p \notin Z$ then take $r = 0$ and if $p \notin D$ then take $m_i = 0$ for all i).

Theorem 2.4. *Suppose that Y is a nonsingular projective three dimensional variety over an algebraically closed field of characteristic zero. Let J be a finite set. Suppose that $\{V_j\}_{j\in J}$ is an open cover of Y and E_j for $j \in J$ are reduced SNC divisors on V_j. Let $\overline{E}_j$ be the Zariski closure of E_j in Y, and let $\tilde{E}_0 = (\sum_{j\in J} \overline{E}_j)_{\mathrm{red}}$. Then there exists a proper birational morphism $\pi : \tilde{Y} \to Y$ such that $\tilde{Y}$ is nonsingular, $\pi^*(\tilde{E}_0)_{\mathrm{red}}$ is a SNC divisor on $\tilde{Y}$, and π has a factorization*

$$\tilde{Y} = Y_n \xrightarrow{\pi_n} Y_{n-1} \to \cdots \to Y_i \xrightarrow{\pi_i} Y_{i-1} \to \cdots \xrightarrow{\pi_1} Y_0 = Y$$

such that each π_i is the blowup of a nonsingular center $Z_{i-1} \subset Y_{i-1}$ which is either a point or a curve. Let $\Pi_i = \pi_1 \circ \cdots \circ \pi_i : Y_i \to Y$. For $j \in J$, let $V_{i,j} = \Pi_i^{-1}(V_j)$, $\pi_{i,j} = (\pi_i|_{V_{i,j}}) : V_{i,j} \to V_{i-1,j}$, $\Pi_{i,j} = (\Pi_i|_{V_{i,j}}) : V_{i,j} \to V_j$, $E_{i,j} = \Pi_{i,j}^*(E_j)_{\mathrm{red}}$ and $\overline{E}_{i,j}$ be the Zariski closure of $E_{i,j}$ in Y_i.

Let $\tilde{E}_i = (\sum_{j\in J} \overline{E}_{i,j})_{\mathrm{red}}$, a divisor on Y_i. We further have that

1) *$E_{i,j}$ is a SNC divisor on $V_{i,j}$ for all i, j, and $Z_i \cap V_{ij}$ makes SNCs with E_{ij} on V_{ij} for all i, j. (Although possibly $Z_i \cap E_{ij} \neq \emptyset$ but $Z_i \cap V_{ij} \not\subset E_{ij}$).*
2) *$\tilde{E}_i \subseteq \Pi_i^*(\tilde{E}_0)_{\mathrm{red}}$ for all i.*

The remainder of this subsection is devoted to the proof of Theorem 2.4. At the end, we will provide all possible local equations of a nonsingular curve $C \subset Y$ such that for all $j \in J$, $C \cap V_j$ makes SNCs with E_j (Remark–Definition 2.8).

Example 2.5. This example shows that the inclusion of 2) of the conclusions of Theorem 2.4 will not in general be an equality. Let Y be a nonsingular 3-fold. Suppose that $S \subset Y$ is a singular surface and Z is the singular locus of S.

Let $\{V_1 = Y \setminus Z, V_2 = Y\}$ which is clearly an open cover of Y and consider the SNC divisors $E_1 = S \setminus Z$ on V_1 and $E_2 = 0$ on V_2. Then $\widetilde{E}_0 = S$. Embedded resolution of singularities $\pi_1 : Y_1 \to Y$ of S has an exceptional locus F which appears in $\pi_1^*(\widetilde{E}_0)$. However, $V_{1,1} \cap F = \emptyset$ and $E_{1,2} = 0$ on $V_{1,2}$. So, F is not contained in $\overline{E}_{1,1} + \overline{E}_{1,2}$.

Lemma 2.6. *Suppose that A is the strict transform on Y_i of an irreducible component of $\tilde{E}_0$. If $Z_i \cap A \neq \emptyset$ then $Z_i \subset A$.*

Proof. Let D be the strict transform of $\tilde{E}_0$ on Y_i. Let r be the maximum multiplicity of points of D on Y_i. Since Z_i is permissible for the resolution algorithm, $Z_i \subset D$ and D has order r at all points of D. Let $B_1, \ldots, B_r$ be the irreducible components of D containing Z_i and let $A, C_1, \ldots, C_s$ be the irreducible components of D which contain q but do not contain Z_i. Let Q be the generic point of Z_i. Then the multiplicity satisfies

$$\begin{array}{lll} r & = & \nu_q(D) = \nu_q(A) + \sum \nu_q(C_i) + \sum \nu_q(B_i) \\ & > & \sum \nu_q(B_i) \\ & \geq & \sum \nu_Q(B_i) \text{ by upper semicontinuity of multiplicity (Appendix A.19 [7])} \\ & = & \nu_Q(D) = r \end{array}$$

giving a contradiction. □

Lemma 2.7. *Suppose that X is a nonsingular variety and $D = A + B$ is a SNC divisor on X where A and B have no irreducible components in common. Suppose that Z is a nonsingular subvariety of X such that Z makes SNCs with A and if C is an irreducible component of B then either $Z \subset C$ or $Z \cap C = \emptyset$. Then Z makes SNCs with D.*

Proof. Suppose that $q \in Z \cap \operatorname{Supp}(D)$. Let $R = \mathcal{O}_{X,q}$ and $P = \mathcal{I}_{Z,q}$. Let $\mathfrak{m}$ be the maximal ideal of R and let $\mathfrak{m}'$ be the maximal ideal of $R/P = \mathcal{O}_{Z,q}$. We have a short exact sequence of $L = R/\mathfrak{m}$ vector spaces

$$0 \to P/P \cap \mathfrak{m}^2 \cong (P + \mathfrak{m}^2)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}'/(\mathfrak{m}')^2 \to 0.$$

Let $x_t = 0$ for $1 \leq t \leq r$ be local equations at q of the irreducible components of A which contain q and let $x_t = 0$ for $r+1 \leq t \leq s$ be local equations at q of the irreducible components of B which contain q. Since $A + B$ is a SNC divisor,

(2.1) the classes $\overline{x}_1, \ldots, \overline{x}_s$ of $x_1, \ldots, x_s$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent.

After possibly reindexing the x_i, we may assume that $x_1, \ldots, x_c \notin P$ and $x_{c+1}, \ldots, x_r \in P$. By assumption, we have that $x_{r+1}, \ldots, x_s \in P$. Since A makes SNCs with Z, $x_1, \ldots, x_r$ can be extended to a regular system of parameters

$$x_{c+1}, \ldots, x_r, y_1, \ldots, y_a, x_1, \ldots, x_c, z_1, \ldots, z_b$$

of R such that $\{\overline{x}_{c+1}, \ldots, \overline{y}_a\}$ is a basis of $P + \mathfrak{m}^2/\mathfrak{m}^2$ and $\{\overline{x}_{c+1}, \ldots, \overline{z}_b\}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$. In particular,

$$\operatorname{Span}(\overline{x}_1, \ldots, \overline{x}_c) \cap (P + \mathfrak{m}^2)/\mathfrak{m}^2 = \{0\}. \quad (2.2)$$

By (2.1), $\overline{x}_{c+1}, \ldots, \overline{x}_s$ are linearly independent, so they can be extended to a basis $\{\overline{x}_{c+1}, \ldots, \overline{x}_s, \sigma_1, \ldots, \sigma_d\}$ of $(P+\mathfrak{m}^2)/\mathfrak{m}^2$. By (2.2),

$$\overline{x}_{c+1}, \ldots, \overline{x}_s, \sigma_1, \ldots, \sigma_d, \overline{x}_1, \ldots, \overline{x}_c$$

are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, so we can extend them to a basis

$$\{\overline{x}_{c+1}, \ldots, \overline{x}_s, \sigma_1, \ldots, \sigma_d, \overline{x}_1, \ldots, \overline{x}_c, \tau_1, \ldots, \tau_e\}$$

of $\mathfrak{m}/\mathfrak{m}^2$. Let $u_1, \ldots, u_d \in P$ be such that the class of $\overline{u}_i$ in $P/P \cap \mathfrak{m}^2 \cong (P+\mathfrak{m}^2)/\mathfrak{m}^2$ is σ_i for $1 \le i \le d$ and let $v_1, \ldots, v_e \in \mathfrak{m}$ be such that $\overline{v}_i = \tau_i$ for $1 \le i \le e$. Then (by definition)

$$x_{c+1}, \ldots, x_s, u_1, \ldots, u_d, x_1, \ldots, x_c, v_1, \ldots, v_e \tag{2.3}$$

is a regular system of parameters in R. Let I be the ideal

$$I = (x_{c+1}, \ldots, x_s, u_1, \ldots, u_d).$$

We have that

$$s - c + d = \dim R - \dim R/I \tag{2.4}$$

by Proposition A.4 [2]. Let $\mathfrak{n}$ be the maximal ideal of R/I. From the exact sequence

$$0 \to (I+\mathfrak{m}^2)/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0$$

of L-vector spaces, we see that

$$\begin{aligned} \dim \mathfrak{n}/\mathfrak{n}^2 &= \dim \mathfrak{m}/\mathfrak{m}^2 - \dim(I+\mathfrak{m}^2)/\mathfrak{m}^2 \\ &= \dim \mathfrak{m}/\mathfrak{m}^2 - (s-c+d) \\ &= \dim \mathfrak{m}/\mathfrak{m}^2 - (\dim R - \dim R/I) \text{ by (2.4)} \\ &= \dim R/I \end{aligned}$$

since R is a regular local ring. Thus R/I is a regular local ring; in particular, I is a prime ideal. Since I and P have the same height, $I = P$. Thus by (2.3), Z makes SNCs with $D = A + B$ at q. □

Now we give the proof of Theorem 2.4.

Proof of Theorem 2.4. We first prove 1). We will establish 1) by induction on i that $E_{i,j}$ is a SNC divisor on $V_{i,j}$ and $Z_i \cap V_{i,j}$ makes SNCs with $E_{i,j}$ at all points of $Z_i \cap V_{i,j}$. Since $E_{i+1,j}$ will be a SNC divisor if $Z_i \cap V_{i,j}$ makes SNCs with $E_{i,j}$, we may assume by induction that $E_{i,j}$ is a SNC divisor. Let $Z = Z_i \cap V_{i,j}$. We must show that Z makes SNCs with $E_{i,j}$. We decompose $E_{i,j} = A + B$ where B is the strict transform of E_j on V_j to $V_{i,j}$ and A is the sum of exceptional components of $\Pi_{i,j}$. Since Z_i is permissible for the resolution datum of the algorithm, Z_i makes SNCs with A. By Lemmas 2.6 and 2.7, we conclude that Z makes SNCs with $E_{i,j}$.

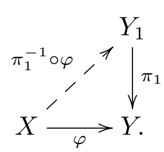
And clearly, 2) follows since $Z_i \subseteq \Pi_i^*(\tilde{E}_0)_{\mathrm{red}}$ for all i, as Z_i is a permissible center for the algorithm (Definition 5.21 [7]). □

Remark–Definition 2.8. Suppose that $\varphi : X \to Y$ is a morphism of nonsingular 3-folds, which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$ and $C \subset Y$ is a nonsingular curve such that for all $j \in J$, $C \cap V_j$ makes SNCs with E_j. If $q \in E_j \cap C$ for some j, then there exist algebraic permissible parameters u, v, w at q for E_j such that one of the following holds.

1) q is a 3-point of E_j (which has local equation $uvw = 0$ at q) and $u = v = 0$ are local equations of C at q and, C is called a **2^+-curve** for E_j at q.
2) q is a 2-point of E_j (which has local equation $uv = 0$ at q) and $u = v = 0$ are local equations of C at q and, C is called a **2-curve** for E_j at q.
3) q is a 2-point of E_j (which has local equation $uv = 0$ at q) and $u = w = 0$ are local equations of C at q and, C is called a **1^+-curve** for E_j at q.
4) q is a 1-point of E_j (which has local equation $u = 0$ at q) and $u = v = 0$ are local equations of C at q and, C is called a **1-curve** for E_j at q.
5) q is a 1-point of E_j (which has local equation $u = 0$ at q) and $v = w = 0$ are local equations of C at q and, C is called a **0^+-curve** for E_j at q.

If $q \in (C \cap V_j) \setminus E_j$, i.e., q is a 0-point for E_j, there exist regular parameters u, v, w at q such that $u = v = 0$ are local equations of C at q and C is called a **0-curve** for E_j at q.

2.3. **Extended Strategy for the Proof.** Suppose that $\varphi : X \to Y$ is a locally toroidal morphism of nonsingular 3-folds with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$ and $\pi_1 : Y_1 \to Y$ is the blow up of a permissible center $Z \subset Y$ satisfying the conclusions 1), 2) of the Theorem 2.4. Then we will obtain



The locus of indeterminant points of the rational map $\pi_1^{-1} \circ \varphi$ is the set $W_Z(X)$ defined in Definition 2.9 below. Due to the toroidal forms of Proposition 2.2, we will describe $W_Z(X)$ explicitly in Proposition 2.10.

To resolve the indeterminancy of the rational map $X \dashrightarrow Y_1$, we will provide a careful algorithm for principalization of monomial ideals in Definition 2.11 so that the resulting morphism, after resolution of indeterminancy, is again locally toroidal. We must be very careful about how we principalize, as most resolutions of indeterminancy will not have this property – see Example 2.14.

Definition 2.9. Suppose that $\varphi : X \to Y$ is a dominant morphism of nonsingular varieties and $Z \subset Y$ is a nonsingular subvariety. Define

$$\mathbf{W_Z(X) = \{p \in X \mid \mathcal{I}_Z\mathcal{O}_{X,p} \text{is not invertible}\}}$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z in $\mathcal{O}_Y$.

Suppose that X is a 3-fold. We will say that $W_Z(X)$ is **SNC** if the reduced ideal sheaf $\mathcal{I}_{W_Z(X)}$ has the property that for every close point $p \in W_Z(X)$, there are regular parameters x, y, z in $\mathcal{O}_{X,p}$ such that one of the following forms hold:

(N.1) $\mathcal{I}_{W_Z(X),p} = (x, y) \cap (x, z) \cap (y, z)$,
(N.2) $\mathcal{I}_{W_Z(X),p} = (x, z) \cap (y, z)$,
(N.3) $\mathcal{I}_{W_Z(X),p} = (x, z)$,
(N.4) $\mathcal{I}_{W_Z(X),p} = (x, y, z)$.

Proposition 2.10. *Suppose that $\varphi : X \to Y$ is a morphism of nonsingular 3-folds which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Let $Z \subset Y$ be a point q or a nonsingular curve C such that for all $j \in J$, $C \cap V_j$ makes SNCs with E_j. Then $W_Z(X)$ is SNC.*

Proof. First, suppose that Z is a point $q \in Y$ and $\mathcal{I}_q$ is the ideal sheaf of q in $\mathcal{O}_Y$. Since X is nonsingular we have the factorization $\mathcal{I}_q\mathcal{O}_X = \mathcal{J}\mathcal{K}$ where $\mathcal{J}$ is an invertible ideal sheaf and $\dim \mathcal{O}_X/\mathcal{K} < \dim X - 1$ (Lemma 15.8 [10]). Then $\sqrt{\mathcal{K}} = \mathcal{I}_{W_q(X)}$ and $\mathrm{Supp}(\mathcal{O}_X/\mathcal{K}) = W_q(X)$.

Let $p \in W_q(X)$ which is obviously contained in $\varphi^{-1}(q)$. Suppose that $p \in U_j$ for some $j \in J$.

If $q \notin E_j$, then φ is smooth at p since φ is locally toroidal. So there exist regular parameters x, y, z in $\mathcal{O}_{X,p}$ and regular parameters u, v, w in $\mathcal{O}_{Y,q}$ such that $u = x$, $v = y$ and $w = z$. Thus

$$\mathcal{I}_q\mathcal{O}_{X,p} = (u, v, w)\mathcal{O}_{X,p} = (x, y, z) = \mathcal{K}_p$$

and $\mathcal{I}_{W_q(X),p} = \sqrt{\mathcal{K}_p} = (x, y, z)$. So, (N.4) holds for p.

If $q \in E_j$ for some $j \in J$, then p must lie in $D_j \subset U_j$ and, since $\varphi_j : U_j \to V_j$ is toroidal with respect to E_j and D_j, by Proposition 2.2, one of the toroidal forms (T1) through (T6) holds.

Suppose that (T1) holds for $q \in E_j$ and $p \in W_q(X) \subseteq \varphi^{-1}(q) \subset D_j$, then there exist algebraic permissible parameters u, v, w at q for E_j and (formal) permissible parameters x, y, z at p for D_j such that p is a 3-point of D_j with local equation $xyz = 0$, q is a 3-point for E_j with local equation $uvw = 0$, and

$$u = x^a y^b z^c$$
$$v = x^d y^e z^f$$
$$w = x^g y^h z^i,$$

where $a, b, c, d, e, f, g, h, i \in \mathbb{N}$ and $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0$.

In addition, D_j is a SNC divisor on U_j and there exist regular parameters $\bar{x}, \bar{y}, \bar{z}$ in $\mathcal{O}_{X,p}$ such that $\bar{x}\bar{y}\bar{z} = 0$ is a local equation of D_j. So there exists unit series $\delta_x, \delta_y, \delta_z \in \hat{\mathcal{O}}_{X,p}$ such that, after possibly interchanging the variables, $x = \delta_x\bar{x}$, $y = \delta_y\bar{y}$ and $z = \delta_z\bar{z}$.

Set $\delta_u = \delta_x^a\delta_y^b\delta_z^c$, $\delta_v = \delta_x^d\delta_y^e\delta_z^f$ and $\delta_w = \delta_x^g\delta_y^h\delta_z^i$. Then we have

$$u = \delta_u\bar{x}^a\bar{y}^b\bar{z}^c$$
$$v = \delta_v\bar{x}^d\bar{y}^e\bar{z}^f$$
$$w = \delta_w\bar{x}^g\bar{y}^h\bar{z}^i.$$

In fact, δ_u, δ_v and δ_w are units in $\mathcal{O}_{X,p}$ since, for instance,

$$\delta_u = \frac{u}{\bar{x}^a\bar{y}^b\bar{z}^c} \in QF(\mathcal{O}_{X,p}) \cap \widehat{\mathcal{O}}_{X,p} = \mathcal{O}_{X,p}$$

by Lemma 2.1 [5]. So, $\bar{u} = \delta_u^{-1}u$, $\bar{v} = \delta_v^{-1}v$ and $\bar{w} = \delta_w^{-1}w$ are regular algebraic permissible parameters at q and we have

$$\mathcal{I}_q\mathcal{O}_{X,p} = (\bar{u},\bar{v},\bar{w})\mathcal{O}_{X,p} = (\bar{x}^a\bar{y}^b\bar{z}^c, \bar{x}^d\bar{y}^e\bar{z}^f, \bar{x}^g\bar{y}^h\bar{z}^i).$$

Let $d_x = \min\{a,d,g\}$, $d_y = \min\{b,e,h\}$ and $d_z = \min\{c,f,i\}$, then

$$\mathcal{I}_q\mathcal{O}_{X,p} = (\bar{x}^{d_x}\bar{y}^{d_y}\bar{z}^{d_z})\mathcal{K}_p.$$

Since $\mathcal{I}_q\mathcal{O}_{X,p}$ is a monomial ideal, so are $\mathcal{K}_p$ and $\mathcal{I}_{W_q(X),p} = \sqrt{\mathcal{K}_p}$. Therefore, $\mathcal{I}_{W_q(X),p}$ must be one of the ideals in (N.1) through (N.4). Geometrically, there exists an affine neighborhood U_p of p such that $U_p \cap W_q(X)$ is a finite union of coordinate subspaces, i.e., vector subspaces of $\mathfrak{k}^n$ defined by setting some subset of variables $\bar{x}, \bar{y}, \bar{z}$ equal to zero (page 440, Proposition 1. [12]).

The proof is similar when one of the forms (T2) through (T6) holds, also when Z is a nonsingular curve $C \subset Y$ such that, for all $j \in J$, $C \cap V_j$ makes SNCs with E_j. □

Definition 2.11. Suppose that $\varphi : X \to Y$ is a morphism of nonsingular 3-folds, which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Let $Z \subset Y$ be a point or a nonsingular curve C such that for all $j \in J$, $C \cap V_j$ makes SNCs with E_j. A **principalization sequence** of Z is a sequence

$$\cdots \to X_n \xrightarrow{\lambda_n} X_{n-1} \to \cdots \to X_i \xrightarrow{\lambda_i} X_{i-1} \to \cdots X_1 \xrightarrow{\lambda_1} X$$

such that each $\lambda_i : X_i \to X_{i-1}$ is the blow up of a nonsingular curve or point in $W_Z(X_{i-1})$ satisfying the following conditions.

1) $W_Z(X_i)$ is SNC for all i.
2) Since X is nonsingular, we have a factorization $\mathcal{I}_Z\mathcal{O}_X = \mathcal{J}\mathcal{I}_0$ where $\mathcal{I}_Z$ is the ideal sheaf of Z in $\mathcal{O}_Y$, $\mathcal{J}$ is an invertible ideal sheaf and $\dim \mathcal{O}_X/\mathcal{I}_0 < \dim X - 1$ (Lemma 15.8 [10]). Thus $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}_0) = W_Z(X)$. Let $\mathcal{I}_i$ be the weak transform of $\mathcal{I}_0$ (page 65 [7]), so that $\mathrm{Supp}(\mathcal{O}_X/\mathcal{I}_i) = W_Z(X_i)$. Let

$$r_i = \max\{\nu_p(\mathcal{I}_i) \mid p \in X_i\}$$

where $\nu_p(\mathcal{I}_i)$ is the order of $(\mathcal{I}_i)_p$ in $\mathcal{O}_{X_i,p_i}$ (Definition A.17 [7]). Then for all i, Z_i is an irreducible component of maximal dimension of

$$MaxW_Z(X_i) = \{p \in X_i \mid \nu_p(\mathcal{I}_i) = r_i\}.$$

Example 2.14 shows that we have to be careful in our construction of a principalization of an ideal sheaf in order to obtain a locally toroidal morphism. This is the reason for the condition 2) in the statement of Definition 2.11.

Definition 2.12. The composition of a principalization sequence and a toroidal form is called a **quasi-toroidal form**.

In Section 3, we will apply this algorithm to resolve the indeterminancy of the rational map $\pi_1^{-1}\circ\varphi : X \dashrightarrow Y_1$ considering Z to be different types of points (Definition 2.1) and different types of permissible curves (Remark–Definition 2.8). Meanwhile we will provide a thorough list of quasi-toroidal forms. We will see that any principalization sequence λ of Z which is actually a resolution of indeterminancy of $X \dashrightarrow Y_1$ is finite.

Then we will prove, in Section 4, that the induced morphism $\phi_1 : X_1 \to Y_1$ which gives the commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_1} & Y_1 \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

is locally toroidal with respect to the modified local toroidal structure.

Iterative use of this process results in Theorem 2.13 below, which is proven in subsection 4.2. Finally, in the proof of our main result Theorem 1.2, we will prove that $\tilde{\varphi}$ constructed in Theorem 2.13 is actually our desired toroidal morphism.

Theorem 2.13. *Suppose that $\varphi : X \to Y$ is a morphism of nonsingular 3-folds which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Let $\overline{E}_j$ be the Zariski closure of E_j in Y, and let $\tilde{E}_0 = (\sum_{j\in J}\overline{E}_j)_{\text{red}}$. Then there exist proper birational morphisms $\pi : \widetilde{Y} \to Y$ and $\lambda : \widetilde{X} \to X$ such that $\widetilde{Y}$ and $\widetilde{X}$ are nonsingular, $\pi^*(\tilde{E}_0)_{\text{red}}$ is a SNC divisor on $\widetilde{Y}$ and, $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ is locally toroidal with respect to $\widetilde{\mathcal{L}} = \{\tilde{U}_j, \tilde{D}_j, \tilde{V}_j, \tilde{E}_j\}_J$ where $\tilde{U}_j = \lambda^{-1}(U_j)$, $\tilde{V}_j = \pi^{-1}(V_j)$, $\tilde{D}_j = (\lambda|_{\tilde{U}_j})^*(D_j)_{\text{red}}$ and $\tilde{E}_j = (\pi|_{\tilde{V}_j})^*(E_j)_{\text{red}}$. Furthermore, there exists a commutative diagram*

$$(\mathcal{LT}) \qquad \begin{array}{ccccccccc} \widetilde{X} = X_n & \xrightarrow{\lambda_n} & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_1 & \xrightarrow{\lambda_1} & X \\ {\scriptstyle\tilde{\varphi}=\phi_n}\downarrow & & {\scriptstyle\phi_{n-1}}\downarrow & & & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\varphi} \\ \widetilde{Y} = Y_n & \xrightarrow[\pi_n]{} & Y_{n-1} & \longrightarrow & \cdots & \longrightarrow & Y_1 & \xrightarrow[\pi_1]{} & Y \end{array}$$

such that each π_i is the blow up of a nonsingular center $Z_{i-1} \subset Y_{i-1}$ which is either a point or a curve, and each λ_i is a principalization sequence of Z_{i-1} with $W_{Z_{i-1}}(X_i) = \emptyset$.

Let $\Pi_i = \pi_1 \circ \cdots \circ \pi_i : Y_i \to Y$ and $\Lambda_i = \lambda_1 \circ \cdots \circ \lambda_i : X_i \to X$. For $j \in J$, let $V_{i,j} = \Pi_i^{-1}(V_j)$, $\Pi_{i,j} = (\Pi_i|_{V_{i,j}}) : V_{i,j} \to V_j$, $E_{i,j} = \Pi_{i,j}^*(E_j)_{\text{red}}$, and let

$U_{i,j} = \Lambda_i^{-1}(U_j)$, $\Lambda_{i,j} = (\Lambda_i|_{U_{i,j}}) : U_{i,j} \to U_j$, $D_{i,j} = \Lambda_{i,j}^*(D_j)_{\rm red}$. *We further have that, for all* i, j,

1) $D_{i,j}$ *is a SNC divisor on* $U_{i,j}$ *and* $E_{i,j}$ *is a SNC divisor on* $V_{i,j}$.
2) $\phi_i : X_i \to Y_i$ *is locally toroidal w.r.t.* $\mathcal{L}_i = \{U_{i,j}, D_{i,j}, V_{i,j}, E_{i,j}\}_J$.

Example 2.14. This example shows that if $\varphi : X \to Y$ is locally toroidal and $Z \subset Y$ is a nonsingular curve such that $C \cap V_j$ makes SNCs with E_i for all j, then most sequences of blow ups of points and nonsingular curves which principalize $\mathcal{I}_Z\mathcal{O}_X$ will not lead to a resolution of indeterminacy $\phi_1 : X_1 \to Y_1$ (where Y_1 is the blow up of Z) such that ϕ_1 is locally toroidal. This is why we need the restriction 2) of Definition 2.11. For our example, we consider the following germ of a locally toroidal map $\varphi : X \to Y$.

Suppose that C is a 2-curve at the point $q \in Y$ which is a 2-point of E_j. So, there exist algebraic permissible parameters u, v, w at q for E_j such that $uv = 0$ is a local equation of E_j at q and $u = v = 0$ are local equations of C at q. We consider $p \in \varphi^{-1}(q)$ to be a 2-point of D_j and that there exist (formal) permissible parameters x, y, z at p such that $xy = 0$ is a local equation of D_j at p and

$$u = x^2y,\ v = xy^3,\ w = z.$$

Suppose that $\pi : Y_1 \to Y$ is the blow up of C, and we want to resolve the indeterminancy. We note that

$$(u, v)\hat{\mathcal{O}}_{X,p} = (x^2y, xy^3) = (xy)(x, y^2).$$

So, $W_C(X) \cap U_j$ contains the curve $x = y = 0$. Now, the point p which has local equations $x = y = z = 0$ makes SNCs with $W_C(X)$. If we blow it up to get $\lambda_1 : X_1 \to X$, we also have that $D_{1,j} = (\lambda_1|_{U_j})^*(D_j)_{\rm red}$ is a SNC divisor on U_j.

Consider the point $p_1 \in \lambda_1^{-1}(p)$ which has regular parameters x_1, y_1, z_1 defined by

$$x = x_1,\ y = x_1y_1,\ z = x_1z_1.$$

Then $D_{1,j}$ has the local equation $x_1y_1 = 0$ at p_1, so x_1, y_1, z_1 are permissible parameters at p_1. Substituting into u, v, w, we obtain

$$u = x_1^3y_1,\ v = x_1^4y_1^3,\ w = x_1z_1.$$

Thus the rational map $\phi_{1,j} : U_{1,j} \dashrightarrow V_{1,j}$ is a morphism at p_1, and $q_1 = \phi_{1,j}(p_1)$ has regular parameters $u_1 = u$, $v_1 = \frac{v}{u}$, $w_1 = w$. These are permissible parameters at q_1 for $E_{1,j} = (\pi|_{V_j})^*(E_j)_{\rm red}$ and $u_1v_1 = 0$ is a local equation of $E_{1,j}$ at q_1. However, local equations of $\phi_{1,j}$ at p_1 are

$$u_1 = x_1^3y_1,\ v_1 = x_1y_1^2,\ w_1 = x_1z_1$$

which is not toroidal with respect to $D_{1,j}$ and $E_{1,j}$.

3. **Principalization**

This Section is devoted to proving that any principalization sequence (Definition 2.11) of a permissible center Z in the resolution algorithm is finite. That is Theorem 3.3 and the proof is based on a detailed analysis of the principalization sequences of all types of permissible centers.

Suppose that $\varphi : X \to Y$ is a locally toroidal morphism of nonsingular 3-folds with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Let

$$(\mathcal{P}) \qquad \cdots \to X_n \xrightarrow{\lambda_n} X_{n-1} \to \cdots \to X_i \xrightarrow{\lambda_i} X_{i-1} \to \cdots X_1 \xrightarrow{\lambda_1} X$$

be a principalization sequence of a point $q \in Y$ or a nonsingular curve $C \subset Y$ such that $C \cap V_j$ makes SNCs with E_j for all $j \in J$, and let $\Lambda_i = \lambda_1 \circ \cdots \lambda_i$.

3.1. **Analysis of Principalization Sequences of Points.** In the following Lemma, we study the sequence ($\mathcal{P}$) for all possibilities of a center that is a point $q \in Y$ (Definition 2.1).

Lemma 3.1. *Suppose $j \in J$ and $q \in V_j$. There exist algebraic permissible parameters u, v, w at q for E_j such that for all i, if $p \in (\varphi \circ \Lambda_i)^{-1}(q) \cap \Lambda_i^{-1}(U_j)$, then*

I. *$D_{i,j} = (\Lambda_i|_{U_j})^*(D_j)_{\mathrm{red}}$ is a SNC divisor in a neighborhood of p, and there exist formal permissible parameters x, y, z at p for $D_{i,j}$ such that*

- (i) *If q is a 3-point for E_j, and $uvw = 0$ is a local equation of E_j at q, one of the following forms holds at p.*
  - ($\mathfrak{qt}1$) *p is a 3-point for $D_{i,j}$, $xyz = 0$ is a local equation of $D_{i,j}$ at p and (T1) holds.*
  - ($\mathfrak{qt}2$) *p is a 2-point for $D_{i,j}$, $xy = 0$ is a local equation of $D_{i,j}$ at p and (T2) holds.*
  - ($\mathfrak{qt}3$) *p is a 1-point for $D_{i,j}$, $x = 0$ is a local equation of $D_{i,j}$ at p and (T3) holds.*
- (ii) *If q is a 2-point for E_j, and $uv = 0$ is a local equation of E_j at q, one of the following forms holds at p.*
  - ($\mathfrak{qt}4$) *p is a 2-point and $xy = 0$ is a local equation of $D_{i,j}$ at p, and*
  $$u = x^a y^b,\ v = x^d y^e,\ w = x^g y^h (z + \alpha)$$
  with $ae - bd \neq 0$, $g \leqslant \min\{a, d\}, h \leqslant \min\{b, e\}$, and $\alpha \in \mathfrak{k}$.
  - ($\mathfrak{qt}5$) *p is a 3-point and $xyz = 0$ is a local equation of $D_{i,j}$ at p, and*
  $$u = x^a y^b z^c,\ v = x^d y^e z^f,\ w = x^g y^h z^i$$
  where $g = \min\{a, d, g\}, h = \min\{b, e, h\}$ and $i = \min\{c, f, i\}$, and
  $$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.$$
  - ($\mathfrak{qt}6$) *p is a 1-point and $x = 0$ is a local equation of $D_{i,j}$ at p, and*
  $$u = x^a,\ v = x^d (y + \alpha),\ w = x^g (z + \beta)$$

with $a, d > 0$, $\alpha, \beta \in \mathfrak{k}$, $\alpha \neq 0$ *and* $g \leqslant \min\{a, d\}$.

($\mathfrak{qt}7$) *p is a 2-point and $xz = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a z^a,\ v = x^d z^d (y + \alpha),\ w = x^g z^{g+1}$$

with $a, d > 0$, $0 \neq \alpha \in \mathfrak{k}$ *and* $g + 1 \leqslant \min\{a, d\}$.

(iii) *If q is a 1-point for E_j, and $u = 0$ is a local equation of E_j at q, one of the following forms holds at p.*

($\mathfrak{qt}8$) *p is a 1-point and $x = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a,\ v = x^{a'}(y + \alpha),\ w = x^{a'}(z + \beta)$$

where $a, a' \in \mathbb{N}$ satisfy $a > 0$ and $a' \leqslant a$, and $\alpha, \beta \in \mathfrak{k}$.

($\mathfrak{qt}9$) *p is a 2-point and $xy = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a y^a,\ v = x^{a'} y^{a'+1},\ w = x^{a'} y^{a'+1}(z + \alpha)$$

with $\alpha \in \mathfrak{k}$ and $a, a' \in \mathbb{N}$ satisfy $a > 0$ and $a' + 1 \leqslant a$.

($\mathfrak{qt}10$) *p is a 2-point and $xz = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a z^a,\ v = x^{a'} y z^{a'+1},\ w = x^{a'} z^{a'+1}$$

where $a, a' \in \mathbb{N}$ satisfy $a > 0$ and $a' + 1 \leqslant a$.

(iv) *If q is a 0-point for E_j, i.e., $q \in V_j \setminus E_j$, one of the following forms holds at p.*

($\mathfrak{qt}11$) *p is a 0-point for $D_{i,j}$, and $u = x, v = y, w = z$.*

($\mathfrak{qt}12$) *p is a 0-point for $D_{i,j}$, and*

$$u = x, v = x(y + \alpha), w = x(z + \beta) \text{ with } \alpha, \beta \in \mathfrak{k}.$$

($\mathfrak{qt}13$) *p is a 0-point for $D_{i,j}$, and*

$$u = xy, v = y, w = y(z + \alpha) \text{ with } \alpha \in \mathfrak{k}.$$

($\mathfrak{qt}14$) *p is a 0-point for $D_{i,j}$, and $u = xz, v = yz, w = z$.*

II. *$W_q(X_i)$ is SNC for all i. Precisely, if $p \in W_q(X_i)$, then there exist (formal) permissible parameters x, y, z at p for $D_{i,j}$ such that one of the following possibilities holds. The weak transform of $\mathcal{I}_0$ defined by 2) of Definition 2.11 on X_i is denoted by $\mathcal{I}_i$ which satisfies $\sqrt{\mathcal{I}_i} = \mathcal{I}_{W_q(X_i)}$.*

($\mathfrak{qt}1.\mathfrak{np}$) *We are in the case ($\mathfrak{qt}1$) and $\hat{\mathcal{I}}_{W_q(X_i),p}$ is one of the ideals (N.1) through (N.4).*

($\mathfrak{qt}2.\mathfrak{np}$) *We are in the case ($\mathfrak{qt}2$) and $\hat{\mathcal{I}}_{W_q(X_i),p} = (x, y)$.*

($\mathfrak{qt}4.\mathfrak{np}$) *We are in the case ($\mathfrak{qt}4$) with equations*

$$u = x^a y^b,\ v = x^d y^e,\ w = x^g y^h z$$

where $ae - bd \neq 0$, and

$$g < \min\{a, d\} \text{ or } h < \min\{b, e\} \text{ or } (a - d)(b - e) < 0.$$

Then $\mathcal{I}_i$ satisfies

$$\hat{\mathcal{I}}_{i,p} = (x^{\min\{a-g,d-g\}}, z) \cap (x^{\max\{a-g,d-g\}}, y^{\max\{b-h,e-h\}}, z) \cap (y^{\min\{b-h,e-h\}}, z).$$

In particular, if $g = \min\{a,d\}, h = \min\{b,e\}$,

$$\hat{\mathcal{I}}_{i,p} = (x^{\max\{a-g,d-g\}}, y^{\max\{b-h,e-h\}}, z).$$

Hence $\hat{\mathcal{I}}_{W_q(X_i),p}$ *is one of the ideals (N.2), (N.3) or (N.4).*

($\mathfrak{qt}6.\mathfrak{np}$) *We are in the case* ($\mathfrak{qt}6$) *with equations*

$$u = x^a,\ v = x^d(y+\alpha),\ w = x^g z$$

where $0 \neq \alpha \in \mathfrak{k}$ *and* $g < \min\{a,d\}$. *In addition,* $\hat{\mathcal{I}}_{W_q(X_i),p} = (x,z)$.

($\mathfrak{qt}8.\mathfrak{np}$) *We are in the case* ($\mathfrak{qt}8$) *with equations*

$$u = x^a,\ v = x^{a'}y,\ w = x^{a'}z \text{ and } a' < a.$$

Further, $\mathcal{I}_i$ *satisfies* $\hat{\mathcal{I}}_{i,p} = (x^{a-a'}, y, z)$ *with* $a - a' > 0$. *Thus* $\hat{\mathcal{I}}_{W_q(X_i),p} = (x,y,z)$.

($\mathfrak{qt}11.\mathfrak{np}$) *We are in the case* ($\mathfrak{qt}11$)*, and* $\mathcal{I}_i$ *satisfies* $\hat{\mathcal{I}}_{i,p} = (x,y,z) = \hat{\mathcal{I}}_{W_q(X_i),p}$.

Proof. By Proposition 2.2 and Proposition 2.10, X satisfies the conclusions of the lemma. If we prove conclusion I, then, by an argument similar to that of Proposition 2.10, we will obtain conclusion II. So, it remains to show that conclusion I holds. Inductively, we assume that the conclusions hold for Λ_{i-1} and we prove them for Λ_i.

Suppose that $p \in (\varphi\circ\Lambda_i)^{-1}(q) \cap \Lambda_i^{-1}(U_j) = \lambda_i^{-1}((\varphi\circ\Lambda_{i-1})^{-1}(q) \cap \Lambda_{i-1}^{-1}(U_j))$ and let $T_{i-1} \subseteq W_q(X_{i-1})$ be the center of λ_i.

We can assume that $p \in \lambda_i^{-1}(T_{i-1})$ since λ_i is an isomorphism out of the center, i.e., at points $p \in X_i \setminus \lambda_i^{-1}(T_{i-1})$. Let $\bar{p} = \lambda_i(p) \in T_{i-1}$.

By the induction hypothesis, there exist formal permissible parameters $\bar{x}, \bar{y}, \bar{z}$ at $\bar{p}$ for $D_{i-1,j}$ such that one of the cases ($\mathfrak{qt}1.\mathfrak{np}$) through ($\mathfrak{qt}11.\mathfrak{np}$) of the conclusion II of the lemma holds. All of the cases are similar. We will work out in detail the case when ($\mathfrak{qt}4.\mathfrak{np}$) holds for $\bar{p}$ (so $q \in V_j$ is a 2-point for E_j, $\bar{p}$ is a 2-point for $D_{i-1,j}$, and we are in the case ($\mathfrak{qt}4$)).

Suppose that ($\mathfrak{qt}4.\mathfrak{np}$) holds for $\bar{p}$, then, by the induction hypothesis, the weak transform $\mathcal{I}_{i-1}$ of $\mathcal{I}_0$ on X_{i-1} satisfies

$$\hat{\mathcal{I}}_{i-1,\bar{p}} = (\bar{x}^{\min\{a-g,d-g\}}, \bar{z}) \cap (\bar{x}^{\max\{a-g,d-g\}}, \bar{y}^{\max\{b-h,e-h\}}, \bar{z}) \cap (\bar{y}^{\min\{b-h,e-h\}}, \bar{z})$$

where $ae - bd \neq 0$, and

$$g < \min\{a,d\} \text{ or } h < \min\{b,e\} \text{ or } (a-d)(b-e) < 0.$$

We note that $\hat{\mathcal{I}}_{i-1,\bar{p}}$ has order $r_{i-1} = 1$ at $\bar{p}$. The center T_{i-1} is the point $\bar{p}$ if and only if $\min\{a-g, d-g\} = \min\{b-h, e-h\} = 0$, and then $\hat{\mathcal{I}}_{i-1,\bar{p}} = (\bar{x}^{\max\{a-g,d-g\}}, \bar{y}^{\max\{b-h,e-h\}}, \bar{z})$. In this case, $\bar{x} = \bar{y} = \bar{z} = 0$ are clearly formal local equations of T_{i-1} at $\bar{p}$. Hence there exist permissible parameters x, y, z at $p \in \lambda_i^{-1}(\bar{p})$ for $D_{i,j}$ such that one of the following equations holds.

($\mathfrak{pb}1$) $$\bar{x} = x, \bar{y} = x(y+\alpha), \bar{z} = x(z+\beta),\ \alpha,\beta \in \mathfrak{k} \text{ or,}$$

($\mathfrak{pb}2$) $$\bar{x} = xy, \bar{y} = y, \bar{z} = y(z+\alpha),\ \alpha \in \mathfrak{k} \text{ or,}$$

($\mathfrak{pb}3$) $$\bar{x} = xz, \bar{y} = yz, \bar{z} = z.$$

However, if $g < \min\{a,d\}$, i.e., $\min\{a-g, d-g\} > 0$, r_{i-1} is 1 at all points of the curve with local equations $\bar{x} = \bar{z} = 0$ at $\bar{p}$, or if $\min\{b-h, e-h\} > 0$, this order is 1 at all points of the curve which has local equations $\bar{y} = \bar{z} = 0$ at $\bar{p}$. Since T_{i-1} is an irreducible component of maximal dimension of $\mathrm{Max} W_q(X_{i-1})$ due to condition 2) of Definition 2.11, and $W_q(X_{i-1})$ is SNC, if $g < \min\{a,d\}$ or $h < \min\{b,e\}$, we have that T_{i-1} is a curve and either $\bar{x} = \bar{z} = 0$, or $\bar{y} = \bar{z} = 0$ are formal local equations of T_{i-1} at $\bar{p}$.

After possibly permuting $\bar{x}, \bar{y}$, we can assume that $g < \min\{a,d\}$ and $\bar{x} = \bar{z} = 0$ are local equations of T_{i-1} at $\bar{p}$. Thus there exist permissible parameters x, y, z at $p \in \lambda_i^{-1}(\bar{p})$ for $D_{i,j}$ such that one of the following equations holds.

$$(\mathfrak{cb}3) \qquad \bar{x} = x, \bar{y} = y, \bar{z} = x(z+\alpha),\ \alpha \in \mathfrak{k} \text{ or,}$$

$$(\mathfrak{cb}4) \qquad \bar{x} = xz, \bar{y} = y, \bar{z} = z.$$

Since T_{i-1} makes SNCs with $D_{i-1,j}$, we have that $D_{i,j} = (\lambda_i|_{U_{i,j}})^*(D_{i-1,j})_{\mathrm{red}}$ is a SNC divisor.

Suppose that T_{i-1} is a point. This case only happen if $\min\{a-g, d-g\} = 0$ and $\min\{b-h, e-h\} = 0$, so that $g = \min\{a,d\}$ and $h = \min\{b,e\}$. In this case, since ($\mathfrak{qt}4.\mathfrak{np}$) holds for $\bar{p}$, we must have $(a-d)(b-e) < 0$. Then, after possibly interchanging $\bar{x}$ and $\bar{y}$ (as $\bar{y}\bar{x} = \bar{x}\bar{y} = 0$ is a local equation of $D_{i-1,j}$ at $\bar{p}$), we must have $g = a = \min\{a,d\}$, $h = e = \min\{b,e\}$, $g < d$ and $h < b$ and so

$$(3.1) \qquad g + h + 1 \leq \min\{a+b, d+e\}.$$

($\mathfrak{qt}4.\mathfrak{np}.1$) Suppose that ($\mathfrak{pb}1$) holds with $\alpha = \beta = 0$. Then p is a 2-point of $D_{i,j}$ with local equation $xy = 0$, and

$$u = x^{a+b}y^b$$
$$v = x^{d+e}y^e$$
$$w = x^{g+h+1}y^h z$$

where

$$\det\begin{pmatrix} a+b & b \\ d+e & e \end{pmatrix} = \det\begin{pmatrix} a & b \\ d & e \end{pmatrix} \neq 0.$$

So, we are in the case ($\mathfrak{qt}4$) since (3.1) holds. (Similarly, if ($\mathfrak{pb}1$) holds with $\alpha = 0, \beta \neq 0$ or, ($\mathfrak{pb}2$) holds, we are in the case ($\mathfrak{qt}4$)).

($\mathfrak{qt}4.\mathfrak{np}.2$) If ($\mathfrak{pb}1$) holds with $\alpha \neq 0, \beta = 0$, then $x^2(y+\alpha) = 0$ is a local equation of divisor whose support is $D_{i,j}$ and this implies that $x = 0$ is a local equation of $D_{i,j}$ and p is a 1-point of $D_{i,j}$. In addition,

$$u = x^{a+b}(y+\alpha)^b$$
$$v = x^{d+e}(y+\alpha)^e$$
$$w = x^{g+h+1}(y+\alpha)^h z$$

where $a+b, d+e>0$ since $ae-bd\neq 0$. Hence

$$\det\begin{pmatrix} a+b & 0 \\ g+h+1 & 1 \end{pmatrix} = a+b>0$$

and so, there exist (unique) $\gamma_1,\gamma_2\in\mathbb{Q}$ such that

$$\begin{pmatrix} a+b & 0 \\ g+h+1 & 1 \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} b \\ h \end{pmatrix}. \tag{3.2}$$

Then we can set

$$\begin{aligned}
\tilde{x} &= x(y+\alpha)^{\gamma_1}\\
\tilde{z} &= z(y+\alpha)^{\gamma_2}\\
\tilde{\alpha} &= \alpha^{e-\gamma_1(d+e)}\\
\tilde{y} &= (y+\alpha)^{e-\gamma_1(d+e)}-\tilde{\alpha}
\end{aligned}$$

which satisfy

$$\begin{aligned}
\begin{vmatrix} \frac{\partial\tilde{x}}{\partial x} & \frac{\partial\tilde{x}}{\partial y} & \frac{\partial\tilde{x}}{\partial z} \\ \frac{\partial\tilde{y}}{\partial x} & \frac{\partial\tilde{y}}{\partial y} & \frac{\partial\tilde{y}}{\partial z} \\ \frac{\partial\tilde{z}}{\partial x} & \frac{\partial\tilde{z}}{\partial y} & \frac{\partial\tilde{z}}{\partial z} \end{vmatrix}_{(0,0,0)} &= \begin{vmatrix} \alpha^{\gamma_1} & 0 & 0 \\ 0 & (e-\gamma_1(d+e))\alpha^{e-\gamma_1(d+e)-1} & 0 \\ 0 & 0 & \alpha^{\gamma_2} \end{vmatrix}\\
&= (e-\gamma_1(d+e))\alpha^{\gamma_1+\gamma_2+e-\gamma_1(d+e)-1}\neq 0
\end{aligned}$$

since $\alpha\neq 0$ and $e-\gamma_1(d+e)\neq 0$. (Otherwise, if $e-\gamma_1(d+e)=0$, due to the first equation in (3.2), we have that

$$\begin{pmatrix} a+b & b \\ d+e & e \end{pmatrix}\begin{pmatrix} \gamma_1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which contradicts $ae-bd\neq 0$).

So, we obtain permissible parameters $\tilde{x},\tilde{y},\tilde{z}$ at p for $D_{i,j}$ such that $\tilde{x}=0$ is a local equation of $D_{i,j}$ at p and

$$\begin{aligned}
u &= \tilde{x}^{a+b}\\
v &= \tilde{x}^{d+e}(\tilde{y}+\tilde{\alpha})\\
w &= \tilde{x}^{g+h+1}\tilde{z}
\end{aligned}$$

with $a+b, d+e>0$ since $ae-bd\neq 0$, $\tilde{\alpha}\neq 0$ since $\alpha\neq 0$ and, $g+h+1\leq \min\{a+b,d+e\}$ due to (3.1). So, we are in the case ($\mathfrak{qt}6$).

($\mathfrak{qt}4.\mathfrak{np}.3$) If ($\mathfrak{pb}1$) holds and both α,β are nonzero, then $x^3(y+\alpha)(z+\beta)=0$ is a local equation of a divisor whose support is $D_{i,j}$ which implies that $x=0$ is a local equation of $D_{i,j}$ at p and p is a 1-point of $D_{i,j}$. Further,

$$\begin{aligned}
u &= x^{a+b}(y+\alpha)^b\\
v &= x^{d+e}(y+\alpha)^e\\
w &= x^{g+h+1}(y+\alpha)^h(z+\beta)
\end{aligned}$$

where $a+b, d+e>0$ since $ae-bd\neq 0$. So, we can set

$$\begin{aligned}
\tilde{x} &= x(y+\alpha)^{\frac{b}{a+b}}\\
\tilde{\alpha} &= \alpha^{-(\frac{b}{a+b})(d+e)+e} = \alpha^{\frac{ae-bd}{a+b}}\\
\tilde{y} &= (y+\alpha)^{\frac{ae-bd}{a+b}} - \tilde{\alpha}\\
\tilde{\beta} &= \alpha^{-(\frac{b}{a+b})(g+h+1)+h}\beta = \alpha^{\frac{(ah-bg)-b}{a+b}}\beta\\
\tilde{z} &= (y+\alpha)^{\frac{(ah-bg)-b}{a+b}}(z+\beta) - \tilde{\beta}
\end{aligned}$$

which satisfy

$$\begin{aligned}
\left|\begin{array}{ccc} \frac{\partial\tilde{x}}{\partial x} & \frac{\partial\tilde{x}}{\partial y} & \frac{\partial\tilde{x}}{\partial z}\\ \frac{\partial\tilde{y}}{\partial x} & \frac{\partial\tilde{y}}{\partial y} & \frac{\partial\tilde{y}}{\partial z}\\ \frac{\partial\tilde{z}}{\partial x} & \frac{\partial\tilde{z}}{\partial y} & \frac{\partial\tilde{z}}{\partial z}\end{array}\right|_{(0,0,0)} &= \left|\begin{array}{ccc} \alpha^{\frac{b}{a+b}} & 0 & 0\\ 0 & (\frac{ae-bd}{a+b})\alpha^{\frac{ae-bd}{a+b}-1} & 0\\ 0 & (\frac{ah-bg-b}{a+b})\alpha^{\frac{ah-bg-b}{a+b}-1}\beta & \alpha^{\frac{ah-bg-b}{a+b}}\end{array}\right|\\
&= (\frac{ae-bd}{a+b})\alpha^{\frac{ae-bd+ah-bg}{a+b}-1}\neq 0
\end{aligned}$$

since $\alpha\neq 0$ and $ae-bd\neq 0$.

Therefore, we obtain permissible parameters $\tilde{x},\tilde{y},\tilde{z}$ at p for $D_{i,j}$ such that $\tilde{x}=0$ is a local equation of $D_{i,j}$ at p and

$$\begin{aligned}
u &= \tilde{x}^{a+b}\\
v &= \tilde{x}^{d+e}(\tilde{y}+\tilde{\alpha})\\
w &= \tilde{x}^{g+h+1}(\tilde{z}+\tilde{\beta})
\end{aligned}$$

with $a+b, d+e>0$ since $ae-bd\neq 0$, $\tilde{\alpha},\tilde{\beta}\neq 0$ since $\alpha,\beta\neq 0$ and, $g+h+1\leq \min\{a+b,d+e\}$ due to (3.1). So, we are in the case ($\mathfrak{qt}6$).

($\mathfrak{qt}4.\mathfrak{np}.4$) If ($\mathfrak{pb}3$) holds, then p is a 3-point of $D_{i,j}$ with local equation $xyz=0$ and,

$$\begin{aligned}
u &= x^a y^b z^{a+b}\\
v &= x^d y^e z^{d+e}\\
w &= x^g y^h z^{g+h+1}
\end{aligned}$$

where

$$\det\begin{pmatrix} a & b & a+b\\ d & e & d+e\\ g & h & g+h+1\end{pmatrix} = \det\begin{pmatrix} a & b & 0\\ d & e & 0\\ g & h & 1\end{pmatrix} = ae-bd\neq 0,$$

$g\leq\min\{a,d\}$ and $h\leq\min\{b,e\}$ since ($\mathfrak{qt}4.\mathfrak{np}$) holds for $\bar{p}$, and $g+h+1\leq \min\{a+b,d+e\}$ due to (3.1). So, we are in the case ($\mathfrak{qt}5$).

Now, suppose that T_{i-1} is the curve with local equations $\bar{x}=\bar{z}=0$ at $\bar{p}$ (so that $g<\min\{a,d\}$). Then one of the equations ($\mathfrak{cb}3$), or ($\mathfrak{cb}4$) holds.

($\mathfrak{qt}4.\mathfrak{np}$.5) Suppose that ($\mathfrak{cb}3$) holds, then p is a 2-point of $D_{i,j}$ with local equation $xy = 0$ and,

$$u = x^a y^b$$
$$v = x^d y^e$$
$$w = x^{g+1} y^h (z + \alpha)$$

where $ae - bd \neq 0$, $g + 1 \leq \min\{a, d\}$ since $g < \min\{a, d\}$, $h \leq \min\{b, e\}$ and $\alpha \in \mathfrak{k}$. So, we are in the case ($\mathfrak{qt}4$).

($\mathfrak{qt}4.\mathfrak{np}$.6) If ($\mathfrak{cb}4$) holds, then p is a 3-point of $D_{i,j}$ with local equation $xyz = 0$ and,

$$u = x^a y^b z^a$$
$$v = x^d y^e z^d$$
$$w = x^g y^h z^{g+1}$$

where $g \leq \min\{a, d\}, h \leq \min\{b, e\}$, and $g+1 \leq \min\{a, d\}$ since $g < \min\{a, d\}$. So, we are in the case ($\mathfrak{qt}5$).

Therefore, in the case when ($\mathfrak{qt}4.\mathfrak{np}$) holds for $\bar{p}$, and similarly, in other cases, the conclusion I of the lemma holds for all i. In all the cases, we have that $W_q(X_i)$ is SNC, by an argument similar to that of Proposition 2.10. ☐

3.2. **Analysis of Principalization Sequences of Curves.** In the following Lemma, we study the sequence ($\mathcal{P}$) for all possibilities of a permissible center that is a nonsingular curve $C \subset Y$ – see Remark–Definition 2.8.

Lemma 3.2. *Suppose that $j \in J$ and $q \in C \cap V_j$. There exist algebraic permissible parameters u, v, w at q for E_j such that for all i, if p lies in $(\varphi \circ \Lambda_i)^{-1}(q) \cap \Lambda_i^{-1}(U_j)$, then*

I. *$D_{i,j} = (\Lambda_i|_{U_j})^*(D_j)_{\text{red}}$ is a SNC divisor in a neighborhood of p, and there exist formal permissible parameters x, y, z at p for $D_{i,j}$ such that*

- (i) *If q is a 3-point for E_j and $uvw = 0$ is a local equation of E_j at q, and C is a 2^+-curve for E_j at q such that $u = v = 0$ are local equations of C at q, then one of the quasi-toroidal forms ($\mathfrak{qt}1$), ($\mathfrak{qt}2$) or ($\mathfrak{qt}3$) of Lemma 3.1 holds at p.*
- (ii) *If q is a 2-point for E_j and $uv = 0$ is a local equation of E_j at q, and C is a 2-curve for E_j at q such that $u = v = 0$ are local equations of C at q, then one of the following forms holds at p.*
  - ($\mathfrak{qt}15$) *p is a 2-point and $xy = 0$ is a local equation of $D_{i,j}$ at p and (T4) holds.*
  - ($\mathfrak{qt}16$) *p is a 1-point and $x = 0$ is a local equation of $D_{i,j}$ at p and (T5) holds.*
- (iii) *If q is a 2-point for E_j and $uv = 0$ is a local equation of E_j at q, and C is a 1^+-curve for E_j at q such that $u = w = 0$ are local equations of C at q, then one of the following forms holds at p.*

(qt17) *p is a 2-point and $xy = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a y^b,\ v = x^d y^e,\ w = x^g y^h (z+\alpha)$$

with $ae - bd \neq 0$ and $g \leqslant a, h \leqslant b$ and, $\alpha \in \mathfrak{k}$.

(qt18) *p is a 3-point and $xyz = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a y^b z^c,\ v = x^d y^e z^f,\ w = x^g y^h z^i$$

where $g \leqslant a$, $h \leqslant b$ and $i \leqslant c$ and $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.$

(qt19) *p is a 1-point and $x = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a,\ v = x^d(y+\alpha),\ w = x^g(z+\beta)$$

where $a, d > 0$, $\alpha, \beta \in \mathfrak{k}$, $\alpha \neq 0$ and $g \leqslant a$.

(qt20) *p is a 2-point and $xz = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a z^a,\ v = x^d z^d(y+\alpha),\ w = x^g z^{g+1}$$

with $a, d > 0$, $0 \neq \alpha \in \mathfrak{k}$ and $g + 1 \leqslant a$.

(iv) *If q is a 1-point for E_j and $u = 0$ is a local equation of E_j at q, and C is a 1-curve for E_j at q such that $u = v = 0$ are local equations of C at q, then one of the following forms holds at p.*

(qt21) *p is a 1-point and $x = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a,\ v = x^{a'}(y+\alpha),\ w = z$$

where $a, a' \in \mathbb{N}$ satisfy $a > 0$ and $a' \leqslant a$, and $\alpha \in \mathfrak{k}$.

(qt22) *p is a 2-point and $xy = 0$ is a local equation of $D_{i,j}$ at p, and*

$$u = x^a y^a,\ v = x^{a'} y^{a'+1},\ w = z$$

where $a, a' \in \mathbb{N}$ satisfy $a > 0$ and $a' + 1 \leqslant a$.

(v) *If q is a 1-point for E_j and $u = 0$ is a local equation of E_j at q, and C is a 0^+-curve for E_j at q such that $v = w = 0$ are local equations of C at q, then one of the following forms holds at p.*

(qt23) *p is a 1-point for $D_{i,j}$, $x = 0$ is a local equation of $D_{i,j}$ at p and (T6) holds.*

(qt24) *p is a 1-point for $D_{i,j}$, $x = 0$ is a local equation of $D_{i,j}$ at p and,*

$$u = x^a, v = y, w = y(z+\alpha) \text{ with } a > 0 \text{ and } \alpha \in \mathfrak{k}.$$

(qt25) *p is a 1-point for $D_{i,j}$, $x = 0$ is a local equation of $D_{i,j}$ at p and,*

$$u = x^a, v = yz, w = z \text{ with } a > 0.$$

(vi) *If q is a 0-point for E_j, i.e., $q \in (C \cap V_j) \setminus E_j$, and C is a 0-curve for E_j at q such that $u = v = 0$ are local equations of C at q, then (qt11) or one of the following forms holds at p.*

(qt26) *p is a 0-point for $D_{i,j}$ and $u = x, v = x(y+\alpha), w = z$ with $\alpha \in \mathfrak{k}$.*

(qt27) *p is a 0-point for $D_{i,j}$ and $u = xy, v = y, w = z$.*

II. *$W_C(X_i)$ is SNC for all i. Precisely, if $p \in W_C(X_i)$, then there exist (formal) permissible parameters x, y, z at p (for $D_{i,j}$) such that one of the following possibilities holds. The weak transform of $\mathcal{I}_0$ defined by 2) of Definition 2.11 on X_i is denoted by $\mathcal{I}_i$ which satisfies $\sqrt{\mathcal{I}_i} = \mathcal{I}_{W_C(X_i)}$.*

($\mathfrak{qt1.cnp}$) *We are in the case (a.1) with $(a-d)(b-e) < 0$ or, $(a-d)(c-f) < 0$ or, $(b-e)(c-f) < 0$. Further, at most two of these three conditions can hold and, after possibly permuting the parameters x, y, z, we can assume $(a-d)(b-e) < 0$ and $(a-d)(c-f) \leqslant 0$. Then $\mathcal{I}_i$ satisfies*

$$\hat{\mathcal{I}}_{i,p} = (x^{|a-d|}, y^{|b-e|}z^{|c-f|}) = (x^{|a-d|}, y^{|b-e|}) \cap (x^{|a-d|}, z^{|c-f|}).$$

($\mathfrak{qt2.cnp}$) *We are in the case (a.2) with $(a-d)(b-e) < 0$, and $\mathcal{I}_i$ satisfies $\hat{\mathcal{I}}_{i,p} = (x^{|a-d|}, y^{|b-e|})$.*

($\mathfrak{qt15.np}$) *We are in the case (b.1) with equations*

$$u = x^a y^b,\ v = x^d y^e,\ w = z$$

where $(a-d)(b-e) < 0$. Further, $\mathcal{I}_i$ satisfies $\hat{\mathcal{I}}_{i,p} = (x^{|a-d|}, y^{|b-e|})$.

($\mathfrak{qt17.np}$) *We are in the case (c.1) with equations*

$$u = x^a y^b,\ v = x^d y^e,\ w = x^g y^h z$$

where $g < a$ or $h < b$ and, $ae - bd \neq 0$. So, $\mathcal{I}_i$ satisfies

$$\hat{\mathcal{I}}_{i,p} = (x^{a-g}, z) \cap (y^{b-h}, z)$$

with $a - g > 0$ or $b - h > 0$.

($\mathfrak{qt19.np}$) *We are in the case (c.3) with equations*

$$u = x^a,\ v = x^d(y + \alpha),\ w = x^g z$$

where $a, d > 0$, $0 \neq \alpha \in \mathfrak{k}$, $g < a$, and $\mathcal{I}_i$ satisfies $\hat{\mathcal{I}}_{i,p} = (x^{a-g}, z)$ with $a - g > 0$.

($\mathfrak{qt21.np}$) *We are in the case (d.1) with equations*

$$u = x^a,\ v = x^{a'} y,\ w = z$$

and $a' < a$. In addition, $\mathcal{I}_i$ satisfies $\hat{\mathcal{I}}_{i,p} = (x^{a-a'}, y)$ with $a - a' > 0$.

($\mathfrak{qt23.np}$) *We are in the case (e.1), and $\mathcal{I}_i$ satisfies $\hat{\mathcal{I}}_{i,p} = (y, z)$.*

($\mathfrak{qt11.cnp}$) *We are in the case (f.1), and $\mathcal{I}_i$ satisfies $\hat{\mathcal{I}}_{i,p} = (x, y)$.*

Proof. The proof is completely similar to that of Lemma 3.1. By Proposition 2.2, after changing the parameters u, v, w, if necessary, X satisfies the conclusion I of the lemma. Conclusion II also holds for X by Proposition 2.10. If we prove conclusion I, then, by an argument similar to that of Proposition 2.10, we will obtain conclusion II. So, it remains to show that conclusion I holds. Inductively, we assume that the conclusions hold for Λ_{i-1} and we prove them for Λ_i.

Suppose that $p \in (\varphi \circ \Lambda_i)^{-1}(q) \cap \Lambda_i^{-1}(U_j) = \lambda_i^{-1}((\varphi \circ \Lambda_{i-1})^{-1}(q) \cap \Lambda_{i-1}^{-1}(U_j))$ and let $T_{i-1} \subseteq W_C(X_{i-1})$ be the center of λ_i.

We can assume that $p \in \lambda_i^{-1}(T_{i-1})$ since λ_i is an isomorphism out of the center, i.e., at points $p \in X_i \setminus \lambda_i^{-1}(T_{i-1})$. Let $\bar{p} = \lambda_i(p) \in T_{i-1}$. By the induction hypothesis, there exist formal permissible parameters $\bar{x}, \bar{y}, \bar{z}$ at $\bar{p}$ for $D_{i-1,j}$ such that one of the cases ($\mathfrak{qt1.cnp}$) through ($\mathfrak{qt11.cnp}$) of conclusion II of the lemma holds. All of the cases are similar. We will work out in detail the case when ($\mathfrak{qt23.np}$) holds for $\bar{p}$ (so $q \in C \cap V_j$ is a 1-point for E_j, C is a 0^+-curve for E_j at q, and $\bar{p}$ is a 1-point for $D_{i-1,j}$, and we are in the case ($\mathfrak{qt23}$)).

Suppose that ($\mathfrak{qt23.np}$) holds for $\bar{p}$, then, by the induction hypothesis, the weak transform $\mathcal{I}_{i-1}$ of $\mathcal{I}_0$ on X_{i-1} satisfies $\hat{\mathcal{I}}_{i-1,\bar{p}} = (\bar{y}, \bar{z})$.

We note that $\hat{\mathcal{I}}_{i-1,\bar{p}}$ has order 1 at $\bar{p}$ as well as all points of the curve that has local equations $\bar{y} = \bar{z} = 0$ at $\bar{p}$. Since T_{i-1} must be an irreducible component of maximal dimension of $\mathrm{Max}W_C(X_{i-1})$ due to condition 2) of Definition 2.11, we have that T_{i-1} is a curve and $\bar{y} = \bar{z} = 0$ are local equations of T_{i-1} at $\bar{p}$. Thus there exist permissible parameters x, y, z for $D_{i,j}$ at $p \in \lambda_i^{-1}(\bar{p})$ such that one of the following equations hold.

$$(\mathfrak{cb5}) \qquad \bar{x} = x, \bar{y} = y, \bar{z} = y(z + \alpha),\ \alpha \in \mathfrak{k} \text{ or,}$$

$$(\mathfrak{cb6}) \qquad \bar{x} = x, \bar{y} = yz, \bar{z} = z.$$

In addition, since T_{i-1} makes SNCs with $D_{i-1,j}$, we have that $D_{i,j} = (\lambda_i|_{U_{i,j}})^*(D_{i-1,j})_{\mathrm{red}}$ is a SNC divisor.

By substituting ($\mathfrak{cb5}$) and ($\mathfrak{cb6}$) in ($\mathfrak{qt23.np}$), we obtain ($\mathfrak{qt24}$) and ($\mathfrak{qt25}$) respectively, and in both cases $x = 0$ is a local equation of $D_{i,j}$ at p.

Therefore, in the case when ($\mathfrak{qt23.np}$) holds for $\bar{p}$, and similarly, in other cases, the conclusion I of the lemma holds for all i. In all the cases, we have that $W_C(X_i)$ is SNC, by an argument similar to that of Proposition 2.10. ☐

3.3. **Principalization Sequences Are Finite.** In this subsection we prove the following theorem, that is, any principalization sequence is finite.

Theorem 3.3. *Suppose that $\varphi : X \to Y$ is a morphism of nonsingular 3-folds, which is locally toroidal with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$. Let $Z \subset Y$ be a point or a nonsingular curve C such that $C \cap V_j$ makes SNCs with E_j for all $j \in J$. Then any principalization sequence of Z, obtained by successive blow ups of centers satisfying the conditions of Definition 2.11, will terminate after a finite number $n \geqslant 0$ of blow ups with $W_Z(X_n) = \emptyset$.*

Proof. We will show that any principalization sequence must terminate with $W_Z(X_n) = \emptyset$ after some finite number n of iterations.

Suppose that the algorithm of Definition 2.11 does not end in a finite number of steps with $W_Z(X_n) = \emptyset$. Then the algorithm produces an infinite sequence

$$\cdots \to X_n \xrightarrow{\lambda_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\lambda_1} X,$$

and there exist $r > 0$, a positive integer n_0 and $p_n \in X_n$ for $n \geq n_0$ such that for all $n \geq n_0$, $\lambda_{n+1}(p_{n+1}) = p_n$ and $\nu_{p_n}(\mathcal{I}_n) = r$ by Lemma 6.4 [7] where $\mathcal{I}_n$ is the weak transform of $\mathcal{I}_0$ on X_n and $\sqrt{\mathcal{I}_n} = \mathcal{I}_{W_Z(X_n)}$, and $\mathcal{I}_0$ is defined by 2) of Definition 2.11.

Let $R_i = \hat{\mathcal{O}}_{X_{n_0+i},p_{n_0+i}}$ and $J_i = (\mathcal{I}_{n_0+i})_{p_{n_0+i}} R_i$ for $i \geq 0$. We have that $\nu_{R_i}(J_i) = r$ for all i. Without loss of generality, we may reindex the R_i so that $R_i \neq R_{i+1}$ for all i and we have an induced sequence

$$R_0 \to R_1 \to R_2 \to \cdots \ . \tag{3.3}$$

We can show that this leads to a contradiction, by considering how the quasi toroidal forms of $\varphi \circ \Lambda_i$ transform under blowups. All of the cases are similar. We will work out in detail the most involved case which is when $\varphi \circ \Lambda_{n_0}$ has the quasi toroidal form ($\mathfrak{qt1}$) of the conclusions of Lemma 3.1 at p_{n_0} (so the center Z is a point $q \in Y$). We assume that this is the case, and we will derive a contradiction.

There are regular parameters x_0, y_0, z_0 in R_0 such that

$$\mathcal{I}_q R_0 = (x_0^a y_0^b z_0^c, x_0^d y_0^e z_0^f, x_0^g y_0^h z_0^i).$$

Let $d_x = \min\{a, d, g\}$, $d_y = \min\{b, e, h\}$ and $d_z = \min\{c, f, i\}$. Then

$$J_0 = \frac{1}{x_0^{d_x} y_0^{d_y} z_0^{d_z}} \mathcal{I}_q R_0 = (x_0^{a'} y_0^{b'} z_0^{c'}, x_0^{d'} y_0^{e'} z_0^{f'}, x_0^{g'} y_0^{h'} z_0^{i'})$$

where $a' = a - d_x$, $b' = b - d_y$, $c' = c - d_z$, $d' = d - d_x$, $e' = e - d_y$, $f' = f - d_z$, $g' = g - d_x$, $h' = h - d_y$ and $i' = i - d_z$. After permuting $x_0^{a'} y_0^{b'} z_0^{c'}$ and $x_0^{d'} y_0^{e'} z_0^{f'}$ and $x_0^{g'} y_0^{h'} z_0^{i'}$, we may assume that

$$r = \nu_{R_0}(J_0) = \operatorname{ord}(x_0^{a'} y_0^{b'} z_0^{c'}) = a' + b' + c' \geq 1.$$

So, after permuting x_0, y_0, z_0, we can assume that $c' > 0$. Hence

$$\frac{\partial^{r-1}}{\partial x_0^{a'} \partial y_0^{b'} \partial z_0^{c'-1}} x_0^{a'} y_0^{b'} z_0^{c'} = a'! b'! (c'-1)! z_0 \in \hat{\Delta}^{r-1}(J_0).$$

Thus $H = \mathbf{V}(z_0) \subset \operatorname{Spec}(R_0)$ is a nonsingular hypersurface such that $\mathcal{I}_H = (z_0)$ is contained in $\hat{\Delta}^{r-1}(J_0)$ (Definition 6.1 [7]). By Lemma 6.21 [7], H satisfies the conditions of Definition 6.7 [7] and $z_0 = 0$ is a formal hypersurface of maximal contact for $\operatorname{Sing}(J_0, r)$. So each R_i has regular parameters x_i, y_i, z_i such that one of the following equations holds.

$$x_i = x_{i+1}, y_i = x_{i+1}(y_{i+1} + \alpha_{i+1}), z_i = x_{i+1} z_{i+1}, \ \alpha_{i+1} \in \mathfrak{k} \text{ or,} \tag{3.4}$$

$$x_i = x_{i+1} y_{i+1}, y_i = y_{i+1}, z_i = y_{i+1} z_{i+1} \tag{3.5}$$

if T_i is the point p_i,

$$x_i = x_{i+1}, y_i = y_{i+1}, z_i = x_{i+1} z_{i+1} \tag{3.6}$$

if T_i is the curve with the formal local equations $x_i = z_i = 0$ at p_i and

$$x_i = x_{i+1}, y_i = y_{i+1}, z_i = y_{i+1} z_{i+1} \tag{3.7}$$

if T_i is the curve with the formal local equations $y_i = z_i = 0$ at p_i.

Let $S_i = R_i/z_iR_i$ so that S_i has regular parameters x_i, y_i, and (3.3) induces an infinite sequence

$$S_0 \to S_1 \to S_2 \to \cdots .$$

Let $K_0 = C(J_0)$ be the coefficient ideal of J_0 on H (Definition 6.22 [7]) and inductively define

$$K_{i+1} = \frac{1}{h^{r!}} K_i S_{i+1} \text{ for } i \geq 0$$

where $h = 0$ is a local equation of the exceptional locus of $\mathrm{Spec}(S_{i+1}) \to \mathrm{Spec}(S_i)$. (If (3.4) or (3.6) holds, then $h = x_{i+1}$ and if (3.5) or (3.7) holds then $h = y_{i+1}$).

By formula (6.17) [7], we have that

$$\mathrm{Sing}(K_i, r!) = \mathrm{Sing}(J_i, r);$$

in particular, $\nu_{S_i}(K_i) \geq r!$ for all i. In addition, K_i is a principal ideal for $i \gg 0$ by Theorem 4.11 [7], so there exist n_1 and a_i, b_i such that $K_i = (x_i^{a_i} y_i^{b_i})$ for $i \geq n_1$. We now establish the formula

$$a_{i+1} + b_{i+1} < a_i + b_i \tag{3.8}$$

for $i \geq n_1$ which leads to contradiction.

First suppose that there are infinitely many i such that λ_{i+1} is the blow up of the point p_i. For these i, $S_{i+1} \to S_i$ is a quadratic transform and we have equations

$$x_i = x_{i+1}, y_i = x_{i+1}(y_{i+1} + \alpha_{i+1}) \text{ or } x_i = x_{i+1}y_{i+1}, y_i = y_{i+1}.$$

Note that by the algorithm (Definition 2.11), we only blow up the point p_i if

$$\nu_{(R_i)_{(x_i,z_i)}}(J_i) < r \text{ and } \nu_{(R_i)_{(y_i,z_i)}}(J_i) < r,$$

and the conditions $\nu_{(R_i)_{(x_i,z_i)}}(J_i) < r$ and $\nu_{(S_i)_{(x_i)}}(K_i) < r!$ are equivalent by formula (6.17) [7], also $\nu_{(R_i)_{(y_i,z_i)}}(J_i) < r$ and $\nu_{(S_i)_{(y_i)}}(K_i) < r!$ are equivalent. Thus if we blow up the point p_i, then $K_i = (x_i^{a_i} y_i^{b_i})$ with $a_i = \nu_{(S_i)_{(x_i)}}(K_i) < r!$ and $b_i = \nu_{(S_i)_{(y_i)}}(K_i) < r!$. Further, (3.4) or (3.5) holds. If (3.4) holds, then

$$K_{i+1} = (x_{i+1}^{a_i+b_i-r!}(y_{i+1} + \alpha_{i+1})^{b_i}),\ \alpha_{i+1} \in \mathfrak{k}.$$

If $\alpha_{i+1} \neq 0$, then $K_{i+1} = (x_{i+1}^{a_i+b_i-r!})$ and we have $a_{i+1} = a_i + b_i - r!$ and $b_{i+1} = 0$ and clearly (3.8) holds.

If $\alpha_{i+1} = 0$, we have $a_{i+1} = a_i + b_i - r!$ and $b_{i+1} = b_i$, and

$$a_{i+1} + b_{i+1} = a_i + b_i - r! + b_i = (a_i + b_i) - (r! - b_i) < a_i + b_i,$$

which establishes (3.8). If (3.5) holds, then the same argument shows that (3.8) also holds.

Now, suppose that there are infinitely many i such that λ_{i+1} is the blow up of a curve containing p_i. By the algorithm, we only blow up a curve if $\nu_{(R_i)_{(x_i,z_i)}}(J_i) = r$ or $\nu_{(R_i)_{(y_i,z_i)}}(J_i) = r$ (equivalently, $a_i = \nu_{(S_i)_{(x_i)}}(K_i) = r!$ or $b_i = \nu_{(S_i)_{(y_i)}}(K_i) = r!$).

If local equations of T_i at p_i are $x_i = z_i = 0$, so that (3.6) holds, then $\nu_{(R_i)_{(x_i,z_i)}}(J_i) = r$, and

$$K_{i+1} = (x_{i+1}^{a_i - r!} y_{i+1}^{b_i})$$

and (3.8) holds. A similar argument shows that if local equations of T_i at p_i are $y_i = z_i = 0$, so that (3.7) holds, then (3.8) also holds.

But (3.8) is in contradiction to the assumption $p_i \in \mathrm{Sing}(K_i, r!)$ for all i, which implies that $a_i + b_i \geq r!$ for all i. Therefore, $W_Z(X_n) = \emptyset$ after some finite number n of iterations, in the case when $\varphi \circ \Lambda_{n_0}$ has quasi-toroidal form ($\mathtt{qt1}$) at p_{n_0}, and similarly in all the other cases. □

4. **Toroidalization**

In this Section, we first prove that principalization sequences obtained from the algorithm of Definition 2.11 have the property that the resulting morphism, after resolution of indeterminancy, is again locally toroidal with respect to the modified local structure. Then, summing up all our arguments, we deduce Theorem 2.13 and consequently prove our main result Theorem 1.2.

4.1. **Local Toroidalization.** Suppose that $\varphi : X \to Y$ is a locally toroidal morphism of nonsingular 3-folds with respect to $\mathcal{L} = \{U_j, D_j, V_j, E_j\}_J$ and $\pi : Y_1 \to Y$ is the blow up of $Z \subset Y$ where Z is a point $q \in V_j$ or a nonsingular curve C such that $C \cap V_j$ makes SNCs with E_j, for all $j \in J$. Let $\lambda : X_1 \to X$ be a principalization sequence of Z such that $W_Z(X_1) = \emptyset$. For $j \in J$, let $U_{1,j} = \lambda^{-1}(U_j)$ and $V_{1,j} = \pi^{-1}(V_j)$, and let $D_{1,j} = (\lambda|_{U_{1,j}})^*(D_j)_{\mathrm{red}}$ and $E_{1,j} = (\pi|_{V_{1,j}})^*(E_j)_{\mathrm{red}}$. Let $\phi_{1,j}$ be the morphism giving a commutative diagram

$$\begin{array}{ccc} U_{1,j} & \xrightarrow{\phi_{1,j}} & V_{1,j} \\ {\scriptstyle \lambda|_{U_{1,j}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi|_{V_{1,j}}} \\ U_j & \xrightarrow[\varphi_j]{} & V_j \end{array} .$$

In this subsection we will verify that $\phi_{1,j} : U_{1,j} \to V_{1,j}$ is toroidal with respect to $D_{1,j}$ and $E_{1,j}$, for all $j \in J$. Consequently, we will prove the following theorem.

Theorem 4.1. *There exists a sequence of blowups of nonsingular subvarieties $\lambda : X_1 \to X$ such that λ is a resolution of indeterminancy of the rational map $X \dashrightarrow Y_1$ and the induced morphism $\phi_1 : X_1 \to Y_1$ which gives the commutative diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_1} & Y_1 \\ {\scriptstyle \lambda}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

is locally toroidal w.r.t. $\mathcal{L}_{X_1} = \{U_{1,j}, D_{1,j}, V_{1,j}, E_{1,j}\}_J$.

The remainder of this subsection is devoted to the proof of Theorem 4.1.

Lemma 4.2. *For all $j \in J$, we have that $U_{1,j} \setminus D_{1,j} \to V_{1,j} \setminus E_{1,j}$ is smooth and $D_{1,j}$, $E_{1,j}$ are SNC divisors in $U_{1,j}$ and $V_{1,j}$ respectively.*

Proof. First we note that $E_{1,j}$ is a SNC divisor since E_j is and $Z \cap V_j$ makes SNCs with E_j, and $D_{1,j}$ is a SNC divisor by Lemma 3.1 if Z is a point, and by Lemma 3.2 in case Z is a curve. In addition, since π is the blow up of Z and λ is a principalization sequence of Z centered at $W_Z(X) \subseteq \varphi^{-1}(Z)$, we have that

$$\begin{array}{rcl} V_{1,j} \setminus \pi^{-1}(Z \cap V_j) & \cong & V_j \setminus Z \text{ and,} \\ U_{1,j} \setminus (\varphi \circ \lambda)^{-1}(Z \cap V_j) & \cong & U_j \setminus \varphi^{-1}(Z \cap V_j). \end{array} \tag{4.1}$$

Suppose that $j \in J$. We now verify that $U_{1,j} \setminus D_{1,j} \to V_{1,j} \setminus E_{1,j}$ is smooth. Let $p_1 \in U_{1,j} \setminus D_{1,j}$, then $q_1 = \phi_{1,j}(p_1) \in V_{1,j} \setminus E_{1,j}$. Hence $q^* = \pi(q_1) = \varphi\lambda(p_1) \in V_j \setminus E_j$ and $p^* = \lambda(p_1) \in \varphi^{-1}(q^*) \subset U_j \setminus D_j$ which implies that $U_j \to V_j$ is smooth above a neighborhood of q^*.

Suppose that $Z \cap V_j \subset E_j$, i.e., Z is a point in E_j, or it is a 2^+-curve, a 2-curve, a 1^+-curve or a 1-curve for E_j, or $Z \cap V_j = \emptyset$. So $q^* \notin Z \cap V_j$ since $q^* \notin E_j$, and then $p^* \notin \varphi_j^{-1}(Z \cap V_j) \subset D_j$. Thus $U_{1,j} \setminus D_{1,j} \to V_{1,j} \setminus E_{1,j}$ is smooth at p_1 due to the isomorphisms (4.1), and since $U_j \setminus D_j \to V_j \setminus E_j$ is smooth at p^*. In particular, if $Z \cap V_j = \emptyset$, then $U_{1,j} = U_j$ and $V_{1,j} = V_j$, and so

$$U_{1,j} \setminus D_{1,j} = U_j \setminus D_j \to V_j \setminus E_j = V_{1,j} \setminus E_{1,j}$$

is smooth.

Suppose that $Z \cap V_j \not\subseteq E_j$, i.e., Z is a point in $V_j \setminus E_j$, or it is a 0^+-curve, or a 0-curve for E_j. Either $q^* \notin Z \cap V_j$ or $q^* \in (Z \cap V_j) \setminus E_j$.

If $q^* \notin Z \cap V_j$, then $p^* \notin \varphi_j^{-1}(Z \cap V_j)$ and $U_{1,j} \setminus D_{1,j} \to V_{1,j} \setminus E_{1,j}$ is smooth at p_1 due to the isomorphisms (4.1), and since $U_j \setminus D_j \to V_j \setminus E_j$ is smooth at p^*.

Suppose that $q^* \in (Z \cap V_j) \setminus E_j$. Since $U_j \to V_j$ is smooth in a neighborhood of q^* and $\dim U_j = \dim V_j = 3$, we have that $\hat{\mathcal{O}}_{V_j,q^*} \xrightarrow{\sim} \hat{\mathcal{O}}_{U_j,p^*}$. Since λ is a principalization sequence with $W_C(X_1) = \emptyset$, p_1 is actually a point above p^* in the blow up of $\varphi_j^{-1}(Z \cap V_j)$, which is a finite number of points if Z is a point, and it is a curve when Z is a curve. So, $\hat{\mathcal{O}}_{V_{1,j},q_1} \cong \hat{\mathcal{O}}_{U_{1,j},p_1}$. Thus $U_{1,j} \to V_{1,j}$ is smooth in a neighborhood of p_1.

Therefore, $U_{1,j} \setminus D_{1,j} \to V_{1,j} \setminus E_{1,j}$ is smooth. □

Lemma 4.3. *Suppose that $j \in J$ and $q \in Z \cap E_j$. Then $\phi_{1,j} : U_{1,j} \to V_{1,j}$ is toroidal at each point $p_1 \in (\varphi\lambda)^{-1}(q) \cap U_{1,j}$ with respect to $E_{1,j}$ and $D_{1,j}$.*

Proof. Let $p_1 \in (\varphi\lambda)^{-1}(q) \cap U_{1,j}$, then $p_1 \in D_{1,j}$ and $q_1 = \phi_{1,j}(p_1)$ lies in $\pi^{-1}(q) \cap V_{1,j} \subset E_{1,j}$ since $q \in Z \cap E_j$. We will use the criteria of Proposition 2.2 to show that $\phi_{1,j}$ is toroidal at p_1, by considering the quasi-toroidal form of $\varphi\lambda$ at p_1.

Since $q \in Z \cap E_j$, $\varphi\lambda$ has one of the quasi-toroidal forms of the conclusions (i), (ii), or (iii) of Lemma 3.1 if Z is a point, and when Z is a curve, it is one of the quasi-toroidal forms of the conclusions (i) through (v) of Lemma 3.2.

Further, the quasi-toroidal form that holds for $\varphi\lambda$ at p_1 satisfies $\mathcal{I}_Z\mathcal{O}_{X_1,p_1}$ is principal since $W_Z(X_1)=\emptyset$. All of the cases are similar. We will work out in detail the case ($\mathfrak{qt}4$) of the conclusion (ii) of Lemma 3.1 (so the center $Z=\{q\}$ is a 2-point for E_j).

Suppose that q is a 2-point for E_j, and there exist algebraic permissible parameters u,v,w at q for E_j, and (formal) permissible parameters x,y,z at p_1 for $D_{1,j}$ such that the quasi-toroidal form ($\mathfrak{qt}4$) holds for $\varphi\lambda$ at p_1. Then $xy=0$ is a local equation of $D_{1,j}$ at p_1 and, after possibly permuting u,v, we have the following possibilities for $\mathcal{I}_q\hat{\mathcal{O}}_{X_1,p_1}$ to be principal.

($\mathfrak{qt}4.1$) $\min\{a,d\}=a=g$ and $\min\{b,e\}=b=h$ and $\mathcal{I}_q\hat{\mathcal{O}}_{X_1,p_1}=(x^ay^b)$. So there exist permissible parameters u_1,v_1,w_1 at $q_1\in\pi^{-1}(q)$ for $E_{1,j}$ such that

$$u=u_1, v=u_1v_1, w=u_1(w_1+\alpha) \text{ with } \alpha\in\mathfrak{k},$$

and $u_1v_1=0$ is a local equation of $E_{1,j}$ at q_1, and

$$\begin{aligned} u_1 &= u = x^ay^b\\ v_1 &= \frac{v}{u} = x^{d-a}y^{e-b}\\ w_1 &= \frac{w}{u}-\alpha = z \end{aligned}$$

where $a(e-b)-(d-a)b\neq 0$ since $ae-bd\neq 0$. So, toroidal form (T4) holds.

($\mathfrak{qt}4.2$) $(g,h)\neq(0,0)$, $\alpha\neq 0$ and $\mathcal{I}_q\hat{\mathcal{O}}_{X_1,p_1}=(x^gy^h(z+\alpha))$. So there exist permissible parameters u_1,v_1,w_1 at $q_1\in\pi^{-1}(q)$ for $E_{1,j}$ such that $u=u_1w_1, v=v_1w_1, w=w_1$ and $u_1v_1w_1=0$ is a local equation of $E_{1,j}$ at q_1 and

$$\begin{aligned} u_1 &= \frac{u}{w} = x^{a-g}y^{b-h}(z+\alpha)^{-1}\\ v_1 &= \frac{v}{w} = x^{d-g}y^{e-h}(z+\alpha)^{-1}\\ w_1 &= w = x^gy^h(z+\alpha). \end{aligned}$$

Since $ae-bd\neq 0$,

$$\operatorname{rank}\begin{pmatrix} a-g & b-h\\ d-g & e-h\\ g & h\end{pmatrix} = \operatorname{rank}\begin{pmatrix} a & b\\ d & e\\ g & h\end{pmatrix}=2$$

and, after possibly permuting u_1,v_1,w_1, we may assume $\det\begin{pmatrix} a-g & b-h\\ d-g & e-h\end{pmatrix}$ is nonzero. So there exist $\gamma_1,\gamma_2\in\mathbb{Q}$ such that

$$\begin{pmatrix} a-g & b-h\\ d-g & e-h\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}=\begin{pmatrix}-1\\ -1\end{pmatrix} \tag{4.2}$$

and we can set

$$\begin{aligned}\tilde{x} &= x(z+\alpha)^{\gamma_1}\\ \tilde{y} &= y(z+\alpha)^{\gamma_2}\\ \tilde{\alpha} &= \alpha^{1-(g\gamma_1+h\gamma_2)}\\ \tilde{z} &= (z+\alpha)^{1-(g\gamma_1+h\gamma_2)} - \tilde{\alpha}\end{aligned}$$

which satisfy

$$\begin{aligned}\left|\begin{array}{ccc} \frac{\partial\tilde{x}}{\partial x} & \frac{\partial\tilde{x}}{\partial y} & \frac{\partial\tilde{x}}{\partial z}\\ \frac{\partial\tilde{y}}{\partial x} & \frac{\partial\tilde{y}}{\partial y} & \frac{\partial\tilde{y}}{\partial z}\\ \frac{\partial\tilde{z}}{\partial x} & \frac{\partial\tilde{z}}{\partial y} & \frac{\partial\tilde{z}}{\partial z}\end{array}\right|_{(0,0,0)} &= \left|\begin{array}{ccc} \alpha^{\gamma_1} & 0 & 0\\ 0 & \alpha^{\gamma_2} & 0\\ 0 & 0 & (1-(g\gamma_1+h\gamma_2))\alpha^{-(g\gamma_1+h\gamma_2)}\end{array}\right|\\ &= (1-(g\gamma_1+h\gamma_2))\alpha^{\gamma_1+\gamma_2-(g\gamma_1+h\gamma_2)} \neq 0\end{aligned}$$

since $\alpha \neq 0$ and $1-(g\gamma_1+h\gamma_2) \neq 0$. (Otherwise, if $(g\gamma_1+h\gamma_2) = 1$, the equation (4.2) implies that

$$\begin{pmatrix} a & b\\ d & e\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

and since $ae-bd \neq 0$, $\gamma_1 = \gamma_2 = 0$ which is in contradiction with the fact that γ_1, γ_2 are solutions to the equation (4.2).

Therefore, we obtain permissible parameters $\tilde{x}, \tilde{y}, \tilde{z}$ at p_1 for $D_{1,j}$ such that $\tilde{x}\tilde{y} = 0$ is a local equation of $D_{1,j}$ at p_1 and

$$\begin{aligned}u_1 &= \tilde{x}^{a-g}\tilde{y}^{b-h}\\ v_1 &= \tilde{x}^{d-g}\tilde{y}^{e-h}\\ w_1 &= \tilde{x}^{g}\tilde{y}^{h}(\tilde{z}+\tilde{\alpha})\end{aligned}$$

where $(a-g)(e-h)-(b-h)(d-g) \neq 0$ and $\tilde{\alpha} \neq 0$ since $\alpha \neq 0$. Thus toroidal form (T2) holds for p_1.

Therefore, in the the case when the quasi-toroidal form ($\mathfrak{qt}4$) holds for $\varphi\lambda$ at p_1, and similarly in other cases, $\phi_{1,j} : U_{1,j} \to V_{1,j}$ is toroidal at $p_1 \in (\varphi\lambda)^{-1}(q) \cap U_{1,j}$ where $q \in Z \cap E_j$ with respect to $E_{1,j}$ and $D_{1,j}$. □

We now give the proof of Theorem 4.1.

Proof of Theorem 4.1. We first construct, by Theorem 3.3, a principalization sequence

$$\lambda : X_1 = X_{1,n} \xrightarrow{\lambda_n} X_{1,n-1} \to \cdots \to X_{1,1} \xrightarrow{\lambda_1} X_0 = X$$

of Z satisfying $W_Z(X_1)=\emptyset$, so that the rational map $X_1 \dashrightarrow Y_1$ is a morphism, say ϕ_1, and we have a commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi_1} & Y_1 \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\varphi]{} & Y\ . \end{array}$$

We want to show that $\phi_1 : X_1 \to Y_1$ is locally toroidal with respect to $\mathcal{L}_{X_1} = \{U_{1,j}, D_{1,j}, V_{1,j}, E_{1,j}\}_J$. Suppose that $j \in J$. We must show that $\phi_{1,j} : U_{1,j} \to V_{1,j}$ is toroidal with respect to the divisors $E_{1,j}$ and $D_{1,j}$.

By Lemma 4.2, $E_{1,j}$ and $D_{1,j}$ are SNC divisors and $U_{1,j} \setminus D_{1,j} \to V_{1,j} \setminus E_{1,j}$ is smooth. So, by Proposition 2.2, it just remains to verify that one of the forms (T1) through (T6) holds for each $p_1 \in D_{1,j} \subset U_{1,j}$ and $q_1 = \phi_{1,j}(p_1)$ lies in $E_{1,j} \subset V_{1,j}$.

since $\varphi_j : U_j \to V_j$ is toroidal, and due to the isomorphisms (4.1) in the proof of Lemma 4.2, we only need to show that $\phi_{1,j}$ is toroidal at each point $p_1 \in D_{1,j}$ with $\phi_{1,j}(p_1) = q_1 \in \pi^{-1}(Z \cap V_j)$ and therefore $q = \pi(q_1) \in Z \cap E_j$ since $q_1 \in E_{1,j}$. This is accomplished by the proof of Lemma 4.3. ☐

4.2. **Proof of the Main Theorem.** This subsection is devoted to the proofs of Theorem 2.13 and finally our main result Theorem 1.2.

Proof of Theorem 2.13. Since $\varphi : X \to Y$ is locally toroidal, it clearly satisfies the conclusions of the theorem. Let

$$\pi : \widetilde{Y} = Y_n \xrightarrow{\pi_n} Y_{n-1} \to \cdots \to Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y$$

be an embedded resolution of $\tilde{E}_0 \subset Y$ satisfying the conclusions of Theorem 2.4, where each π_i is the blow up of a nonsingular center $Z_{i-1} \subset Y_{i-1}$ which is either a point or a curve. Due to the conclusion 1) of Theorem 2.4, $E_{i,j}$ is a SNC divisor on $V_{i,j}$ for all i, j, and $Z_i \cap V_{i,j}$ makes SNCs with $E_{i,j}$ on $V_{i,j}$ for all i, j.

Assume that we have constructed the commutative diagram

$$\begin{array}{ccccccccc} X_{i-1} & \longrightarrow & \cdots & \longrightarrow & X_2 & \xrightarrow{\lambda_2} & X_1 & \xrightarrow{\lambda_1} & X \\ {\scriptstyle\phi_{i-1}}\downarrow & & & & {\scriptstyle\phi_2}\downarrow & & {\scriptstyle\phi_1}\downarrow & & {\scriptstyle\varphi}\downarrow \\ Y_{i-1} & \longrightarrow & \cdots & \longrightarrow & Y_2 & \xrightarrow[\pi_2]{} & Y_1 & \xrightarrow[\pi_1]{} & Y \end{array}$$

with $i < n+1$, satisfying the conclusions of the theorem. Then we consider the blowup $\pi_i : Y_i \to Y_{i-1}$ of $Z_{i-1} \subset Y_{i-1}$ in the resolution sequence. By Theorem 3.3, there exists a principalization sequence of Z_{i-1}

$$\lambda_i : X_i = X_{i,k_i} \to X_{i,k_i-1} \to \cdots \to X_{i,1} \to X_{i-1}$$

with $W_{Z_{i-1}}(X_i) = \emptyset$, for some finite number $k_i \in \mathbb{N}$. Further, $D_{i,j} = (\lambda_i|_{U_{i,j}})^*(D_{i-1,j})_{\rm red}$ is a SNC divisor on $U_{i,j} = \lambda_i^{-1}(U_{i-1,j})$ for all j by Lemma 3.1 in case Z_{i-1} is a point, and by Lemma 3.2 if Z_{i-1} is a curve.

Therefore, we gain the morphism $X_i \to Y_i$, say ϕ_i, and the commutative diagram of morphisms

$$\begin{array}{ccc} X_i & \xrightarrow{\phi_i} & Y_i \\ \lambda_i \downarrow & & \downarrow \pi_i \\ X_{i-1} & \xrightarrow[\phi_{i-1}]{} & Y_{i-1} \end{array}$$

where $\phi_i : X_i \to Y_i$ is locally toroidal with respect to $\mathcal{L}_{X_i} = \{U_{i,j}, D_{i,j}, V_{i,j}, E_{i,j}\}_J$ by Theorem 4.1.

In sum, iterative use of this process results in the diagram ($\mathcal{LT}$) which satisfies the conclusions of the theorem. □

Now we give the proof of the Main Theorem.

Proof of Theorem 1.2. Due to Theorem 2.13, there exist proper birational morphisms $\pi : \widetilde{Y} \to Y$ and $\lambda : \widetilde{X} \to X$ such that $\widetilde{Y}$ and $\widetilde{X}$ are nonsingular, and we have the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\varphi}} & \widetilde{Y} \\ \lambda \downarrow & & \downarrow \pi \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

such that $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ is locally toroidal with respect to $\widetilde{\mathcal{L}} = \{\tilde{U}_j, \tilde{D}_j, \tilde{V}_j, \tilde{E}_j\}_J$, i.e., $\tilde{\varphi}_j = \tilde{\varphi}|_{\tilde{U}_j} : \tilde{U}_j \to \tilde{V}_j$ is toroidal with respect to $\tilde{D}_j$ and $\tilde{E}_j$ for all $j \in J$, where $\tilde{U}_j = \lambda^{-1}(U_j)$, $\tilde{V}_j = \pi^{-1}(V_j)$, $\tilde{D}_j = (\lambda|_{\tilde{U}_j})^*(D_j)_{\text{red}}$ and $\tilde{E}_j = (\pi|_{\tilde{V}_j})^*(E_j)_{\text{red}}$. Further, $\widetilde{E} = \pi^*(\tilde{E}_0)_{\text{red}}$, which contains $\left(\sum_{j\in J} \overline{\tilde{E}_j}\right)_{\text{red}}$, is a SNC divisor on $\widetilde{Y}$ where $\overline{\tilde{E}_j}$ is the Zariski closure of $\tilde{E}_j$ in $\widetilde{Y}$. We will prove that $\tilde{\varphi}$ is in fact toroidal with respect to $\widetilde{E}$ and $\widetilde{D} = \tilde{\varphi}^*(\widetilde{E})_{\text{red}}$ which is a SNC divisor on $\widetilde{X}$ as well.

We first verify that $\widetilde{X} \setminus \widetilde{D} \to \widetilde{Y} \setminus \widetilde{E}$ is smooth. Let $p \in \widetilde{X} \setminus \widetilde{D}$, then $q = \tilde{\varphi}(p) \notin \widetilde{E}$, and hence $q \notin \tilde{E}_j$ for all $j \in J$. So $p \notin \tilde{\varphi}^*(\tilde{E}_j)_{\text{red}} = \tilde{\varphi}_j^*(\tilde{E}_j)_{\text{red}} = \tilde{D}_j$ for all j. Suppose that $p \in \tilde{U}_j$ for some $j \in J$. The morphism $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{D}_j$ and $\tilde{E}_j$ by the fact that $\tilde{\varphi}$ is locally toroidal with respect to $\widetilde{\mathcal{L}} = \{\tilde{U}_j, \tilde{D}_j, \tilde{V}_j, \tilde{E}_j\}_J$. So $\tilde{\varphi}$ is smooth at p since $\tilde{\varphi}_j$ is. Thus $\tilde{\varphi} : \widetilde{X} \setminus \widetilde{D} \to \widetilde{Y} \setminus \widetilde{E}$ is smooth.

Now suppose that $p \in \widetilde{D}$ and $q = \tilde{\varphi}(p) \in \widetilde{E}$. We must show that there exist (algebraic) permissible parameters u, v, w at q for $\widetilde{E}$ and (formal) permissible parameters x, y, z at p for $\widetilde{D}$ such that one of the forms (T1) through (T6) in Proposition 2.2 holds.

First, we note that there exist $j \in J$ such that $p \in \tilde{U}_j$. Then $q \in \tilde{V}_j$ and we will use the fact that $\tilde{\varphi}|_{\tilde{U}_j} = \tilde{\varphi}_j : \tilde{U}_j \to \tilde{V}_j$ is toroidal with respect to $\tilde{E}_j$ and

$\tilde{D}_j$ to prove that $\tilde{\varphi}$ is also toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$. Meanwhile, we show $\widetilde{D}$ is a SNC divisor on $\widetilde{X}$.

If $q \notin \tilde{E}_j$, then $\tilde{\varphi}_j$ is smooth in a neighborhood of p, and so is $\tilde{\varphi}$. Hence we have $\hat{\mathcal{O}}_{\widetilde{Y},q} \xrightarrow{\sim} \hat{\mathcal{O}}_{\widetilde{X},p}$ since $\dim X = \dim Y = 3$. Thus $\widetilde{D}$ is a SNC divisor at p since $\widetilde{E}$ is a SNC divisor (at q). So we can assume that $q \in \tilde{E}_j$.

Suppose that $q = \tilde{\varphi}(p)$ is a 1-point for $\widetilde{E}$, then it is a 1-point for $\tilde{E}_j$ as well, and $\widetilde{E}, \tilde{E}_j$ are equal in a neighborhood of q. So $\widetilde{D}, \tilde{D}_j$ are also equal in a neighborhood of p. Hence $\widetilde{D}$ is a SNC divisor at p. In addition, by Proposition 2.2, there exist (algebraic) permissible parameters u, v, w at q for $\tilde{E}_j$ (hence for $\widetilde{E}$) and (formal) permissible parameters x, y, z at p for $\tilde{D}_j$ (so, for $\widetilde{D}$) such that $u = 0$ is a local equation of both $\tilde{E}_j, \widetilde{E}$ at q, $x = 0$ is a local equation of both $\tilde{D}_j, \widetilde{D}$ at p and (T6) holds for p since $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{E}_j$ and $\tilde{D}_j$. Thus $\tilde{\varphi}$ is toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$.

Suppose that q is a 2-point for $\widetilde{E}$, then q can be a 2-point or a 1-point for $\tilde{E}_j$. If q is a 2-point for $\tilde{E}_j$, we can argue exactly as before to see that $\widetilde{D}$ has SNCs at p as $\tilde{D}_j$ has, and p is either a 2-point or a 1-point for both $\tilde{D}_j$ and $\widetilde{D}$. In addition, (T4) holds for p in case it is a 2-point, and (T5) holds for p in case it is a 1-point since $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{E}_j$ and $\tilde{D}_j$. So $\tilde{\varphi}$ is toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$.

Now suppose that q is a 2-point for $\widetilde{E}$, but it is a 1-point for $\tilde{E}_j$. By Proposition 2.2, since $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{E}_j$ and $\tilde{D}_j$, there exist (algebraic) permissible parameters u, v, w at q for $\tilde{E}_j$ and (formal) permissible parameters x, y, z at p for $\tilde{D}_j$ such that $u = 0$ is a local equation of $\tilde{E}_j$ at q, $x = 0$ is a local equation of $\tilde{D}_j$ at p and (T6) holds for p, i.e.,

$$u = x^a, v = y, w = z \quad \text{with} \quad a > 0.$$

We can change the parameters v, w to obtain new permissible parameters $u, \tilde{v}, \tilde{w}$ at q for both $\tilde{E}_j$ and $\widetilde{E}$ such that $u\tilde{v} = 0$ is a local equation of $\widetilde{E}$ at q and $u = 0$ remains as a local equation of $\tilde{E}_j$ at q. By the formal inverse function theorem, the expansions

$$\begin{array}{rcl} \tilde{v} & = & \alpha_1 u + \beta_1 v + \gamma_1 w + \text{higher degree terms in } u, v \text{ and } w \\ \tilde{w} & = & \alpha_2 u + \beta_2 v + \gamma_2 w + \text{higher degree terms in } u, v \text{ and } w \end{array} \tag{4.3}$$

of $\tilde{v}, \tilde{w}$ in $\hat{\mathcal{O}}_{\widetilde{Y},q} \cong \mathfrak{k}[[u, v, w]]$, with $\alpha_i, \beta_i, \gamma_i \in \mathfrak{k}$, satisfy

$$0 \neq \det \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{w}}{\partial u} \\ \frac{\partial u}{\partial v} & \frac{\partial \tilde{v}}{\partial v} & \frac{\partial \tilde{w}}{\partial v} \\ \frac{\partial u}{\partial w} & \frac{\partial \tilde{v}}{\partial w} & \frac{\partial \tilde{w}}{\partial w} \end{pmatrix} = \det \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & \beta_1 & \beta_2 \\ 0 & \gamma_1 & \gamma_2 \end{pmatrix} = \det \begin{pmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}.$$

In addition, $\tilde{D}_j$ has SNCs at p and there exist algebraic permissible parameters $\bar{x}, \bar{y}, \bar{z}$ in $\mathcal{O}_{\widetilde{X},p}$ such that $\bar{x} = 0$ is a local equation of $\tilde{D}_j$ at p and so, there exists a unit $\delta_x \in \hat{\mathcal{O}}_{\widetilde{X},p}$ such that $x = \delta_x \bar{x}$ (Recall that $x = 0$ is also a local equation of $\tilde{D}_j$ at p).

By substituting (T6) in (4.3), we have

$$\tilde{v} = \alpha_1 x^a + \beta_1 y + \gamma_1 z + \text{higher degree terms in } x, y \text{ and } z$$
$$\tilde{w} = \alpha_2 x^a + \beta_2 y + \gamma_2 z + \text{higher degree terms in } x, y \text{ and } z$$

and

$$\det\begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial \tilde{v}}{\partial x} & \frac{\partial \tilde{v}}{\partial y} & \frac{\partial \tilde{v}}{\partial z} \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & \frac{\partial \tilde{w}}{\partial z} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial \tilde{v}}{\partial x} & \frac{\partial \tilde{v}}{\partial y} & \frac{\partial \tilde{v}}{\partial z} \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & \frac{\partial \tilde{w}}{\partial z} \end{pmatrix} = \beta_1\gamma_2 - \gamma_1\beta_2 \bmod \hat{\mathfrak{m}}_p$$

is a unit in $\hat{\mathcal{O}}_{\widetilde{X},p}$. Thus $x, \tilde{y} = \tilde{v}, \tilde{z} = \tilde{w}$ are formal regular parameters at p. Since $x = \delta_x \bar{x}$ and $\tilde{y}, \tilde{z} \in \mathcal{O}_{\widetilde{X},p}$ then we have that $\bar{x}, \tilde{y}, \tilde{z}$ are regular parameters in $\mathcal{O}_{\widetilde{X},p}$ and a local equation for $\widetilde{D}$ at p is

$$0 = u\tilde{v} = x^a \tilde{y} = \delta_x^a \bar{x}^a \tilde{y}.$$

Since $a > 0$, $\bar{x}\tilde{y} = 0$ is a local equation of $\widetilde{D}$ at p, showing that $\widetilde{D}$ is a SNC divisor at p.

We have an expression

$$u = x^a,\ \tilde{v} = \tilde{y},\ \tilde{w} = \tilde{z} \text{ with } \det\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a > 0.$$

So toroidal form (T4) holds for p and $\tilde{\varphi}$ is toroidal with respect to $\widetilde{E}$ and $\widetilde{D}$.

Suppose that q is a 3-point for $\widetilde{E}$, then q can be a 3-point, a 2-point or a 1-point for $\tilde{E}_j$. If q is a 3-point for $\tilde{E}_j$, then $\widetilde{E}, \tilde{E}_j$ are equal in a neighborhood of q and $\widetilde{D}, \tilde{D}_j$ are also equal in a neighborhood of p. Hence D is a SNC divisor at p. In addition, $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{E}_j$ and $\tilde{D}_j$ and, by Proposition 2.2, there exist (algebraic) permissible parameters u, v, w at q for $\tilde{E}_j$ (hence for $\widetilde{E}$) and (formal) permissible parameters x, y, z at p for $\tilde{D}_j$ (so for $\widetilde{D}$) such that $uvw = 0$ is a local equation of both $\tilde{E}_j, \widetilde{E}$ at q and, either $xyz = 0$ is a local equation of both $\tilde{D}_j, \widetilde{D}$ at p and (T1) holds for p, or $xy = 0$ is a local equation of both $\tilde{D}_j, \widetilde{D}$ at p and (T2) holds for p, or $x = 0$ is a local equation of both $\tilde{D}_j, \widetilde{D}$ at p and (T3) holds for p. Thus $\tilde{\varphi}$ is toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$.

Suppose that q is a 3-point for $\widetilde{E}$, but it is a 2-point for $\tilde{E}_j$. Since $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{E}_j$ and $\tilde{D}_j$, by Proposition 2.2, there exist (algebraic) permissible parameters u, v, w at q for $\tilde{E}_j$ and (formal) permissible parameters x, y, z at p for $\tilde{D}_j$ such that $uv = 0$ is a local equation of $\tilde{E}_j$ at q and one of the toroidal forms (T4) or (T5) holds.

We can change the parameter w to obtain new permissible parameters $u, v, \tilde{w}$ at q for both $\tilde{E}_j$ and $\widetilde{E}$ such that $uv\tilde{w} = 0$ is a local equation of $\widetilde{E}$ at q and $uv = 0$ remains as a local equation of $\tilde{E}_j$ at q. By the formal inverse function theorem, the expansion

$$\tilde{w} = \alpha u + \beta v + \gamma w + \text{higher degree terms in } u, v \text{ and } w$$

of $\tilde{w}$ in $\hat{\mathcal{O}}_{\widetilde{Y},q} \cong \mathfrak{k}[[u,v,w]]$, with $\alpha, \beta, \gamma \in \mathfrak{k}$, satisfies

$$0 \neq \det \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial v}{\partial u} & \frac{\partial \tilde{w}}{\partial u} \\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial v} & \frac{\partial \tilde{w}}{\partial v} \\ \frac{\partial u}{\partial w} & \frac{\partial v}{\partial w} & \frac{\partial \tilde{w}}{\partial w} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & \gamma \end{pmatrix} = \gamma.$$

We first suppose that (T4) holds for permissible parameters u, v, w at q for $\tilde{E}_j$ and, for permissible parameters x, y, z at p for $\tilde{D}_j$. So, $xy = 0$ is a local equation of $\tilde{D}_j$ at p and, $u = x^a y^b, v = x^d y^e, w = z$ with $ae - bd \neq 0$. In addition, $\tilde{D}_j$ is a SNC divisor on $\tilde{U}_j$ at p and there exist algebraic permissible parameters $\bar{x}, \bar{y}, \bar{z}$ in $\mathcal{O}_{\widetilde{X},p}$ such that $\bar{x}\bar{y} = 0$ is a local equation of $\tilde{D}_j$ at p and so, after possibly permuting x, y, there exist units $\delta_x, \delta_y \in \hat{\mathcal{O}}_{\widetilde{X},p}$ such that $x = \delta_x \bar{x}$ and $y = \delta_y \bar{y}$. Then we have

$$\tilde{w} = \alpha x^a y^b + \beta x^d y^e + \gamma z + \text{higher degree terms in } x, y \text{ and } z$$

and

$$\det \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & \frac{\partial \tilde{w}}{\partial z} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & \frac{\partial \tilde{w}}{\partial z} \end{pmatrix} = \frac{\partial \tilde{w}}{\partial z} = \gamma \bmod \hat{\mathfrak{m}}_p$$

is a unit in $\hat{\mathcal{O}}_{\widetilde{X},p}$. Thus $x, y, \tilde{z} = \tilde{w}$ are formal regular parameters at p. Since $x = \delta_x \bar{x}$ and $y = \delta_y \bar{y}$ and $\tilde{z} \in \mathcal{O}_{\widetilde{X},p}$, we have that $\bar{x}, \bar{y}, \tilde{z}$ are regular parameters in $\mathcal{O}_{\widetilde{X},p}$ and a local equation for $\widetilde{D}$ at p is

$$0 = uv\tilde{w} = x^{a+d} y^{b+e} \tilde{z} = \delta_x^{a+d} \delta_y^{b+e} \bar{x}^{a+d} \bar{y}^{b+e} \tilde{z}.$$

Now $a + d > 0$ and $b + e > 0$ since $ae - bd \neq 0$ and $a, b, d, e \geqslant 0$. So $\bar{x}\bar{y}\tilde{z} = 0$ is a local equation of $\widetilde{D}$ at p, showing that $\widetilde{D}$ is a SNC divisor at p. We have an expression

$$u = x^a y^b, \ v = x^d y^e, \ \tilde{w} = \tilde{z} \text{ with } ae - bd \neq 0.$$

So these parameters and equations satisfy all the conditions of toroidal form (T1) since

$$\det \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} a & b \\ d & e \end{pmatrix} = ae - bd \neq 0.$$

Thus $\tilde{\varphi}$ is toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$.

Now suppose that (T5) holds for permissible parameters u, v, w at q for $\tilde{E}_j$ and permissible parameters x, y, z at p for $\tilde{D}_j$. So, $x = 0$ is a local equation of $\tilde{D}_j$ at p and, $u = x^a, v = x^d(y + \sigma), w = z$ with $0 \neq \sigma \in \mathfrak{k}$ and $a, d > 0$. In addition, $\tilde{D}_j$ is a SNC divisor on $\tilde{U}_j$ at p and there exist algebraic permissible parameters $\bar{x}, \bar{y}, \bar{z}$ in $\mathcal{O}_{\widetilde{X},p}$ such that $\bar{x} = 0$ is a local equation of $\tilde{D}_j$ at p and so, there exists a unit $\delta_x \in \hat{\mathcal{O}}_{\widetilde{X},p}$ such that $x = \delta_x \bar{x}$. In this case, we have

$$\tilde{w} = \alpha x^a + \beta x^d (y + \sigma) + \gamma z + \text{higher degree terms in } x, y \text{ and } z$$

and, the same as before,

$$\det\begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & \frac{\partial \tilde{w}}{\partial z} \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial \tilde{w}}{\partial x} & \frac{\partial \tilde{w}}{\partial y} & \frac{\partial \tilde{w}}{\partial z} \end{pmatrix} = \frac{\partial \tilde{w}}{\partial z} = \gamma \bmod \hat{\mathfrak{m}}_p$$

is a unit in $\hat{\mathcal{O}}_{\widetilde{X},p}$. So, $x, y, \tilde{z} = \tilde{w}$ are formal regular parameters at p. Since $x = \delta_x \bar{x}$ and $\tilde{z} \in \mathcal{O}_{\widetilde{X},p}$, we have that $\bar{x}, \tilde{z}$ are linearly independent in $\mathfrak{m}_p/\mathfrak{m}_p^2$. Thus they extend to a regular system of parameters in $\mathcal{O}_{\widetilde{X},p}$, say $\bar{x}, \tilde{z}, \tilde{y}$, and

$$0 = uv\tilde{w} = x^{a+d}(y+\sigma)\tilde{z} = \delta_x^{a+d}\bar{x}^{a+d}(y+\sigma)\tilde{z}$$

is a local equation of $\widetilde{D}$ at p. We note that $a + d > 0$ since $a, d > 0$, so, $\bar{x}\tilde{z} = 0$ is a local equation of $\widetilde{D}$ at p, which shows that $\widetilde{D}$ is a SNC divisor at p.

We have an expression

$$u = x^a,\ \tilde{w} = \tilde{z},\ v = x^d(y+\sigma) \text{ with } a, d > 0,\ \sigma \neq 0,$$

and here $x\tilde{z} = 0$ is a (formal) local equation of $\widetilde{D}$ at p. So these parameters and equations satisfy all the condition of toroidal form (T2) since $(d, 0) \neq (0, 0)$ and

$$\det\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a > 0.$$

Thus $\tilde{\varphi}$ is toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$.

Finally, suppose that q is a 3-point for $\widetilde{E}$, but it is a 1-point for $\tilde{E}_j$. By Proposition 2.2, there exist (algebraic) permissible parameters u, v, w at q for $\tilde{E}_j$ and (formal) permissible parameters x, y, z at p for $\tilde{D}_j$ such that $u = 0$ is a local equation of $\tilde{E}_j$, $x = 0$ is a local equation of $\tilde{D}_j$ at p and (T6) holds for p since $\tilde{\varphi}_j$ is toroidal with respect to $\tilde{E}_j$ and $\tilde{D}_j$.

We can change the parameters v, w to obtain new permissible parameters $u, \tilde{v}, \tilde{w}$ at q for both $\tilde{E}_j$ and $\widetilde{E}$ such that $u\tilde{v}\tilde{w} = 0$ is a local equation of $\widetilde{E}$ at q and $u = 0$ remains as a local equation of $\tilde{E}_j$ at q.

Completely similar to the case when q is a 2-point for $\widetilde{E}$, but it is a 1-point for $\tilde{E}_j$, we see that $\bar{x}, \tilde{v}, \tilde{w}$ are regular parameters in $\mathcal{O}_{\widetilde{X},p}$, where $\bar{x} = 0$ is an algebraic local equation of $\tilde{D}_j$, such that $\bar{x}\tilde{v}\tilde{w} = 0$ is a local equation of $\widetilde{D}$ at p since $u\tilde{v}\tilde{w} = 0$ is a local equation of $\widetilde{E}$ at $q = \tilde{\varphi}(p)$ and $u = x^a = \delta_x^a \bar{x}^a$ for some unit $\delta_x \in \hat{\mathcal{O}}_{\widetilde{X},p}$ and $0 < a \in \mathbb{N}$. So $\widetilde{D}$ is a SNC divisor at p.

Also, $x, \tilde{v}, \tilde{w}$ are permissible parameters at p for $\widetilde{D}$ such that $x\tilde{v}\tilde{w} = 0$ is a local equation of $\widetilde{D}$ at p and toroidal form (T1) holds for these parameters at p and $u, \tilde{v}, \tilde{w}$ at q for $\widetilde{E}$ since $a > 0$. Thus $\tilde{\varphi}$ is toroidal at p with respect to $\widetilde{E}$ and $\widetilde{D}$.

Therefore, $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ is a toroidal morphism with respect to $\widetilde{E}, \widetilde{D}$. □

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