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FINITE GROUPS WITH X-QUASIPERMUTABLE SUBGROUPS OF PRIME POWER ORDER

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Abstract. Let H, L and X be subgroups of a finite group G. Then H is said to be X-permutable with L if for some $x \in X$ we have $AL^x = L^x A$. We say that H is X-*quasipermutable* (X_S-*quasipermutable*, respectively) in G provided G has a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (with all Sylow subgroups, respectively) V of B such that $(|H|, |V|) = 1$. In this paper, we analyze the influence of X-quasipermutable and X_S-quasipermutable subgroups on the structure of G. Some known results are generalized.

Keywords: X-quasipermutable subgroup, Sylow subgroup, p-soluble group, p-supersoluble group, p-nilpotent group.
MSC(2010): Primary: 20D10; Secondary: 20D15, 20D20.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, p and q are always supposed to be primes.

If $AB = BA$, then A is said to *permute* with B; if $G = AB$, then B is called a *supplement* of A to G; if $AB^x = B^x A$ for at least one element $x \in X$, then A is said to X-*permute* with B [4].

A large number of researches are connected with the study of subgroups H of G such that H permutes with some subgroups of its supplement B in G. If, for example, H X-permutes with all subgroups of B, then H is called X-*semipermutable* in G [5]; if H permutes with all Sylow subgroups of B, then H is called SS-*quasinormal* in G [10]. Subgroups with a condition of this kind have been useful in the analysis of many aspects of the theory of finite groups.

In this paper, we introduce and analyze some applications of the following concepts that cover the conditions of X-semipermutability and SS-quasinormality.

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Definition 1.1. Let H and X be subgroups of G. Then we say that H is *X-quasipermutable* (*X_S-quasipermutable*, respectively) in G provided G has a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (with all Sylow subgroups, respectively) V of B such that $(|H|, |V|) = 1$.

We say also that H is *quasipermutable* (*S-quasipermutable*, respectively) if H is 1-quasipermutable (1_S-quasipermutable, respectively) in G.

Before continuing, consider two examples.

Example 1.2. (i) Let q divide $p-1$ and $G = Q \rtimes (P \rtimes H)$, where $|H| = q$, $P = C_{PH}(P)$ is a group of order p and Q is a simple $\mathbb{F}_q[PH]$-module which is faithful for PH. Let $X = F(G) = Q$. It is clear that H is X-quasipermutable in G. We shall show that H is not S-quasipermutable in G and so H is not quasipermutable in G. Assume that $G = N_G(H)B$ for some subgroup B of G such that H permutes with all Sylow p-subgroup of B. It is clear that p does not divide $|N_G(H)|$, $N_Q(P) = 1$ and $N_Q(H) \neq Q$. Let $a \in Q \backslash N_Q(H)$. Then $a = bn$ for some $n \in N_G(H)$ and $b \in B$. Thus $HP^a = H(P^b)^n = (P^b)^n H = P^a H$. Hence $PH^{a^{-1}} = H^{a^{-1}}P$, so $H^{a^{-1}} \leq N_G(P)$. Therefore $H^{a^{-1}} \leq PH$ and so $H = (H^{a^{-1}})^x$ for some $x \in P$. But then $a^{-1}x \in N_G(H) \leq QH$, which implies that $x = 1$. Thus $a^{-1}, a \in N_Q(H)$. This contradiction shows that H is not S-quasipermutable in G.

(ii) It is clear that every X-semipermutable subgroup is X-quasipermutable and every SS-quasipermutable subgroup is S-quasipermutable. We shall show that the inverse statements are not true in general. Indeed, let $G = PH \rtimes P_1$, where $P = \langle y \rangle$ and H are the groups defined above, and $P_1 = \langle y_1 \rangle$ is a group of order p. Then H evidently is quasipermutable in G, and for every $z \in G$, $\langle yy_1 \rangle^z H \neq H \langle yy_1 \rangle^z$. Thus H is not G-semipermutable in G.

Now, let $G = C_7 \rtimes \mathrm{Aut}(C_7)$, where C_7 is a group of order 7. Let H and S be the subgroups of order 2 and 3, respesctively, in $\mathrm{Aut}(C_7)$. Then H is quasipermutable in G. Assume that H is SS-quasinormal in G. For any supplement B of H to G we have $C_7S \leq B$, so for every $1 \neq x \in C_7$ we have $HS^x = S^xH$, which implies that $G = H^G \leq N_G(S)$ (see (i)), so $S \leq C_G(C_7)$. This contradiction shows that H is not SS-quasinormal in G.

Our main goal here is to prove the following result.

Theorem 1.3. *Let P be a Sylow p-subgroup of G and $X = F(G)$.*

(I) If P is X_S-quasipermutable in G, then G is p-soluble and P^G is soluble.

(II) If P is X-quasipermutable in G, then the following statements hold:

(i) $P' \leq O_p(G)$. If, in addition, $N_G(P)$ is p-nilpotent, then the focal subgroup $G' \cap P$ of G is contained in $O_p(G)$.

(ii) $l_p(G) \leq 2$.

(iii) If $p > q$ for all primes q dividing $|G : N_G(P)|$, then P is normal in G.

(iv) If for some prime $q \neq p$ a Hall p'-subgroup of G is q-supersoluble, then G is q-supersoluble.

Corollary 1.4. *(See Main result in [2]) Let P be a Sylow p-subgroup of G. If P is 1-semipermutable in G, then the following statements hold:*

(i) G is p-soluble and $P' \leq O_p(G)$.

(ii) $l_p(G) \leq 2$.

(iii) If for some prime $q \neq p$ a Hall p'-subgroup of G is q-supersoluble, then G is q-supersoluble.

Corollary 1.5. *Let $p_1 > \cdots > p_t$ be the set of all primes dividing $|G|$, $\pi_i = \{p_1, \ldots, p_i\}$ and H_i a Hall π_i-subgroup of G. Let $X = F(G)$. If for every $k \leq i$, H_k is X-quasipermutable in G, then H_i is normal in G.*

Proof. By Theorem 1.3(II)(iii), H_1 is normal in G, and H_k/H_1 is X-quasipermutable in G/H_1 (see Lemma 2.2(1) below). Hence H_k/H_1 is normal in G/H_1 by induction, so H_i is normal in G. □

Theorem 5.4 in [5] is equivalent to the following special case of Corollary 1.5.

Corollary 1.6. *Let $p_1 > \cdots > p_t$ be the set of all primes dividing $|G|$, $\pi_i = \{p_1, \ldots, p_i\}$ and H_i a Hall π_i-subgroup of G. Let $X = F(G)$. If for every $k \leq i$, H_k is X-semipermutable in G, then H_i is normal in G.*

We use $\mathfrak{M}_\phi(G)$ to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathfrak{M}_\phi(G)$.

On the base of Theorem 1.3 we prove also the following results.

Theorem 1.7. *Let P be a Sylow p-subgroup of G and $X = F(G)$. Suppose that every number V of some fixed $\mathfrak{M}_\phi(P)$ is X_S-quasipermutable in G.*

(i) If $|P| > p$, then G is p-supersoluble.

(ii) If $(p-1, |G|) = 1$, then G is p-nilpotent.

Corollary 1.8. *(See Theorem 1.1 in [10]) Let P be a Sylow p-subgroup of G, where p be the smallest prime dividing $|G|$. If every number V of some fixed $\mathfrak{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 1.9. *Let P be a Sylow p-subgroup of G and $X = F(G)$. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathfrak{M}_\phi(P)$ is X_S-quasipermutable in G, then G is p-nilpotent.*

Proof. If $|P| = p$, then G is p-nilpotent by Burnside's theorem [6, IV, 2.6]. Otherwise, G is p-supersoluble by Theorem 1.7. The hypothesis holds for $G/O_{p'}(G)$ (see Lemma 2.2(1) below) and so in the case when $O_{p'}(G) \neq 1$, $G/O_{p'}(G)$ is p-nilpotent by induction, which implies the p-nilpotency of G. Therefore we may assume that $O_{p'}(G) = 1$. But then, by Lemma 2.7(3) below, P is normal in G. Hence G is p-nilpotent by hypothesis. □

From Corollary 1.9 we get

Corollary 1.10. *(See Theorem 1.2 in [10]) Let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathfrak{M}_\phi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Theorem 1.11. *If every Sylow subgroup of G is $F(G)$-quasipermutable in G, then G is supersoluble.*

Corollary 1.12. *(See Theorem 5 in [11]) If every Sylow subgroup of G is 1-semipermutable in G, then G is supersoluble.*

2. **Preliminaries**

The first lemma is evident.

Lemma 2.1. *Let A, B and X be subgroups of G and N a normal subgroup of G. If A X-permutes with B, then AN/N (XN/N)-permutes with BN/N. Hence in the case when $X \leq N$, AN/N permutes with BN/N.*

Lemma 2.2. *Let H and X be subgroups of G and N a normal subgroup of G. Suppose that H is X-quasipermutable (X_S-quasipermutable, respectively) in G.*

(1) If either H is a Hall subgroup of G or for every prime p dividing $|H|$ and for every Sylow p-subgroup H_p of H we have $H_p \nleq N$, then HN/N is (XN/N)-quasipermutable ($(XN/N)_S$-quasipermutable, respectively) in G/N.

(2) If H is S-quasipermutable in G, then H permutes with some Sylow p-subgroup of G for all primes p such that $(|H|, p) = 1$.

Proof. By hypothesis, there is a subgroup B of G such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (with all Sylow subgroups, respectively) L of B such that $(|H|, |L|) = 1$.

(1) It is clear that $G/N = N_{G/N}(HN/N)(BN/N)$. Let K/N be any subgroup (any Sylow p-subgroup, respectively) of BN/N such that $(|HN/N|, |K/N|) = 1$. Then $K = (K \cap B)N$. Let B_0 be a minimal supplement of $K \cap B \cap N$ to $K \cap B$. Then $K/N = (K \cap B)N/N = B_0(K \cap B \cap N)N/N = B_0N/N$ and $K \cap B \cap N \cap B_0 \leq N \cap B_0 \leq \Phi(B_0)$. Therefore $\pi(K/N) = \pi(B_0)$, so $(|HN/N|, |B_0|) = 1$. It follows that $(|H|, |B_0|) = 1$, so in the case when H is X-qiasipermutable in G, H X-permutes with B_0 and hence HN/N (XN/N)-permutes with $K/N = B_0N/N$. Thus HN/N is (XN/N)-qiasipermutable in G/N.

Finally, suppose that H is X_S-qiasipermutable in G and K/N is a Sylow p-subgroup of BN/N. Then B_0 is a p-group, so $(|H|, p) = 1$ and for some Sylow p-subgroup B_p of B we have $B_0 \leq B_p$. Then $K/N = B_0N/N$ and hence HN/N (XN/N)-permutes with K/N. Thus HN/N is $(XN/N)_S$-qiasipermutable in G/N.

(2) By [6, VI, 4.6], there are Sylow p-subgroups P_1, P_2 and P of $N_G(H)$, B and G, respectively, such that $P = P_1P_2$. Hence H permutes with P. $\square$

Lemma 2.3. *(See Theorem E in [8]) Suppose that $G = AB$ and $P \leq O_p(A)$. Assume that every conjugate of P in A permutes with every Sylow q-subgroup of B for all primes $q \neq p$. Then P^G is soluble and the p-complements in P^G are nilpotent.*

Lemma 2.4. *(See Lemma 2.15 in [3]) Let E be a normal non-identity quasinilpotent subgroup of G. If $\Phi(G) \cap E = 1$, then E is the direct product of some minimal normal subgroups of G.*

Lemma 2.5. *Let H be a subnormal subgroup of G. If H is nilpotent, soluble, or a π-group, then H^G is nilpotent, soluble, a π-group, respectively.*

Proof. See the proof of Theorem 2.2 in [7, Ch. 2]. □

Lemma 2.6. *(O. Kegel [9]) If G has three nilpotent subgroups A_1, A_2 and A_3 whose indices $|G : A_1|$, $|G : A_2|$, $|G : A_3|$ are pairwise coprime, then G is itself nilpotent.*

We shall need in our proofs the following properties of p-supersoluble groups.

Lemma 2.7. *(1) If $G/\Phi(G)$ is p-supersoluble, then G is p-supersoluble [6, IV, 8.6].*

(2) Let N and R be distinct minimal normal subgroups of G. If G/N and G/R are p-supersoluble, then G is p-supersoluble.

(3) Let $A = G/O_{p'}(G)$. Then G is p-supersoluble if and only if $A/O_p(A)$ is an abelian group of exponent dividing $p - 1$, p is the largest prime dividing $|A|$ and $F(A) = O_p(A)$ is a normal Sylow subgroup of A.

Proof. (2) This follows from the G-isomorphism $NR/N \simeq R$.

(3) Since G is p-supersoluble if and only if $G/O_{p'}(G)$ is p-supersoluble, we may assume without loss of generality that $O_{p'}(G) = 1$.

First assume that G is p-supersoluble. In this case $G/C_G(H/K)$ is an abelian group of exponent dividing $p-1$ for any chief factor H/K of G of order divisible by p. On the other hand,

$$O_{p',p}(G) = O_p(G) = \cap\{C_G(H/K) \mid H/K \text{ is a chief factor of } G \text{ and } p \in \pi(H/K)\}$$

by [1, A, 13.2]. Hence $G/O_p(G)$ is an abelian group of exponent dividing $p-1$. Thus p is the largest prime dividing $|G|$ and $F(G) = O_p(G)$ is a normal Sylow p-subgroup of G.

Finally, if $G/O_p(G)$ is an abelian group of exponent dividing $p - 1$, then any chief factor H/K of G below $O_p(G)$ is cyclic by [1, B, 9.8(d)]. Hence G is supersoluble. □

Lemma 2.8. *Let $G = P \rtimes E$, where P is the Sylow p-subgroup of G and E is a Sylow tower group. Suppose that for every Sylow subgroup Q of E there is a subgroup B of P such that $P = N_P(Q)B$ and Q permutes with all subgroups of B. Then G is p-supersoluble.*

Proof. Suppose that this lemma is false and let G be a counterexample of minimal order. It is clear that G is soluble and $|P| > p$. Let $p_1 > \cdots > p_t$ be the set of all prime divisors of $|E|$. Let P_i be a Sylow p_i-subgroup of E.

Let N be a normal subgroup of G. Then the hypothesis holds for G/N, so the choice of G and Lemma 2.7 imply that N is the only minimal normal subgroup of G and $N \nleq \Phi(G)$. Therefore $N = C_G(N) = F(G) = P$ by [1, A, 15.2], so E is a maximal subgroup of G.

Assume that $|\pi(E)| > 2$. Then $t > 2$. Let E_i be a Hall p_i'-subgroup of E. Then the hypothesis holds for PE_i, so PE_i is p-supersoluble by the choice of G. Moreover, since $P = C_G(P)$ we have $O_{p'}(PE_i) = 1$. Therefore PE_i is supersoluble by Lemma 2.7(3), and $F(PE_i) = P$. Thus $PE_i/P \simeq E_i$ is an abelian group of exponent dividing $p-1$. Therefore E has at least three abelian subgroups E_i, E_j and E_k of exponent dividing $p-1$ whose indices $|E : E_i|$, $|E : E_j|$, $|E : E_k|$ are pairwise coprime. But then by Lemma 2.6, E is nilpotent, and every Sylow subgroup of E is an abelian group of exponent dividing $p-1$. Hence E is an abelian group of exponent dividing $p-1$, which implies that $|P| = p$. This contradiction shows that $|\pi(E)| = 2$.

Since E is a Sylow tower group, P_1 is normal in E, so $N_G(P_1) \cap P = 1$. Therefore P_1 permutes with all subgroups of P. If $P \leq N_G(P_2)$, then $PP_2 = P \times P_2$. Hence in this case $P_2 \leq C_G(P) = P$. This contradiction shows that $N_G(P_2) \cap P \neq P$, so there is a non-identity subgroup $B < P$ such that $P_2B = BP_2$. Hence $BE = B(P_1P_2) = (P_1P_2)B = BE$ is a subgroup of G, which contradicts the maximality of $E = P_1P_2$. □

3. **Proofs of the results**

Proof of Theorem 1.3. Suppose that this is false and let G be a counterexample of minimal order. Let $D = P^G$. Then, in view of Proposition 3.1 in [12], $X \neq 1$.

(I) This assertion is a corollary of Lemmas 2.1 and 2.3.

(II) By hypothesis, there is a subgroup B of G such that $G = N_G(P)$ and P X-permutes with B and with every subgroup S of B such that $(p, |S|) = 1$. Then for some $x \in X$ we have $PB^x = B^xP$. Hence $D = P^G = P^{N_G(P)B^x} = P^{B^x} \leq PB^x$, so $D = P(D \cap B^x)$.

(i) Suppose that this assertion is false.

(1) $O_p(D) = 1$, *so* $O_p(N) = 1$ *for any normal subgroup* N *of* G.

Indeed, suppose that $O_p(G) \neq 1$. By Lemma 2.2(1), the hypothesis holds for $G/O_p(G)$. Hence the choice of G implies that

$$P'O_p(G)/O_p(G) \leq (PO_p(G)/O_p(G))' \leq O_p(G/O_p(G)) = 1,$$

and if $N_G(P)$ is p-nilpotent, then

$$(G/O_p(G))' \cap (P/O_p(G)) = (G'O_p(G)/O_p(G)) \cap (P/O_p(G)) =$$
$$= O_p(G)(G' \cap P)/O_p(G) \leq O_p(G/O_p(G)) = 1.$$

Hence we have $P' \leq O_p(G)$ in the former case, and $G' \cap P \leq O_p(G)$ in the case when $N_G(P)$ is p-nilpotent. This contradiction shows that $O_p(G) = 1$, so $O_p(N) = 1$ since every subnormal p-subgroup of G is contained in $O_p(G)$ by Lemma 2.5.

(2) *P is not abelian.*

Suppose that P is abelian. Then in the case when $N_G(P)$ is p-nilpoten, $P \leq Z(N_G(P))$ and so G is p-nilpotent by Burnside's theorem [6, IV, 2.6]. Hence a Hall p'-subgroup E of G is normal in G. Since P is abelian, it follows that $G' \leq E$. Therefore $G' \cap P = 1 \leq O_p(G)$, contrary to our assumption on G. Hence we have (2).

(3) $\Phi(G) \cap X = 1$.

Suppose that $E = \Phi(G) \cap X \neq 1$. In view of Claims (1) and (2), $P' \nleq E$. On the other hand, the choice of G implies that $P'E/E \leq O/E = O_p(G/E)$. Hence p divides $|O|$. Let O_p be a Sylow p-subgroup of O. Then by the Frattini Argument, $G = ON_G(O_p) = (EO_p)N_G(O_p) = N_G(O_p)$ and so $1 < O_p \leq O_p(G)$, contrary to Claim (1).

(4) *X is a minimal normal subgroup of G, so $X \leq D$.*

Lemma 2.4 and Claim (3) imply that X is the direct product of some minimal normal subgroups $N, \dots, R$ of G. Assume that $N \neq X$ and let $V/N = O_p(G/N)$ and $W/R = O_p(G/R)$. Then $V \cap W \leq O_p(G)$, so Claim (1) implies that $V \cap W = 1$. On the other hand, $P'N/N \leq V/N$ and $P'R/R \leq W/R$ by the choice of G and so $1 < P' \leq V \cap W$, a contradiction. Hence $X = N \leq D$ by Assertion (I).

(5) $C_D(X) = X = F(D)$.

In view of Lemma 2.5 and Claim (4), $F(G) = X \leq F(D) \leq F(G)$, so $X = F(D)$ and hence $C_D(X) = X$ by [1, A, 10.6(a)].

(6) $X \nleq B$.

Assume that $X \leq B$ and let X be a q-group. Then by Claim (4), P permutes with all subgroups of X. Hence XP is supersoluble by Lemma 2.8 and $F(XP) = X$ by Claim (5). Hence $P \simeq XP/F(XP)$ is abelian, contrary to Claim (2).

Final contradiction for (i). Since $x \in X \leq D$ by Claim (4), $D = P(D \cap B^x) = P(D \cap B)$. Therefore the hypothesis holds for D, so in the case when $D < G$, $P' \leq O_p(D) = 1$ by Claim (1) and hence P is abelian, contrary to Claim (2). Therefore $D = G = PB$ and so $X \leq B$, contrary to Claim (6).

(ii) Since by Assertion (i), $P' \leq O_p(G)$, the Sylow p-subgroups of $G/O_p(G)$ are abelian. Hence, by [6, VI, 6.6], we have $l_p(G/O_p(G)) \leq 1$. But then $l_p(G) \leq 2$.

(iii) Assume that this assertion is false.

(a) *NP is normal in G for every non-identity normal subgroup N of G, so $O_p(G) = 1$ and X is a p'-group* (Since the hypothesis holds for G/N, this follows from the choice of G and the fact that $O_p(G) \leq P$).

(b) $\Phi(G)\cap X = 1$ (In view of the Frattini Argument, this follows from Claim (a)).

(c) *X is a minimal normal subgroup of G. Hence for some prime $q \neq p$ we have* $X \leq O_q(G) \cap D$ (see Claim (4) in the proof of (i) and use Claim (a)).

(d) *$X \nleq N_G(P)$, so $q < p$.*

Suppose that $X \leq N_G(P)$. Then, in view of Claims (a) and (c), $XP = X \times P$ is normal in G, which implies the normality of P. Thus $X \nleq N_G(P)$ and so $q < p$ by hypothesis.

(e) *$X \nleq B$.*

Suppose that $X \leq B$. Then Claim (c) and Lemma 2.8 imply that XP is supersoluble. But, in view of Lemma 2.7(3), this contradicts to Claim (d).

(f) *$G \neq PB$ and $D = P(D \cap B) \neq G$.*

In view of Claims (c) and (e), we have $G \neq PB$ and $x \in D$, so $D = P(D \cap B^x) = P(D \cap B)^x = P(D \cap B) \neq G$.

Final contradiction for (iii). By Claim (c), $X \leq D$. Hence, by Claim (f), the hypothesis holds for $D = P(D \cap B)$ since $|D : N_D(P)| = |N_G(P)D : N_G(P)|$. Hence in view of Claim (f), the choice of G implies that P is normal in D and so P is normal in G.

(iv) Let N be a minimal normal subgroup of G. Then the hypothesis holds for G/N, so G/N is q-supersoluble by the choice of G. Hence N is the unique minimal normal subgroup of G, $N \nleq O_{q'}(G)$, and $N \nleq \Phi(G)$ by Lemma 2.7. In particular, $N \nleq P$, which implies that N is a p'-group since G is p-soluble by Assertion (I).

Let E be a Hall p'-subgroup of G. Then $N \leq E$, so N is a q-group since E is q-supersoluble by hypothesis. Hence $N = C_G(N) = O_q(G) = X$ by [1, A, 15.2], and $X \leq D$. Therefore $D = P(D \cap B)^x = P(D \cap B)$, so $X \leq B$.

Since E is q-supersoluble, N has a maximal subgroup V such that V is normal in E. On the other hand, $PV \cap N = V$ is normal in PV. Hence $G = PE \leq N_G(V)$, which in view of the minimality of N implies that $V = 1$. Hence $|N| = q$, so G is q-supersoluble. This contradiction completes the proof of Assertion (iv).

The theorem is proved.

Proof of Theorem 1.7. (i) Suppose that this assertion is false and let G be a counterexample of minimal order.

Let $V \in \mathfrak{M}_\phi(P)$ and $D = V^G$. By hypothesis, there is a subgroup B of G such that $G = N_G(V)B$ and V is X-permutable with B and with all Sylow subgroups S of B such that $(p, |S|) = 1$.

(1) *$O_{p'}(N) = 1$ for every subnormal subgroup N of G. Hence $X \leq O_p(G)$.*

Indeed, suppose that for some subnormal subgroup N of G we have $O_{p'}(N) \neq 1$. Then $O_{p'}(G) \neq 1$ by Lemma 2.5, and the hypothesis holds for $G/O_{p'}(G)$ by Lemma 2.2(1). Hence $G/O_{p'}(N)$ is p-supersoluble by the choice of G. Thus G

is p-supersoluble, a contradiction. Therefore $O_{p'}(N) = 1$. Therefore, since X is nilpotent, $X \leq O_p(G)$.

(2) *If L is a minimal normal subgroup of G, then $L \nleq \Phi(P)$.*

Indeed, in the case when $L \leq \Phi(P)$ we have $L \leq \Phi(G)$ and the hypothesis holds for G/L by Lemma 2.2(1). Hence G/L is p-supersoluble by the choice of L. Therefore G is p-supersoluble by Lemma 2.7(1), a contradiction.

(3) *D is soluble, so $O_p(G) \neq 1$.*

Assume that $O_p(G) = 1$. Then in view of Claim (1), $X = 1$. Therefore V permutes with B and with all Sylow subgroups S of B such that $(p, |S|) = 1$. Therefore $D = V^G = V^{N_G(V)B} = V^B \leq VB$, so $D = V(D \cap B)$. Hence V^D is soluble by Lemma 2.3. But Claim (1) implies that $O_{p'}(V^D) = 1$. Hence $O_p(V^D) \neq 1$, and $O_p(V^D) \leq O_p(G)$ by Lemma 2.5. Thus $O_p(G) \neq 1$, a contradiction.

(4) *P is not cyclic.*

Assume that P is cyclic. Claim (3) implies that for some minimal normal subgroup L of G we have $L \leq O_p(G) \leq P$. Then $|L| = p$, and since $L \nleq \Phi(P)$ by Claim (2), we get $L = P$, contrary to the hypothesis.

(5) *Every normal p-soluble subgroup of G is supersoluble and p-closed* (See Claim (5)(a) in the proof of Proposition in [12]).

(6) *G is not p-soluble* (This directly follows from Claim (5)).

Final contradiction for (i). In view of Claim (4), there is a subgroup $W \in \mathfrak{M}_\phi(P)$ such that $V \neq W$. Then $P = VW$. In view of Claims (3) and (6), $P \nleq D$. Hence V is a Sylow subgroup of D, so V is normal in D (and also in G) by Claim (5). Similarly, W is normal in G. Hence P is normal in G, contrary to Claim (6). This final contradiction completes the proof of Assertion (i).

(ii) If $|P| = p$, then G is p-nilpotent by [6, IV, 5.4]. Let $|P| > p$ and H/K any chief factor of G of order divisible by p. Then $|H/K| = p$ by Assertion (i), so $C_G(H/K) = G$ since $(p-1, |G|) = 1$. Hence G is p-nilpotent.

Proof of Theorem 1.11. Suppose that this theorem is false and let G be a counterexample of minimal order. Let p be the largest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then P is normal in G by Theorem 1.3(II)(iii).

Let N be a minimal normal subgroup of G. Then the hypothesis holds for G/N by Lemma 2.2(1). Hence G/N is supersoluble by the choice of G. Moreover, the choice of G and Lemma 2.7 imply that N is the only minimal normal subgroup of G, $N \leq P$ and $N \nleq \Phi(G)$. Hence $G = N \rtimes M$ for some maximal subgroup M of G, $|N| > p$ and $N = C_G(N) = P$ by [1, A, 15.2]. Let Q be any Sylow subgroup of M. Then Q is a Sylow subgroup of G and so, by hypothesis, there is a subgroup B of G such that $G = N_G(Q)B$ and Q X-permutes with every subgroup L of B such that $(q, |L|) = 1$. It is clear that $P = (P \cap N_G(Q))(P \cap B) = N_P(Q)(P \cap B)$, and Q permutes with every subgroup of $P \cap B$ since $X = N = P$. Therefore G is p-supersoluble by Lemma

2.8, which implies that $|N| = p$. This contradiction completes the proof of the result.

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