ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 2, pp. 427–434

Title:

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Author(s):

N. Ashrafi and N. Pouyan

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 2, pp. 427–434
Online ISSN: 1735-8515

THE UNIT SUM NUMBER OF BAER RINGS

N. ASHRAFI* AND N. POUYAN

(Communicated by Omid Ali S. Karamzadeh)

Abstract. In this paper we prove that each element of any regular Baer ring is a sum of two units if no factor ring of R is isomorphic to $\mathbb{Z}_2$ and we characterize regular Baer rings with unit sum numbers ω and ∞. Then as an application, we discuss the unit sum number of some classes of group rings.

Keywords: unit sum number, regular Baer ring, π-regular Baer ring, right perpetual ideal.
MSC(2010): Primary:16U60; Secondary: 16D10, 16S50, 16S34.

1. Introduction

The study of rings generated additively by their units started in 1953-1954 (See, [21–23]). An associative unital ring R is said to have the *n-good*, for a positive integer n, if its every element can be written as a sum of exactly n units of R. The *unit sum number* of a ring R, denoted by $usn(R)$, is the least integer n, if any such integer exists, such that R is n-good. If R has an element that is not a sum of units then we set $usn(R)$ to be ∞, and if every element of R is n-good for some n but R is not n-good for any n, then we set $usn(R) = \omega$. The unit sum number of a module M, denoted by $usn(M)$, is the unit sum number of its endomorphism ring. Recently some authors have been interested in this concept and they could get various results. For additional historical background the reader is referred to the papers [1–3, 6, 14] and [20]. From [11], a ring R is called *π-regular* if for each element $a \in R$ there exist a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^n x a^n$. A π-regular ring R for which the n in the above can be taken to be 1 is called regular.

A ring R is called Baer if the left annihilator of every nonempty subset of R is generated by an idempotent. The concept of a Baer ring was introduced by Kaplansky in order to abstract properties of rings of operators on a Hilbert

Article electronically published on April 30, 2016.
Received: 4 November 2013, Accepted: 10 February 2015.
*Corresponding author.

space in [12]. The definition of Baer is indeed left-right symmetric by [12]. The aim of this paper is to study the unit sum number of Baer rings.

Zelinsky [23] proved that every element in the ring of linear transformations of a vector space V over a division ring D is a sum of two units unless dim $V = 1$ and $D = \mathbb{Z}_2$. Because $End_D(V)$ is a (von-Neumann) regular ring, Zelinsky's result generated quite a bit of interest in regular rings that have the property that every element is a sum of (two) units. This result motivated Skornjakov to ask in [19, Problem 31, p. 167] if $usn(R) \leq \omega$ for a (von Neumann) regular ring R, but one needs to add some condition ensuring that $\mathbb{Z}_2$ is not a factor ring (for example, that $1/2 \in R$), to exclude the exceptional case already noted in the result of Zelinsky. A negative answer to Skornjakov question was given in 1977 by Bergman who constructed a regular algebra R over rationals with $usn(R) = \infty$, as reported by Handelman in [9]. But since $End_D(V)$ is a regular Baer ring [4, Example 1.26], another natural question which arises from Zelinsky's result is the following: *Which regular Baer rings have the property that every element is a sum of two units?*
In this paper, we answer to this question and prove that every element of a regular Baer ring is a sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$. Also we characterize regular Baer rings with unit sum number ω and ∞.

In 2006 Khurana and Srivastava in [13] proved that every element of a right self-injective ring is a sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$. Since each self-injective ring is a Baer ring, some results which obtained by Khurana and Srivastava in [13] would be an application of Theorem 5. Further in section 3 as an another application, we determine the unit sum number of some classes of group rings.

All rings in this paper will have identity element. For a ring R, $J(R)$ will denote the Jacobson radical of R and for integer $n > 1$, we will denote by $M_n(R)$ the ring of $n \times n$ matrices over R. We use $|X|$ and c to denote the cardinality of a set X and the cardinality of the continuum, respectively.

2. The unit sum number of regular Baer rings

Before discussing the main results we need some properties of the unit sum number of rings and modules.

Lemma 2.1. *Let R be a ring, I an ideal of R, $\eta : R \to R/I$ the natural surjection. Then the following statements hold:*
(a) if $a \in R$ is n-good then so is $\eta(a) \in R/I$, and the converse also holds if $I \subseteq J(R)$;
(b) $usn(R/I) \leq usn(R)$ with equality if $I \subseteq J(R)$;
(c) if the ring R is a finite product of the rings $R_1, \ldots, R_n$ then $usn(R) \geq max\{usn(R_1), \ldots, usn(R_n)\}$ with equality holding if the right hand side is either finite or ∞;

(d) *Let D be a division ring. If $|D| \geq 3$ then $usn(D) = 2$, whereas if $|D| = 2$, that is , $D = \mathbb{Z}_2$ the field of two elements, then $usn(\mathbb{Z}_2) = \omega$.*

Proof. See [20, Lemma 2] and [3, F_1]. □

Theorem 2.2. *A ring R is a Baer ring if and only if R itself, regarded as a regular R-module, is a Baer semisimple module.*

Proof. See [8, Theorem 4]. □

For getting the main result of this paper we need some definitions from [8] which we bring them as follows.

Definition 2.3. Let M be a right R-module. Then, we call a (right R-)submodule N of M a *perpetual submodule* of M if for all $x \in N$ and $y \in M, ann_l(x) \subseteq ann_l(y)$ implies $y \in N$.

Definition 2.4. let M be a right R-module. Then
(1) M is called a *Baer simple R-module* if $M \neq 0$, and M contains no perpetual submodules of M other than M itself and (0).
(2) M is called a *Baer semismiple R-module* if every perpetual submodule of M is a direct summand of M.

Theorem 2.5. *Let R be a regular Baer ring then the following conditions are equivalent:*

(1) *Every element of R is a sum of two units.*
(2) *Identity of R is a sum of two units.*
(3) *R has no factor ring isomorphic to $\mathbb{Z}_2$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are obvious.
Next, we show (3) ⇒ (1). Suppose that no factor ring of R is isomorphic to $\mathbb{Z}_2$. Now we show that each element of R is a sum of two units. By the previous theorem R itself, regarded as a regular (left) right R-module, is a regular Baer semisimple module; therefore, by [8, proposition 2] R is the direct sum of a family of Baer simple submodules. This family is not empty. We have $R_R = \bigoplus_{i=1}^{n} M_i$ while the M_i are Baer simple R-submodules of R. Let $R_R = \bigoplus_{j=1}^{r} M_{i_j}^{n_j}$, where $\{M_{i_1}, \ldots, M_{i_r}\}$ is a set of representatives of the isomorphism classes of M_i for $i = 1, \ldots, n$ such that $n_1 + n_2 + \ldots + n_r = n$. Then

$$R \cong End_R(R) \cong End_R(M_{i_1}^{n_1} \oplus \ldots \oplus M_{i_r}^{n_r})$$

$$\cong \begin{pmatrix} Hom(M_{i_1}^{n_1}, M_{i_1}^{n_1}) & Hom(M_{i_1}^{n_1}, M_{i_2}^{n_2}) & \ldots & Hom(M_{i_1}^{n_1}, M_{i_r}^{n_r}) \\ Hom(M_{i_2}^{n_2}, M_{i_1}^{n_1}) & Hom(M_{i_2}^{n_2}, M_{i_2}^{n_2}) & \ldots & Hom(M_{i_2}^{n_2}, M_{i_r}^{n_r}) \\ \vdots & \vdots & \vdots & \\ Hom(M_{i_r}^{n_r}, M_{i_1}^{n_1}) & Hom(M_{i_r}^{n_r}, M_{i_2}^{n_2}) & \ldots & Hom(M_{i_r}^{n_r}, M_{i_r}^{n_r}) \end{pmatrix}$$

Now by this fact that $M_{i_l} \ncong M_{i_{\acute{l}}}$ for $l \neq \acute{l}$ and regularity of $Hom(M_{i_l}, M_{i_{\acute{l}}})$ we have $Hom(M_{i_l}, M_{i_{\acute{l}}}) = 0$; therefore, $Hom(M_{i_l}^{n_l}, M_{i_{\acute{l}}}^{n_{\acute{l}}}) = 0$. So

$$R \cong \begin{pmatrix} Hom(M_{i_1}^{n_1}, M_{i_1}^{n_1}) & 0 & \dots & 0 \\ 0 & Hom(M_{i_2}^{n_2}, M_{i_2}^{n_2}) & \dots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & Hom(M_{i_r}^{n_r}, M_{i_r}^{n_r}) \end{pmatrix}.$$

Thus $R \cong \prod_{j=1}^{r} End_R(M_{i_j}^{n_j}) \cong \prod_{j=1}^{r} M_{n_j}(End_R(M_{i_j}))$. As M_{i_j} is a Baer simple $R-$module for each $1 \leq j \leq r$, so $End_R(M_{i_j})$ is a domain by [8, Theorem 2]. On the other hand $D_j := End_R(M_{i_j})$ is a regular domain, thus is a division ring. Since R has no factor ring isomorphic to $\mathbb{Z}_2$, each element of $M_{n_j}(D_j)$ for all $1 \leq j \leq r$ is a sum of two units. Therefore, the unit sum number of R is 2. □

Recall that if V is a right vector space over a division ring D, then $End_D(V)$ is a regular Baer ring. It is easy to see that the identity of $End_D(V)$ is a sum of two units, except when $dim(V_D) = 1$ and $D = \mathbb{Z}_2$. As a consequence, we get the following result.

Corollary 2.6. (*Zelinsky, [23]*). *Every element of $End_D(V)$ is a sum of two units, except when $dim(V_D) = 1$ and $D = \mathbb{Z}_2$.*

Remark 2.7. Let R be a regular ring and A its lattice of principal right ideals. If A is a complete lattice then R is a Baer ring [4, Corollary 1.22]. Therefore, the unit sum umber of R is 2 if it has no factor isomorph to $\mathbb{Z}_2$.

Now with the following theorem we can determine when the unit sum number of a regular Baer ring can be 2, ω or ∞.

Theorem 2.8. *The unit sum number of a nonzero regular Baer ring R is 2 , ω or ∞. Moreover,*

(1) *$usn(R) = 2$ if and only if R has no factor ring isomorphic to a nonzero Boolean ring.*
(2) *$usn(R) \geq \omega$ if R has a factor ring isomorphic to $\mathbb{Z}_2$. Further if R has a factor ring isomorphic to a nonzero Boolean ring with more than two elements then $usn(R) = \infty$.*

Proof. (1) Since $\mathbb{Z}_2$ is a homomorphic image of every Boolean ring then by Theorem 2.5, it is obvious.

(2) Let R have a factor ring isomorphic to $\mathbb{Z}_2$. In this case $R/I \cong \mathbb{Z}_2$, but as $usn(R/I) \leq usn(R)$, then $usn(R) \geq \omega$. Now if R has a factor isomorphic to a nonzero Boolean ring with more than two elements, then R has a factor ring isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$; therefore, $usn(R) = \infty$.

□

Since every π-regular Baer ring with $|Id(R)| < c$ is a semilocal ring [16, Theorem 5], by the next proposition we can obtain the unit sum number of π-regular Baer rings with $|Id(R)| < c$.

Proposition 2.9. *Let R be a semilocal ring. If R has no factor ring isomorphic to $\mathbb{Z}_2$ then every element of R is a sum of two units.*

Proof. If R is a semilocal ring thus $R/J(R)$ is a semisimple ring; therefore, $R \cong \prod_{l=1}^{r} M_{n_l}(D_l)$ while $M_{n_l}(D_l)$ admits a diagonal reduction i.e., there exist invertible matrices P and Q in $M_{n_l}(D_l)$ such that $PM_{n_l}(D_l)Q$ is a diagonal matrix. So there are invertible matrices U_1, U_2 such that $PM_{n_l}(D_l)Q = U_1 + U_2$. Therefore, $M_{n_l}(D_l) = P^{-1}U_1Q^{-1} + P^{-1}U_2Q^{-1}$. So each element of R can be written as a sum of two units. □

Corollary 2.10. *Let R be a π-regular Baer ring with $|Id(R)| < c$. If R has no factor ring isomorphic to $\mathbb{Z}_2$ then every element of R is a sum of two units.*

Henriksen proved in [10] that for any ring R and $n \geq 2$, $usn(M_n(R)) \leq 3$. Vámos in [20] showed that the unit sum number of a proper matrix ring over an elementary divisor ring is 2. Here we show that the unit sum number of a square matrix ring over a regular ring with finite Goldie dimension is, 2.

Proposition 2.11. *Let R be a regular ring with finite Goldie dimension then $usn(M_n(R)) = 2$. In particular, for finitely generated free R-module F, $usn(F) = 2$.*

Proof. Let $S = M_n(R)$. By [15], S is a regular Baer ring and by Theorem 2.5, $usn(M_n(R)) = 2$. Since $End_R(F) \cong M_n(R)$, for some nonzero integer n, $usn(F) = 2$. □

Recall that the unit sum number of a module is the unit sum number of its endomorphism ring. So the unit sum number of any module, whose endomorphism ring modulo jacobson radical is regular Baer ring, is 2, ω *or* ∞. We list two such classes of modules below.

Corollary 2.12. *Let M_R be an extending module such that its endomorphism ring S is a regular ring. Then unit sun number of M is 2, ω or ∞.*

Proof. If M be an extending module such that its endomorphism ring S is a regular ring then M is a Baer module, and subsequently S is a Baer ring [18, proposition 4.12]. Therefore, the result follows from Corollary 2.8. □

Srivastava and Khurana in [13] proved that every element of a right self-injective ring is a sum of two units if and only if it has no factor ring isomorphic to $\mathbb{Z}_2$ and they extend this result to endomorphism rings of right quasi-continuous modules with finite exchange property. As continuous modules is quasi-continuous modules with finite exchange property [17, Theorem 3.24], they proved that every element in the endomorphism ring of a continuous module, is a sum of two units if no factor of endomorphism ring is isomorphic to $\mathbb{Z}_2$. In this paper we obtain the same result but with a different proof and use the unit sum number of regular Baer rings. As a consequence we get the following result.

Corollary 2.13. *Let M_R be a continuous module. Then each element of endomorphism of M_R is a sum of two units if and only if has no factor isomorphic to $\mathbb{Z}_2$.*

Proof. Let $S = End_R(M)$. If M is a continuous module, by [17, Theorem 3.11 and proposition 3.5], $\overline{S} = S/J(S)$ is a regular right continuous ring, thus $\overline{S}_{\overline{S}}$ is a extending module with regular endomorphism ring. Therefore, by corollary 2.12, $usn(\overline{S}) = 2$ so $usn(M) = 2$ if and only if S has no factor ring isomorphic to $\mathbb{Z}_2$. ☐

3. The unit sum number of some classes of group rings

In [22], Z. Yi and Q. Y. Zhou studied Baer properties of group rings. In this section we discuss the Baer property of group rings and we conclude by showing an application of our result for group rings.

Theorem 3.1. *If $G = H \times K$ is a locally finite group and RG is a Baer group ring, then $usn(RH) = usn(RK) = 2$ if R has no factor ring isomorphic to $\mathbb{Z}_2$.*

Proof. Note that $RG = R(H \times K) \cong (RH)K$. Since $RG = \oplus_{g\in G} Rg$ is a free left R-module with a basis G satisfying the assumption of [22, Theorem 2.1], RH is regular Baer. Similarly RK is regular Baer. Since R has no factor ring isomorphic to $\mathbb{Z}_2$, so $RH(RK)$ has no factor isomorphic to $\mathbb{Z}_2$. Therefore by Theorem 2.5, $usn(RH) = usn(RK) = 2$. ☐

Theorem 3.2. *Let G be a locally finite group. If RH is a regular Baer group ring for every proper subgroup H of G, then RG is regular Baer group ring. Particularly, $usn(RG) = 2$ if R has no factor ring isomorphic to $\mathbb{Z}_2$.*

Proof. Let X be a arbitrary subset of RG. For each $x_\alpha \in X$, let $x_\alpha = \Sigma_{i=1}^{n} a_{\alpha_i} g_{\alpha_i}$ and $H =< \ldots, g_{\alpha_1}, \ldots, g_{\alpha_n}, \ldots >$. By the assumption, RH is Baer. Since $x_\alpha \in RH$, there exist $e \in RH$ such that $l_{RH}(x_\alpha) = RHe$. Since $ex_\alpha = 0$, we have $RGe \subseteq l_{RG}(x_\alpha)$. We next show that the other inclusions also hold.

Let $v \in l_{RG}(x_\alpha)$ and let $\{1, g_1^{'}, g_2^{'}, ...\}$ be a left coset representative of H in G. That is $G = H \cup g_1^{'}H \cup g_2^{'}H \cup \ldots$. Now v can be written as $v = \Sigma g_i^{'} b_i$, where $b_i \in RH$. Since $0 = vx_\alpha = \Sigma g_i^{'}(b_i x_\alpha)$, we obtain that $b_i u x_\alpha = 0$ for all i. So $b_i \in l_{RH}(x_\alpha) = RHe$, and thus $b_i = c_i e$ for some $c_i \in RH$. It follows that $v = \sum g_i^{'} b_i = \sum (g_i^{'} c_i)e \in RGe$. So $l_{RG}(x_\alpha) \subseteq RGe$, and thus $l_{RG}(x_\alpha) = RGe$. Thus RG is a Baer group ring. On the other hand by [5, Theorem 3], RG is a regular group ring. Note that if R has no factor ring isomorphic to $\mathbb{Z}_2$ then the group ring RG also has no factor ring isomorphic to $\mathbb{Z}_2$; therefore, by Theorem 2.5, $usn(RG) = 2$. □

Lemma 3.3. *Let R be a regular Baer ring. If $2^{-1} \in R$ then $usn(RC_2) = 2$ if and only if $usn(R) = 2$.*

Proof. Write $C_2 = \{1, g\}$. Since $2^{-1} \in R$, then $RC_2 \cong R \times R$. From this fact and $usn(RG) \geq uns(R)$ is obvious. □

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(Nahid Ashrafi) Faculty of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran.

E-mail address: nashrafi@semnan.ac.ir & ashrafi49@yahoo.com

(Neda Pouyan) Faculty of Mathematics, Statistics and Computer Science, Semnan University, Semnan, Iran.

E-mail address: n_pouyan@sun.semnan.ac.ir & neda.pouyan@gmail.com