Bulletin of the Iranian Mathematical Society Vol. 33 *No. 1* (*2007*), *pp 1-10.*

FINITE GROUPS WITH p-SYLOW COVERINGS

K. MEHRABADI AND A. IRANMANESH*

Communicated by Jamshid Moori

Abstract. In this paper we characterize the finite groups with an irredundant covering containing some p-Sylow subgroups. In particular we analyze the symmetric and alternating groups, finding their p-elements having a p-subgroup as centralizer.

1. Introduction

A group G is said to have a covering by subgroups if G is the set-theoretic union of its proper subgroups. These subgroups are called components of the covering. The covering is called irredundant if each proper sub-collection of those subgroups fails to cover G. We will always assume that coverings are irredundant. Covering aspects of groups have been studied by many authors from several different perspectives [2, 3, 9, 13]. A more specific question is covering a group by some special subgroups. For example in [4] the coverings of infinte groups consisting of normal subgroups are investigated and in [8] the authors studied the groups having Hall coverings. See also [5] and [6] for components consisting of conjugates of two subgroups. Here we study the finite groups with components containing Sylow subgroups. Let G be a finite group and p a prime, by *a p-Sylow covering of G* we mean an irredundant

MSC(2000): Primary 20D20, 20D50; Secondary 20B30, 20B35, 20D06, 20E45
Keywords: Covering, Sylow subgroups, Symmetric group, Alternating group
Received: 15 February 2007 , Revised: 25 April 2007
∗Corresponding author

covering of G containing some p-Sylow subgroups. Some examples are in order.

Example 1.1. • In the Symmetric group S_3, the family $\{A_3, \langle(1\ 2)\rangle, \langle(1\ 3)\rangle, \langle(2\ 3)\rangle\}$ is a covering of S_3 whose components are Sylow subgroups of S_3.
• Let $G = S_3 \times C_2$. Then G has no 3-Sylow covering, but there is a 2-Sylow covering. Of course this covering consists of all Sylow 2-subgroups of G and the subgroup $A_3 \times C_2$.
• Let $G = C_2 \times C_2 \times C_3$. There is no 2 or 3-Sylow covering of G.

These examples raise the following questions:

(1) Could one cover a given group by its Sylow subgroups?
(2) Let G be a group and p a prime dividing $|G|$. When there exists a p-Sylow covering?
(3) For a given group G, is there a p-Sylow covering for any prime number p dividing $|G|$?

The first problem has been studied independently by Higman ([7]) and Zacher ([14], [15]) when G is soluble, by Suzuki ([11]) in the case of a simple group and by Brandl ([1]) in general. A missing case of the last paper was studied by Jabara and Lucido ([8]). In section 2, we provide a necessary and sufficient condition on G for affirmative answers to the next questions.
If G is a finite group and p a prime, a p-element is an element whose order is a power of p. A p-element in G is called a C_{pp}-*element* if its centralizer in G is a p-subgroup. Note that $a \in G$ is a C_{pp}-element if and only if every conjugate of a is a C_{pp}-element. In section 3, we use the results of section 2 to characterize all symmetric and alternating groups with p-Sylow coverings. Furthermore we find all C_{pp}-elements in these groups. Let τ be a permutation in S_n or A_n; when we use the statements *a cycle in* τ or *a permutation in* τ, we mean a product of some disjoint cycles which appear in the decomposition of τ into the product of disjoint cycles. Supp(τ) is the set of point in $\{1, 2, ..., n\}$ such that τ moves them and Fix(τ) $= \{1, 2, ..., n\} - $Supp($\tau$). All unexplained notation is standard (see [12]). It is clear that we have to restrict ourselves to the groups with a proper Sylow subgroup, therefore the groups in this paper are not p-groups.

2. **Coverings containing Sylow subgroups**

We begin this section by the following theorem providing an answer to the question 2 raised in the previous section.

Theorem 2.1. *Let G be a non-cyclic finite group, and let p be a prime dividing $|G|$. Then G has a p-Sylow covering if and only if there exists a Cpp-element in G.*

Proof. Suppose that G has a covering containing Sylow p-subgroup P. If each $a \in P$ is not a C_{pp}-element, then there exists an element $x \in C_G(a)$ such that $o(x) = q$, where q is a prime distinct from p. Therefore ax belongs to a component of the covering which is not a Sylow p-subgroup. So $(ax)^q = a^q$ belongs to this component and consequently a does. Therefore each element of P belongs to another component of the covering, against the irredundancy of the covering.

Conversely, let $a \in G$ be a C_{pp}-element. Suppose by contradiction that G has no p-Sylow covering. Let A_1 be a Sylow p-subgroup of G containing a. Now we make a covering as follows: First consider $\{A_1\}$ and then add other Sylow p-subgroups to this set one by one until they cover all the p-elements of G. Then for any other prime q with q dividing $|G|$, do the same procedure. By this way we obtain a set like $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ which is an irredundant covering for the elements of G whose orders have at most one prime divisor. Define $\mathcal{B} = \{\langle g\rangle \mid g \in G \setminus \bigcup_{i=1}^{n} A_i\}$. Order $\mathcal{B}$ by inclusion and add the maximal member of each chain in $\mathcal{B}$ to $\mathcal{A}$. Now $\mathcal{A}$ is a covering for G. But we supposed that G has no p-Sylow covering, therefore all Sylow p-subgroups of G can be omitted from $\mathcal{A}$. In particular A_1 is a subset of union of some other members in $\mathcal{A}$. But then the union of some elements of $\mathcal{B}$ includes A_1. Thus there exists a subgroup $B \in \mathcal{B}$ such that $a \in B$. Now by definition of elements of $\mathcal{B}$, we have $B \subseteq C_G(a)$, against the fact that B is not a p-group. □

It is clear that whenever we mention a p-Sylow covering for a given group G, it is not important which Sylow p-subgroup of G is chosen. The following corollary of Theorem 2.1 gives an answer to the third question:

Corollary 2.2. *Let G be a non-cyclic finite group. Then G has a p-Sylow covering for each prime p dividing $|G|$ if and only if for each p, there exists a C_{pp}-element in G.*

3. The symmetric and alternating groups

Let n be a positive integer and let p be a prime. Suppose that α_1 is the largest positive integer with $n \geq p^{\alpha_1}$ and α_2 is the largest positive integer with $n - p^{\alpha_1} \geq p^{\alpha_2}$ and so on. Thus we have $n = p^{\alpha_1} + p^{\alpha_2} + \cdots + p^{\alpha_t} + r$, where t and r are positive integer, $0 \leq r \leq p-1$ and $\alpha_1 \geq \alpha_2 \geq ...\alpha_t \geq 0$. We call this representation the *p-representation of n*. Observe that in any p-representation the maximum numbers of equal exponents α_i is $p-1$. Moreover collecting the p-powers with the same exponent we pass from the p-representation of n to the unique representation $n_{(p)}$ of n to the basis p. Now let $\tau = \tau_1\tau_2\cdots\tau_t \in S_n$, where $\tau_1 = (a_1\ a_2\ \cdots\ a_{p^{\alpha_1}})$, $\tau_2 = (a_{p^{\alpha_1}+1}\ a_{p^{\alpha_1}+2}\ \cdots\ a_{p^{\alpha_1}+p^{\alpha_2}})$, ... $\tau_t = (a_{p^{\alpha_1}+\cdots+p^{\alpha_{t-1}}+1}\ \cdots\ a_{p^{\alpha_1}+\cdots+p^{\alpha_t}})$, such that a_i's are in $\{1, 2, \cdots, n\}$ and pairwise distinct. If a permutation in S_n has the above form we say it has the *p-form* in S_n.

Remark 3.1. Let n be a positive integer and let p be a prime. If a representation $n = p^{a_1} + p^{a_2} + \cdots + p^{a_k} + s$ of n is not the p-representation, then at least p integers among the a_i are equal or $s \geq p$.

The following lemma is a corollary in [12], p 297.

Lemma 3.2. *If τ is a permutation in S_n and decomposition of τ contains exactly a_i i-cycles, with $a_i \geq 0$, then $|C_{S_n}(\tau)| = \prod(a_i!)i^{a_i}$, where $n = a_1 + 2a_2 + ... + na_n$.*

Corollary 3.3. *For each $n \geq 3$, the symmetric group S_n has a C_{22}-element. Moreover $\tau \in S_n$ is a C_{22}-element if and only if τ is a product of 2^k-cycles, with $k \in \mathbb{N}$, such that at most two of them have the same length.*

Proof. By Lemma 3.2, each permutation with 2-form is a C_{22}-element in S_n, so for each n, S_n has a C_{22}-element. The second part follows directly from Lemma 3.2. □

Corollary 3.4. *For each $n \geq 3$, the symmetric group S_n has a 2-Sylow covering.*

Theorem 3.5. *Let G be the symmetric group S_n for $n \geq 3$ and let $p \leq n$ be an odd prime. Then G has a C_{pp}-element if and only if 0 and 1 are the only digits appearing in the representation of n to the basis p. Moreover in this situation, $\tau \in S_n$ is a C_{pp}-element if and only if τ has the p-form in S_n.*

Proof. Suppose G has a C_{pp}-element τ and τ consists of a_i i-cycles with $i = p^{b_i}$. By Lemma 3.2, $\prod_i (a_i!) i^{a_i}$ is a power of p and $a_i's$ are all equal to 1 or 0. Thus $n = p^{b_1} + p^{b_2} + ... + p^{b_k} + r$, where b_i's are pairwise distinct and $r = 0$ or 1. Hence the only digits appearing in the representation of n in basis p are 0 and 1.
Conversely suppose 0 and 1 are the only digits appearing in the representation of n to the basis p. Under this condition, by Lemma 3.2, each permutation with p-form is a C_{pp}-element. On the other hand if the p-element $\tau \in S_n$ has not the p-form, then there are at least p cycles with the same length. Thus Lemma 3.2 completes the proof. □

Corollary 3.6. *Let G be the symmetric group S_n for $n \geq 3$ and let p be an odd prime divisor of $|G|$. Then G has a p-Sylow covering if and only if 0 and 1 are the only digits which appear in the representation of n to the basis p.*

Lemma 3.7. *For any integer number $n \geq 7$, there exists a prime $p \geq 3$ such that $p + 4 \leq n < 3p$.*

Proof. We proceed by induction on n. If $n = 7$, then take $p = 3$. Now let $n > 7$ and p be a prime such that $p + 4 \leq n < 3p$. If $n + 1 < 3p$, then p satisfies the property for $n + 1$. So let $n + 1 = 3p$. Then by Bertrand theorem, there exists a prime q such that $p < q < 2p$. For such a q we have $q + 4 \leq 2p + 3 \leq 3p = n + 1 < 3q$. □

In the Example 1.1, we had a p-Sylow covering of S_3 for each prime p dividing $|G|$. Also we have $4_{(3)} = 11$, and therefore by Corollaries 3.4 and 3.6, S_4 has a p-Sylow covering for each prime p dividing $|G|$ too.

Theorem 3.8. *The groups S_3 and S_4 are the only symmetric groups which have a p-Sylow covering for each prime $p \leq n$.*

Proof. In the last paragraph we saw the result for $n = 3$ and 4. For $n \in \{5, 6, 7, 8\}$, 2 appears in $n_{(3)}$ hence S_n does not have the covering containing a 3-Sylow subgroup. So let $n \geq 9$. By Lemma 3.7, there exists a prime $p > 3$ such that $p + 4 \leq n < 3p$. Let P be a p-Sylow subgroup of S_n. Then $|P| \leq p^2$. If $|P| = p$, then $S_4 \leq C_{S_n}(P)$, hence an element of order 3 centralizes each element in P. So let $|P| = p^2$. If τ is an arbitrary element in P, then τ is a p-cycle or a product of two p-cycles. In both cases, 2 divides $|C_{S_n}(\tau)|$ and the result follows from Corollary 3.6. □

Now we are going to find similar results for the alternating groups.

Lemma 3.9. *Let $2^n + 2^{n-1} + \cdots + 2^2 + 2 = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_s}$, where $a_i \geq 1$. If these two representations are not the same, then at least three of a_i's are equal.*

proof. By induction on n. □

Theorem 3.10. *For $n \geq 3$, the alternating group A_n has a C_{22}-element if and only if n does not have the form $2^{2k} - 1$, for any $k \in \mathbb{N}$. Moreover in this situation, $\tau \in A_n$ is a C_{22}-element if and only if τ is a C_{22}-element in S_n.*

Proof. First suppose that $n = 2^{2k} - 1$, where k is a positive integer. Hence $n - 1 = 2^m + 2^{m-1} + ... + 2$, where m is an odd number. Let τ be a 2-element in A_n. We show that τ is not a C_{22}-element. If $|\mathrm{Fix}(\tau)| > 2$, then τ commutes with a cycle of length 3. If $|\mathrm{Fix}(\tau)| \leq 2$, then since n is an odd number, we get $|\mathrm{Fix}(\tau)| = 1$. Let $\tau = \tau_1 \cdots \tau_s$ be the product of disjoint cycles τ_i of length 2^{a_i}. Then $2^{a_1} + 2^{a_2} + \cdots + 2^{a_s} = n - 1 = 2^m + 2^{m-1} + \cdots + 2$. These two representations of $n - 1$ are not the same, since s is even and m is odd. Hence by Lemma 3.9, three $a_i's$ are equal. Without loss of generality we may assume that τ_1, τ_2 and τ_3 are three disjoint cycles in τ with equal length. Now there exists a $\pi \in S_n$ such that $\pi^3 = \tau_1\tau_2\tau_3$ with $\pi \in C_{S_n}(\tau)$ and by defining $\rho = \pi\tau_j$ with $j \neq 1, 2, 3$, ρ is an element in $C_{A_n}(\tau)$. So 3 divides $|C_{A_n}(\tau)|$.
For the converse suppose that n does not have the form $2^{2k} - 1$ for any $k \in \mathbb{N}$. This condition can be mentioned as: At least one 0 appears in

the representation of n to the basis 2 or the number of 1's is odd. Take a permutation $\tau = \tau_1\tau_2\cdots\tau_t$ in S_n with 2-form. By definition of 2-form, each length of cycle appears at most one time, thus according to Lemma 3.2, we have that τ is a C_{22}-element in S_n. If all the digits of n to the basis 2 are 1 , then $|\mathrm{Fix}(\tau)| = 1$ and the number of cycles in τ are even, so $\tau \in A_n$ is a C_{22}-element. Hence suppose that at least one 0 appears in the representation of n to the basis 2 and $\tau \notin A_n$. Then we have two cases:

Case (i). If we have just one 0 in the right side digit of $n_{(2)}$, i.e., $n_{(2)} = 11\cdots 10$, then we put $\tau' = \tau_1\cdots\tau_{t-1}$. Now τ' is a permutation in A_n with $|\mathrm{Fix}(\tau')| = 2$ and the result follows from Lemma 3.2.

Case (ii). If the digit 0 appears in another position, we define $\alpha_{t+1} = 0$. In this situation, we have at least one α_i, $1 \leq i \leq t$, such that $\alpha_i > \alpha_{i+1} + 1$. Let $\tau' = \tau_1\cdots\tau_{i-1}\tau_i^2\tau_{i+1}\cdots\tau_t$, then $\tau' \in A_n$. We may write ${\tau_i}^2 = \pi_1\pi_2$, where π_1 and π_2 are distinct cycles of length $2^{\alpha_i - 1} > 2^{\alpha_{i+1}}$. Hence $\tau' \in A_n$. Since π_1 and π_2 are the only cycles in τ' with equal length, by Lemma 3.2, the order of $|C_{S_n}(\tau')|$ is a power of 2 and the result follows.

For the second part it is enough to show that if a 2-element $\tau \in A_n$ contains three cycles with the same length, then it is not a C_{22}-element in A_n. This is clear if $|\mathrm{Fix}(\tau)| \geq 3$. Hence let $|\mathrm{Fix}(\tau)| \leq 2$, by defining ρ as above, we conclude that 3 divides $|C_{A_n}(\tau)|$ and we are done. □

Remark 3.11. The set of C_{22}-elements of A_n coinsides with the set of C_{22}-elements of S_n if and only if $n = 2^{2k+1} - 1$ for some $k \in \mathbb{N}$ and it is constituted by the elements in 2-form. Suppose that $n = 2^{2k+1} - 1$, and let τ be a C_{22}-element in S_n. If τ has 2-form in S_n, then τ belongs to A_n and we are finished. Suppose that τ does not have the 2-form. Since n is odd, $|\mathrm{Fix}(\tau)|$ must be odd and since $C_{S_n}(\tau)$ is a 2-group, we get $|\mathrm{Fix}(\tau)| = 1$. If τ is the product of disjoint cycles of lengths $2^{a_1}, ..., 2^{a_s}, 1$, we obtain $2^{a_1} + ... + 2^{a_s} = 2^{2k} + 2^{2k-1} + ... + 2$ and these two representations are different. Thus by Lemma 3.9, there are three cycles with the same length, against the fact that $C_{S_n}(\tau)$ is a 2-group. Now suppose that $n \neq 2^{2k+1} - 1$, and let τ be a C_{22}-element in S_n with 2-form. If τ does not belong to A_n we are finished. Let $\tau \in A_n$. Then $n \neq 2^{2k} - 1$, and hence at least one zero appears in $n_{(2)}$. Define τ' as in the cases (i) or (ii) of the last theorem. Now τ' is a C_{22}-element in S_n which does not belong to A_n.

Corollary 3.12. *For $n \geq 3$, the alternating group A_n has a 2-Sylow covering if and only if n does not have the form $2^{2k} - 1$, for any $k \in \mathbb{N}$.*

Theorem 3.13. *Let $n \geq 4$. For each odd prime p dividing $|A_n|$, A_n has a C_{pp}-element if and only if 0, 1 and 2 are the only digits appearing in the representation of n to the basis p, and 2 appears at most once. Moreover in this situation, $\tau \in A_n$ is a C_{pp}-element, for a prime $p \geq 5$ if and only if τ has the p-form in S_n, and $\tau \in A_n$ is a C_{33}-element if and only if τ is the product of disjoint 3^k-cycles such that their lengths are pairwise different except at most three of them.*

Proof. Let $p \leq n$ be a prime and let $\tau \in A_n$ be a C_{pp}-element consisting of $a_i \geq 1$ i-cycles, where i is a power of p. Remembering that for each $\sigma \in A_n$, $|C_{S_n}(\sigma)| = \epsilon |C_{A_n}(\sigma)|$, with $\epsilon = 1$ or 2, by Lemma 3.2, $|C_{S_n}(\tau)| = \prod_i (a_i!) i^{a_i} = \epsilon p^m$, where m is a positive integer. Now either we have $a_i = 1$ for all i or there exists just one j such that $a_j \neq 1$. In the second case if $p = 3$, then $a_j = 2$ or 3 and if $p \neq 3$, then $a_j = 2$. Thus $n = p^{b_1} + \ldots + p^{b_k}$, where $b_i \in \mathbb{N}$ and at most two of them are equal. Therefore the digits of $n_{(p)}$ consist of 0, 1 and at most one 2.
Conversely, by the hypothesis, in the p-representation of n either $r = 0$, 1 and at most α_j and α_{j+1} are equal or $r = 2$ and α_i's are pairwise distinct. Consider a permutation τ with the p-form in S_n. If 2 does not appear in $n_{(p)}$, then τ is a C_{pp}-element in A_n. Otherwise τ commutes with a cycle of length $2p^{\alpha_j}$ or 2, hence $C_{A_n}(\tau)$ has index 2 in $C_{S_n}(\tau)$. On the other hand, by Lemma 3.2, $|C_{S_n}(\tau)| = 2p^m$, where m is a positive integer, whence $|C_{A_n}(\tau)|$ is a power of p.
Now suppose $n_{(p)}$ has the right digits. Let $p \neq 3$, we know that each permutation in A_n with p-form is a C_{pp}-element in A_n. Now suppose $\tau \in A_n$ does not have the p-form. Therefore there exist at least p cycles with equal length p^{m_1}, so 3 divides $|C_{A_n}(\tau)|$ and τ is not a C_{pp}-element. Now let $p = 3$. If τ has at most three cycles of equal length 3^{m_1}, then the only elements in $C_{S_n}(\tau)$ which are not 3-element are products of cycles of length 3^k and 3^{m_2} cycles of length 2.3^{m_3} where $m_2 + m_3 = m_1$, but these permutations do not belong to A_n. If τ does not satisfy the hypothesis, then it includes four disjoint cycles σ_i, $i = 1, 2, 3, 4$, such that $o(\sigma_1) = o(\sigma_2) = 3^{m_4}$ and $o(\sigma_3) = o(\sigma_4) = 3^{m_5}$ (not necessarily $m_4 \neq m_5$). Now we have two disjoint cycles with lengthes 2.3^{m_4} and 2.3^{m_5} in $C_{S_n}(\tau)$ and their product belongs to A_n so we are done. $\square$

Corollary 3.14. *Let $n \geq 4$ and let $p \leq n$ be an odd prime. Then A_n has a p-Sylow covering if and only if 0,1 and 2 are the only digits appearing in the representation of n to the basis p, and 2 appears at most once.*

Theorem 3.15. *The groups A_n for $n \in \{4, 5, 6, 7\}$ are the only alternating groups which have a p-Sylow covering for each prime p dividing $|A_n|$.*

Proof. For $n \leq 8$ we can directly check the result by Corollaries 3.12 and 3.14. Let $n \geq 9$. If $n - 3$ has a prime divisor $q > 3$, then $n \equiv 3 (\mathrm{mod}\ q)$ and by Corollary 3.14, the q-Sylow covering does not exist. Otherwise 2 and 3 are the only prime divisors of $n-3$. If $n-3$ is not a prime power, then $n - 4$ has an odd prime divisor $r \geq 5$, so $n \equiv 4 (\mathrm{mod}\ r)$ and again the result follows form Corollary 3.14. So let $n - 3$ is a power of 2 or 3. From Lemma 3.7, there exists a prime $p > 3$ such that $p + 4 \leq n < 3p$. Let P be a p-Sylow subgroup of A_n. If $|P| = p$ then the result follows by the same argument as in Theorem 3.8. Let $|P| = p^2$. Suppose that τ is a C_{pp}-element in P. Clearly $|\mathrm{Fix}(\tau)| \leq 2$ and τ is a product of two cycles of length p. So $|C_{S_n}(\tau)| = 2|C_{A_n}(\tau)| = 2p^2.|\mathrm{Fix}(\tau)|!$. Therefore $n = 2p$ or $n = 2p+1$. Now the only possible case is $n-3 = 2p-2 = 2^\alpha$, where α is a positive integer. But in this case $n - 4$ has an odd prime divisor grater than 3 and this completes the proof. □

Acknowledgements

The authors would like to thank Prof. L. Serena for his useful comments and presenting Lemma 3.7 and the referee for corrections and suggestions.

References

[1] R. Brandl, Finite groups all of whose elements are of prime power order, *Boll. Un. Mat. Ital. 18 - A (**57**)* (1981), 491-493.

[2] R. Brandl, Integer polynomials with roots mod p for all primes p, *J. of Algebra* **240** (2001), 822-835.

[3] R. Brandl, D. Bubboloni and I. Hupp, Polynomials with roots mod p for all primes p, *J. Group Theory* **4** (2001), 233-239.

[4] M. A. Brodie, R. F. Chamberlain and L. C. Kappe, Finite coverings by normal subgroups, *Proc. Amer. Math. Soc.,* **104** (3) (1988), 669-671.

[5] D. Bubboloni, Coverings of the symmetric and alternating groups, *Dipartimento di Matematica U. Dini, Universita' di Firenze*, **7** (1998).

[6] D. Bubboloni, M. S. Lucido, Coverings of linear groups, *Comm. Algebra,* **30**(5) (2002), 2143-2159.

[7] G. Higman, Finite groups in which every element has prime power order, *J. London Math. Soc.* **32** (1957), 335-342.
[8] E. Jabara and M. S. Lucido, Finite groups with Hall coverings, *Aust. Math. Soc.* **78**(1) (2005), 1-16.
[9] G. James and M. Liebeck, *Representations and Characters of Groups*, Cambridge University Press, Cambridge, 1993.
[10] L. C. Kappe, Finite coverings by 2-Engles groups, *Bull. Austral. Math. Soc.* **38** (1988), 141-150.
[11] M. Suzuki, On a class of doubly transitive groups, *Ann. Math.* **75** (1962), 105-145.
[12] M. Suzuki, *Group Theory*, vol. I, Springer-Verlag, New York, 1982.
[13] M. J. Tomkinson, Hypercenter-by-finite groups, *Publ. Debrecen* **40** (1992), 313-321.
[14] G. Zacher, Sull' ordine di un gruppo finito risolubile somma dei sottogruppi di Sylow, *Atti. Acc. Naz. Lincei* **20**(8) (1956), 171-174.
[15] G. Zacher, Sui gruppi finiti somma dei loro sottogruppi di Sylow, *Rend. Sem. Mat. Padova* **27** (1957), 267-275.

K. Mehrabadi and **A. Iranmanesh**
Department of Mathematics
Tarbiat Modares University
P.O.Box: 14115-137
Tehran, Iran
e-mail: kmehr@modares.ac.ir
e-mail: iranmana@modares.ac.ir