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EXISTENCE OF GROUND STATES FOR APPROXIMATELY INNER TWO–PARAMETER C_0–GROUPS ON C^*–ALGEBRAS

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Abstract. In this paper, we generalize the definitions of approximately inner C_0-groups and their ground states to the two- parameter case and study necessary and sufficient conditions for a state to be ground state. Also we prove that any approximately inner two- parameter C_0–group must have at least one ground state. Finally some applications are given.
Keywords: Two-parameter group, approximately inner dynamical system, tensor product, ground state.
MSC(2010): Primary: 47D03; Secondary: 46L05, 46L55.

1. Introduction

The theory of n–parameter semigroups of operators which is an extension of one–parameter case has been developed by Hille in 1944. In 1946 Dunford and Segal [4] applied this concept to prove the theorem of Weierstrass. In [7] Hille and Phillip studied X–parameter semigroups. O.A. Ivanova [8] obtained some other results in n–parameter semigroups in 1966.

One can see that the semigroups of operators arise naturally in several areas of applied mathematics including prediction theory of random fields [3, 13]. Such a semigroup of operators can be used to describe the time evolution of a physical system in quantum field theory, statistical mechanics and partial differential equations [1, 15, 16].

Let X be a C^*–algebra and $\{\alpha_t\}_{t\geq 0}$ be strongly continuous one–parameter group of $*$–automorphisms on X. The infinitesimal generator δ of $\{\alpha_t\}_{t\geq 0}$ is defined by:

$$\delta(x) = \lim_{t\to 0} \frac{\alpha_t(x) - x}{t}\ ,\ x \in D(\delta),$$

where $D(\delta) = \{x \in X : \lim_{t\to 0} \frac{\alpha_t(x)-x}{t} \text{ exists}\}$.

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A subset D of $D(\delta)$ is said to be a core for δ if for every $a \in D(\delta)$ there exists a sequence $\{a_n\}$ in D such that $a_n \longrightarrow a$ and $\delta(a_n)$ converges. The two–parameter dynamical system $\{\alpha_{s,t}\}$ is a function $(s,t) \longrightarrow \alpha_{s,t}$ from $\mathbb{R}_+ \times \mathbb{R}_+$ into $B(X)$ such that:

i) $\alpha_{s+s',t+t'} = \alpha_{s,t}\alpha_{s',t'}$,

ii) $\alpha_{0,0} = I$.

The dynamical system is called strongly continuous if $(s,t) \longrightarrow \alpha_{s,t}(x)$ is continuous for all x in X and is called uniformly continuous if $(s,t) \longrightarrow \alpha_{s,t}$ is norm continuous.

Suppose $u_s = \alpha_{s,0}$ and $v_t = \alpha_{0,t}$, then the semigroup property of $\{\alpha_{s,t}\}$ implies that $\alpha_{s,t} = u_s v_t$. It is easy to see that $\alpha_{s,t}$ is strongly (resp. uniformly) continuous if and only if u_s and v_t are strongly (resp. uniformly) continuous.

Let δ_1 and δ_2 be the infinitesimal generators of u_s and v_t respectively. We assume the pair (δ_1, δ_2) as the infinitesimal generator of $\{\alpha_{s,t}\}$. See [10, 16] for more information.

The above considerations motivated M.Janfada and A.Niknam to generalize the concept of one parameter group and its generator to be expensively developed. The importance of two parameter groups of bounded linear operators is closely related to their applications in partially differential equations. More precisely it can be shown that the existence and uniqueness of the 2-abstract Cauchy problem is closely related to two parameter semigroups of operators.(See [9–11], [12])

The paper is organized as follows. In section 2, we introduce the definitions of approximately inner strongly continuous two–parameter group of $*$–automorphisms on a C^*–algebra and ground state for it. In section 3, we state and prove a theorem which provides necessary and sufficient conditions for a state to be ground state. Also we prove that any approximately inner C_0–group must have at least one ground state. In section 4, we introduce a two–parameter C_0–group, such that it is not approximately inner.

2. **Preliminaries**

Hereby we present the definition of ground state for two–parameter group of $*$–automorphisms on C^*–algebras.

Definition 2.1. The C_0-group $\{\alpha_{s,t}\}$ is approximately inner if there are sequences $\{h'_n\}$ and $\{h''_n\}$ of self-adjoint elements of C^*-algebra A , such that;

$$\|e^{i(sh'_n+th''_n)}ae^{-i(sh'_n+th''_n)} - \alpha_{s,t}(a)\| \longrightarrow 0,$$

uniformly on compact subset of $\mathbb{R}_+ \times \mathbb{R}_+$, for each $a \in A$.

Definition 2.2. Suppose $\{\alpha_{s,t}\}$ is a two–parameter C_0–group of $*$–automorphisms of a C^*-algebra A with unit, we say that ω is a ground state of A for group $\{\alpha_{s,t}\}$ if and only if for all $a, b \in A$ the function $(s,t) \longrightarrow \omega(a\alpha_{s,t}(b))$ is continuous and

$$\int_{R^2} h(s,t)\omega(a\alpha_{s,t}(b))d(s,t) = 0,$$

for all continuous L^1-function h whose Fourier transform

$$\hat{h}(\alpha, \beta) = \int_{R^2} h(s,t)e^{-i(s\alpha+t\beta)}d(s,t),$$

vanishes on $(-\infty, 0] \times (-\infty, 0]$.

Now assume that ω is $\alpha_{s,t}$-invariant state, i.e.,

$$\omega(a) = \omega(\alpha_{s,t}(a)),$$

and $(\pi, \mathcal{H}, f_0)$ cyclic $*$-representation of A induced by ω on Hilbert space $\mathcal{H}$ with cyclic vector f_0, so that $\omega(a) =< \pi(a)f_0, f_0 >$. Since ω is $\alpha_{s,t}$-invariant, we may define unitary operator $U_{s,t}$ on $\mathcal{H}$ by the relation;

$$U_{s,t}\pi(a)f_0 = \pi(\alpha_{s,t}(a))f_0,$$

for all $a \in A$ with additional property;

$$U_{s,t}\pi(a)U_{s,t}^{-1} = \pi(\alpha_{s,t}(a)),$$

and

$$U_{s,t}f_0 = f_0.$$

Let $U(s) = U_{s,0}$ and $V(t) = U_{0,t}$, then $U(s)$ and $V(t)$ are strongly continuous one-parameter groups of unitary operators with the properties;

$$U(s)\pi(a)U(s)^{-1} = \pi(\alpha_{s,0}(a)),$$

and

$$U(s)f_0 = f_0,$$

similarly holds for $V(t)$.

It follows from the Stone 's theorem [16], that, there are self-adjoint operators H_1 and H_2 which generate the one-parameter groups U_s and V_t respectively, that is

$$U_s = e^{isH_1} \quad and \quad V_t = e^{itH_2}.$$

Since $U_sf_0 = f_0$ and $V_tf_0 = f_0$ for all real s and t, then we have,

$$f_0 \in D(H_1) \quad and \quad f_0 \in D(H_2),$$

the domains of H_1 and H_2, and $H_1f_0 = 0$ and $H_2f_0 = 0$.

We will show that ω is a ground state for $\{\alpha_{s,t}\}$ if and only if $H_1 \geq 0$ and $H_2 \geq 0$. Let $\{E_1(\lambda); -\infty < \lambda < \infty\}$ and $\{E_2(\lambda'); -\infty < \lambda' < \infty\}$ be the spectral resolution of H_1 and H_2 respectively [6], i.e.,

$$H_1 = \int \lambda dE_1(\lambda) \quad and \quad U(s) = \int e^{is\lambda} dE_1(\lambda),$$

$$H_2 = \int \lambda' dE_2(\lambda') \quad and \quad V(t) = \int e^{it\lambda'} dE_2(\lambda').$$

For $a, b \in A$ and h an L^1-function we have,

$$\begin{aligned}
\int h(s,t)\omega(a\alpha_{s,t}(b))d(s,t) &= \int h(s,t) < \pi(a\alpha_{s,t}(b))f_0, f_0 > d(s,t) \\
&= \int h(s,t) < \pi(a)\pi(\alpha_{s,t}(b))f_0, f_0 > d(s,t) \\
&= \int h(s,t) < \pi(a)U_{s,t}\pi(b)f_0, f_0 > d(s,t) \\
&= \int h(s,t) < \pi(a)U(s)V(t)\pi(b)f_0, f_0 > d(s,t) \\
&= \int h(s,t) < \pi(a) \int e^{i(s\lambda + t\lambda')} dE_1(\lambda)dE_2(\lambda')\pi(b)f_0, f_0 > d(s,t) \\
&= \int h(s,t) \int e^{i(s\lambda + t\lambda')} < dE_1(\lambda)dE_2(\lambda')\pi(b)f_0, \pi(a^*)f_0 > d(s,t) \\
&= \int \widehat{h}(-\lambda, -\lambda') < dE_1(\lambda)dE_2(\lambda')\pi(b)f_0, \pi(a^*)f_0 > .
\end{aligned}$$

We have ω is ground state if and only if the above integral vanishes for all $a, b \in A$ when $\widehat{h}$ vanishes on $(-\infty, 0]$.On the other hands, since $\{\pi(a^*)f_0; a \in A\}$ and $\{\pi(b)f_0; a \in A\}$ are dense in $\mathcal{H}$, we have ω is ground state if and only if the spectral measures $E_1(\lambda)$ and $E_2(\lambda')$ have support on $[0, \infty)$. Therefore, ω is ground state if and only if H_1 and H_2 are positive, i.e., $H_1 \geq 0$ and $H_2 \geq 0$.

3. Main results

The following theorem provides necessary and sufficient conditions for a state to be ground state.

Theorem 3.1. *Suppose $\{\alpha_{s,t}\}$ is a strongly continuous two-parameter group of $*$-automorphisms of a C^*-algebra A with unit and (δ_1, δ_2) is the infinitesimal generator of $\{\alpha_{s,t}\}$ and D_1, D_2 are the cores for δ_1 and δ_2 respectively. Then a state ω is ground state for $\{\alpha_{s,t}\}$ if and only if;*

$$-i\omega(a_j^*\delta_j(a_j)) \geq 0 \ , \quad j = 1, 2$$

for all $a_j \in D_j$.

Proof. Suppose that ω is a ground state for $\{\alpha_{s,t}\}$, let $(\pi, \mathcal{H}, f_0)$ be a cyclic $*$–representation of A induced by ω with cyclic vector $f_0 \in \mathcal{H}$, so that

$$\omega(a) = \langle \pi(a) f_0, f_0 \rangle,$$

for all $a \in A$.

From the previous discussion in section 2, there are strongly continuous one-parameter groups of unitary operators,

$$U_s = e^{isH_1} \quad , V_t = e^{itH_2},$$

with

$$U_s f_0 = f_0 \quad , V_t f_0 = f_0,$$

and

$$U_s \pi(a) U_s^{-1} = \pi(\alpha_{s,0}(a)),$$
$$V_t \pi(a) V_t^{-1} = \pi(\alpha_{0,t}(a)),$$

for all real s, t and $a \in A$.

Since ω is ground state, we have that the generators H_1 and H_2 are positive. i.e., $H_1 \geq 0$ and $H_2 \geq 0$.

Now if $a_j \in D(\delta_j)$, $j = 1, 2$ we have;

$$(is)^{-1}(U_s - I)\pi(a_1) f_0 = (is)^{-1}\pi(\alpha_{s,0}(a_1) - a_1) f_0 \longrightarrow -i\pi(\delta_1(a_1)) f_0,$$

similarly

$$(it)^{-1}(V_t - I)\pi(a_2) f_0 = (it)^{-1}\pi(\alpha_{0,t}(a_2) - a_2) f_0 \longrightarrow -i\pi(\delta_2(a_2)) f_0,$$

as $s \longrightarrow 0$ and $t \longrightarrow 0$. Hence by Stone's theorem we have;

$$\pi(a_j) f_0 \in D(H_j),$$

and

$$H_j \pi(a_j) f_0 = -i\pi(\delta_j(a_j)) f_0 \quad , \; j = 1, 2.$$

Moreover if $a_j \in D_j$ we have

$$\begin{aligned} -i\omega(a_j^* \delta_j(a_j)) &= -i\langle \pi(a_j^*)\pi(\delta_j(a_j)) f_0, f_0 \rangle \\ &= \langle \pi(a_j^*) H_j \pi(a_j), f_0 \rangle \\ &= \langle H_j \pi(a_j) f_0, \pi(a_j) f_0 \rangle \geq 0. \end{aligned}$$

Hence

$$-i\omega(a_j^* \delta_j(a_j)) \geq 0,$$

for all $a_j \in D_j$

Conversely, suppose ω is a state of A so that,

$$-i\omega(a_j^* \delta_j(a_j)) \geq 0 \quad for \quad j = 1, 2 \quad (\star)$$

for all $a_j \in D_j$. We first show that $(\star)$ holds for all $a_j \in D(\delta_j)$.

For each $a_j \in D(\delta_j)$ there exist sequences $\{a_n \in D_1\}$ and $\{b_n \in D_2\}$ such that

$$\|a_n - a_1\| \longrightarrow 0 \quad and \; \|b_n - a_2\| \longrightarrow 0$$

and

$$\|\delta_1(a_n) - \delta_1(a_1)\| \longrightarrow 0,$$
$$\|\delta_2(b_n) - \delta_2(a_2)\| \longrightarrow 0,$$

and therefore;

$$\|a_n^*\delta_1(a_n) - a_1^*\delta_1(a_1)\| \longrightarrow 0,$$

and

$$\|b_n^*\delta_2(b_n) - a_2^*\delta_2(a_2)\| \longrightarrow 0,$$

as $n \longrightarrow \infty$. Thus

$$-i\omega(a_1^*\delta_1(a_1)) = \lim_{n\longrightarrow\infty} -i\omega(a_n^*\delta_1(a_n)) \geq 0,$$

and

$$-i\omega(a_2^*\delta_2(a_2)) = \lim_{n\longrightarrow\infty} -i\omega(b_n^*\delta_2(b_n)) \geq 0,$$

for $a_j \in D(\delta_j)$ and $j = 1, 2$.

Next we will show that ω is $\alpha_{s,t}$-invariant. Since $\alpha_{s,t}(e) = e$ for all s and t, (e is the unit of A), it follows that $e \in D(\delta_1) \cap D(\delta_2)$.
If $a_j \in D(\delta_j)$ and λ is a complex number, we have,
$-i\omega((\lambda e + a_j)^*\delta_j(\lambda e + a_j)) \geq 0$ for $j = 1, 2$, hence $-i\omega(\bar{\lambda}\delta_j(a_j) + a_j^*\delta_j(a_j)) \geq 0$ for all complex number λ. Hence $\omega(\delta_j(a_j)) = 0$, for $a_j \in D(\delta_j)$ and $j = 1, 2$. Let $a_1 \in D(\delta_1)$ and $a_2 \in D(\delta_2)$, since

$$\frac{d}{ds}\omega(\alpha_{s,0}(a_1)) = \omega(\delta_1(a_1)) = 0,$$

and

$$\frac{d}{dt}\omega(\alpha_{0,t}(a_2)) = \omega(\delta_2(a_2)) = 0,$$

hence $\omega(\alpha_{s,0}(a_1)) = \omega(a_1)$ and $\omega(\alpha_{0,t}(a_2)) = \omega(a_2)$ for all s and t. Since $D(\delta_1)$ and $D(\delta_2)$ are norm dense in A therefore for each $a \in A$ we have;

$$\omega(\alpha_{s,0}(a)) = \omega(a),$$

and

$$\omega(\alpha_{0,t}(a)) = \omega(a).$$

Since $\alpha_{0,t}(a) \in A$ and $\alpha_{s,t}(a) = \alpha_{s,0}\alpha_{0,t}(a)$, hence we conclude that

$$\omega(a) = \omega(\alpha_{0,t}(a)) = \omega(\alpha_{s,0}\alpha_{0,t}(a)) = \omega(\alpha_{s,t}(a)),$$

and therefore ω is $\alpha_{s,t}$–invariant.

Let $(\pi, \mathcal{H}, f_0)$ be the cyclic $*$–representation induced by ω and let $s \longrightarrow U_s$ and $t \longrightarrow V_t$ be the strongly continuous one–parameter groups of unitaries defined by the relations;

$$U_s\pi(a)f_0 = \pi(\alpha_{s,0}(a))f_0,$$
$$V_t\pi(a)f_0 = \pi(\alpha_{0,t}(a))f_0,$$

for all real s, t and $a \in A$.

Let H_1 and H_2 be the generators of groups $\{U_s\}$ and $\{V_t\}$ respectively, i.e.,

$$U_s = e^{isH_1} \quad and \quad V_t = e^{itH_2}.$$

To prove that ω is a ground state, we must show that H_1 and H_2 are positive, i.e., $H_1 \geq 0$ and $H_2 \geq 0$.
Suppose $a_j \in D(\delta_j)$ for $j = 1, 2$, then we have
$(is)^{-1}(U_s - I)\pi(a_1)f_0 = (is)^{-1}\pi(\alpha_{s,0}(a_1) - a_1)f_0 \longrightarrow -i\pi(\delta_1(a_1))f_0$,
and
$(it)^{-1}(V_t - I)\pi(a_2)f_0 = (it)^{-1}\pi(\alpha_{0,t}(a_2) - a_2)f_0 \longrightarrow -i\pi(\delta_2(a_2))f_0$,
as $s \longrightarrow 0$ and $t \longrightarrow 0$.

Hence by Stone's theorem we have

$$\pi(a_j)f_0 \in D(H_j),$$

and

$$H_j\pi(a_j)f_0 = -i\pi(\delta_j(a_j))f_0,$$

for all $a_j \in D(\delta_j)$ and $j = 1, 2$.

We have

$$\begin{aligned} \langle H_j\pi(a_j)f_0, \pi(a_j)f_0\rangle &= -i\langle\pi(\delta_j(a_j))f_0, \pi(a_j)f_0\rangle \\ &= -i\langle\pi(a_j^*\delta_j(a_j))f_0, f_0\rangle \\ &= -i\omega(a_j^*\delta_j(a_j)) \geq 0, \end{aligned}$$

for $j = 1, 2$, hence H_j is positive on $\{\pi(D(\delta_j))f_0\}$.

It is enough to show that H_1 is positive on $\mathcal{H}$, the positivity of H_2 is similar. For this, let N_1 be the closure of restriction of H_1 to $\{\pi(D(\delta_1))f_0\}$, i.e., $N_1 = \overline{H_1|\{\pi(D(\delta_1))f_0\}}$. From the above inequality, we have N_1 is positive. Since $N_1 \subset H_1$, we have N_1^* is an extension of $H_1^* = H_1$, hence $N_1 \subset H_1 \subset N_1^*$. We will show that $N_1 = N_1^*$, then we have $N_1 = H_1$.

Since $D(\delta_1)$ is invariant under $\alpha_{s,0}$, we have

$$U_s\{\pi(D(\delta_1))f_0\} = \{\pi(\alpha_{s,0}(D(\delta_1)))f_0\} = \{\pi(D(\delta_1))f_0\}.$$

Since $\{\pi(D(\delta_1))f_0\}$ is a dense linear manifold of $D(N_1)$ invariant under U_s, it follows from Lemma 2 of [20] that N_1 is self–adjoint. Hence $H_1 = N_1$ is positive. □

Next we will show that if a C_0–group is approximately inner, then there would be a ground state for it. First we state a useful lemma.

Lemma 3.2. *Suppose $\{h'_n\}$ and $\{h''_n\}$ be sequences of positive self-adjoint elements of a C^*–algebra A such that zero belongs to spectrums of h'_n and h''_n, then for $n = 1, 2, ...$ there exists a state ω_n on A such that*

$$\omega_n(h'_n) = \omega_n(h'^2_n) = 0,$$

and

$$\omega_n(h''_n) = \omega_n(h''^2_n) = 0.$$

Proof. Since h'_n and h''_n are positive self-adjoint, then by [18] or ([14], p. 306), there exist states γ'_n and γ''_n such that

$$\gamma'_n(h'_n) = \gamma'_n(h'^2_n) = 0,$$

and

$$\gamma''_n(h''_n) = \gamma''_n(h''^2_n) = 0.$$

Let $S(h'_n)$ and $S(h''_n)$ are spaces generated by $\{h'_n, h'^2_n\}$ and $\{h''_n, h''^2_n\}$ respectively. We define states ω'_n and ω''_n on A as follow,

$$\omega'_n(a) = \begin{cases} 0 & a \in S(h''_n) \\ \gamma'_n(a) & a \notin S(h''_n) \end{cases},$$

and

$$\omega''_n(a) = \begin{cases} 0 & a \in S(h'_n) \\ \gamma''_n(a) & a \notin S(h'_n) \end{cases}.$$

Since the set of all states is a convex set, if we define ω_n on A as follow,

$$\omega_n = \frac{1}{2}(\omega'_n + \omega''_n),$$

then ω_n is a state such that satisfies the properties

$$\omega_n(h'_n) = \omega_n(h'^2_n) = 0,$$

and

$$\omega_n(h''_n) = \omega_n(h''^2_n) = 0.$$

□

Theorem 3.3. *Suppose $\{\alpha_{s,t}\}$ be a strongly continuous two-parameter group of $*$–automorphisms of a C^*–algebra A with unit, and $\{\alpha_{s,t}\}$ is approximately inner, then there exists a ground state ω for $\{\alpha_{s,t}\}$.*

Proof. Let $\{\alpha_{s,t}\}$ is approximately inner, then there exist sequences of hermitian elements $\{h_n \in A\}$ and $\{h'_n \in A\}$ such that

$$\|e^{i(sh_n+th'_n)}ae^{-i(sh_n+th'_n)} - \alpha_{s,t}(a)\| \longrightarrow 0,$$

as $n \longrightarrow \infty$ for all $a \in A$. For fixed $a \in A$ the convergence is uniform on compact subset of $\mathbb{R} \times \mathbb{R}$. By adding a multiple of the unit to h_n and h'_n we can arrange them, so that h_n and h'_n are positive and zero is in spectrum of h_n and h'_n, i.e., $0 \in \sigma(h_n)$, $0 \in \sigma(h'_n)$, $h_n \geq 0$ and $h'_n \geq 0$ for $n = 1, 2, ...$, hence it follows from Lemma 3.2 that there is a state ω_n such that,

$$\omega_n(h_n) = \omega_n(h^2_n) = 0,$$

and

$$\omega_n(h'_n) = \omega_n(h'^2_n) = 0.$$

Since the state space of C^*–algebra is compact in the weak $*$–topology [2], there is a state ω which is a cluster point of the sequence $\{\omega_n\}$ in the weak $*$–topology, we will show that ω is ground state. For this suppose h is a continuous L^1–function on $\mathbb{R}^2$ i.e., $h \in L^1(\mathbb{R}^2)$ whose Fourier transform $\hat{h}$ vanishes on the $(-\infty, 0] \times (-\infty, 0]$. Suppose $a, b \in A$, we will show that;

$$\int_{\mathbb{R}^2_+} h(s,t)\omega(a\alpha_{s,t}(b))d(s,t) = 0.$$

For this, let

$$B_0 = \int h(s,t)\alpha_{s,t}(b)d(s,t),$$

and

$$B_n = \int h(s,t)e^{i(sh_n+th'_n)}be^{-i(sh_n+th'_n)}d(s,t).$$

Suppose $\epsilon > 0$, since $h \in L^1(\mathbb{R}^2)$ there exist constants c_1 and c_2 such that,

$$2\|b\| \int_{|s|>c_1, |t|>c_2} |h(s,t)|d(s,t) < \frac{\epsilon}{2},$$

or

$$2\|b\| \int_{|s|,|t|>c} |h(s,t)|d(s,t) < \frac{\epsilon}{2},$$

for $c = \max\{c_1, c_2\}$.

Since $e^{i(sh_n+th'_n)}be^{-i(sh_n+th'_n)}$ converges to $\alpha_{s,t}(b)$ uniformly on $[-c,c] \times [-c,c]$ there is an integer n_0 such that,

$$\|h\|_1\|e^{i(sh_n+th'_n)}be^{-i(sh_n+th'_n)} - \alpha_{s,t}(b)\| < \frac{\epsilon}{2},$$

for $(s,t) \in [-c,c] \times [-c,c]$ and $n \geq n_0$, where $\|h\|_1$ is the L^1–norm of h.

For $n \geq n_0$ and since $\|\alpha_{s,t}(b)\| \leq \|b\|$,([18], 1.2.6), we have

$$\begin{aligned}
\|B_n - B_0\| &\leq \|\int h(s,t)(e^{i(sh_n+th'_n)}be^{-i(sh_n+th'_n)} - \alpha_{s,t}(b))d(s,t)\| \\
&\leq \int_{[-c,c]\times[-c,c]} |h(s,t)|\|e^{i(sh_n+th'_n)}be^{-i(sh_n+th'_n)} - \alpha_{s,t}(b)\|d(s,t) \\
&+ \int_{|s|,|t|>c} |h(s,t)|\|e^{i(sh_n+th'_n)}be^{-i(sh_n+th'_n)} - \alpha_{s,t}(b)\|d(s,t) \\
&< \int_{[-c,c]\times[-c,c]} |h(s,t)|(\|h\|_1^{-1})(\frac{\epsilon}{2})d(s,t) \\
&+ 2\|b\| \int_{|s|,|t|>c} |h(s,t)|d(s,t) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence we have $\|B_n - B_0\| < \epsilon$.

Since ω is a cluster point of sequence ω_n in weak $*$–topology, there is an integer $r \geq n_0$ such that,

$$|\omega_r(aB_0) - \omega(aB_0)| < \epsilon.$$

Next we show that

$$\omega_r(ae^{i(sh_r+th'_r)}be^{-i(sh_r+th'_r)}) = \omega_r(ae^{i(sh_r+th'_r)}b).$$

For this, we have

$$\omega_r(ae^{i(sh_r+th'_r)}be^{-i(sh_r+th'_r)}) = \omega_r(ae^{i(sh_r+th'_r)}b(e + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}(sh_r + th'_r)^n))$$

$$= \omega_r(ae^{i(sh_r+th'_r)}b) + \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\omega_r(ae^{i(sh_r+th'_r)}b(sh_r + th'_r)^n).$$

We will show that the last summation is zero. Set $A = ae^{i(sh_r+th'_r)}b$ and $B = (sh_r + th'_r)$, then by Schwartz inequality [18] and self-adjoint property of $h_r$and h'_r, we have,

$$|\omega_r(AB^n)|^2 \leq \omega_r(AA^*)\omega_r(B^{2n}).$$

It is sufficient to show that $\omega(B^{2n}) = 0$. For this,

$$|\omega_r(B^{2n})|^2 = |\omega_r(BB^{2n-1})|^2 \leq \omega_r(B^2)\omega_r(B^{2(2n-1)}).$$

Since $\omega_r(h) = \omega_r(h^2) = 0$ and $\omega_r(h') = \omega_r(h'^2) = 0$ by using Schwartz inequality we have $\omega_r(B^2) = 0$. Hence $\omega_r(B^{2n}) = 0$ and therefore $\omega(AB^n) = 0$. Finally

$$\omega_r(ae^{i(sh_r+th'_r)}be^{-i(sh_r+th'_r)}) = \omega_r(ae^{i(sh_r+th'_r)}b).$$

Now we have

$$\begin{aligned}\omega_r(aB_r) &= \int h(s,t)\omega_r(ae^{i(sh_r+th'_r)}be^{-i(sh_r+th'_r)})d(s,t)\\ &= \int h(s,t)\omega_r(ae^{i(sh_r+th'_r)}b)d(s,t)\\ &= \sqrt{2\pi}\omega_r(a\hat{h}(-h_r,-h'_r)b) = 0,\end{aligned}$$

where $\hat{h}(-h_r, -h'_r) = 0$, since $\hat{h}$ vanishes at $(-\infty, 0] \times (-\infty, 0]$ and spectrums of $-h_r$ and $-h'_r$ are contained in $(-\infty, 0]$. So

$$\begin{aligned}|\omega(aB_0)| &\leq |\omega(aB_0) - \omega_r(aB_0)|\\ &\quad + |\omega_r(aB_0) - \omega_r(aB_r)| + |\omega_r(aB_r)|\\ &\quad < \epsilon + \|a(B_0 - B_r)\| + 0 < \epsilon + \|a\|\epsilon.\end{aligned}$$

Hence $\omega(aB_0) = 0$, i.e.,

$$\omega(aB_0) = \int h(s,t)\omega(a\alpha_{s,t}(b))d(s,t) = 0,$$

therefore ω is a ground state. □

4. **Applications**

In this section we explain an application of discussed concepts in the form of Theorem 4.3. We first recall the following useful theorem and lemma.

Theorem 4.1. *[15]. Let δ (resp. δ') be the infinitesimal generator of a strongly continuous one–parameter group of $*$–automorphisms of C^*–algebra A (resp.B). Then $\delta \otimes I + I \otimes \delta'$ is closable and its closure is an infinitesimal generator of $A \otimes_S B$.*

In the following, $\delta \otimes \delta'$ denotes the derivation $\delta \otimes I + I \otimes \delta'$.

Lemma 4.2. *Let $\{\alpha_{s,t}\}$ and $\{\beta_{s,t}\}$ be strongly continuous two–parameter group with infinitesimal generators (δ_1, δ_2) and (δ'_1, δ'_2) on C^*–algebra A and B respectively. Then the closure of $(\delta_1 \otimes \delta'_1, \delta_2 \otimes \delta'_2)$ is the infinitesimal generator of $\{\alpha_{s,t} \otimes \beta_{s,t}\}$ on $A \otimes_S B$.*

Proof. straightforward. □

Theorem 4.3. *Suppose $\{\alpha_{s,t}\}$ is strongly continuous two-parameter group of $*$–automorphisms of a C^*–algebra A without ground state. If B is an arbitrary unital C^*–algebra with strongly continuous two-parameter group of $*$–automorphisms $\{\beta_{s,t}\}$, then the two-parameter group of automorphisms $\{\alpha_{s,t} \otimes \beta_{s,t}\}$ of $A \otimes B$ is not approximately inner.*

Proof. Suppose that $\{\alpha_{s,t} \otimes \beta_{s,t}\}$ is approximately inner. Let (δ_1, δ_2) and $(\acute{\delta_1}, \acute{\delta_2})$ be infinitesimal generators of $\{\alpha_{s,t}\}$ and $\{\beta_{s,t}\}$ respectively. Since $\{\alpha_{s,t} \otimes \beta_{s,t}\}$ is approximately inner by theorem 3.3 there would be a ground state ω for $\{\alpha_{s,t} \otimes \beta_{s,t}\}$ on $A \otimes B$. Let ϕ be the state on A defined by

$$\phi(a) = \omega(a \otimes e),$$

$(a \in A)$, where e is the unit of B, Since

$$\begin{aligned} -i\phi(a^*\delta_j(a)) &= -i\omega(a^*\delta_j(a) \otimes e) \\ &= -i\omega((a^* \otimes e)(\delta_j \otimes \acute{\delta_j})(a \otimes e)) \geq 0, \end{aligned}$$

for $j = 1, 2$ and $a \in A$.
Therefore $-i\phi(a^*\delta_j(a)) \geq 0$ for $j = 1, 2$ and $a \in A$.
Hence ϕ is a ground state for $\{\alpha_{s,t}\}$ and therefore $\{\alpha_{s,t} \otimes \beta_{s,t}\}$ is not approximately inner. □

Remark 4.4. Theorem 4.3 suggests to state a generalization of Powers–Sakai conjecture that every two–parameter dynamical systems on UHF algebra is approximately inner (see [17]).

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