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REMARKS ON MICROPERIODIC MULTIFUNCTIONS

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Abstract. It is well known that a microperiodic function mapping a topological group into reals, which is continuous at some point is constant. We introduce the notion of a microperiodic multifunction, defined on a topological group with values in a metric space, and study regularity conditions implying an analogous result. We deal with Vietoris and Hausdorff continuity concepts.

Stability of microperiodic multifunctions is considered, namely we show that an approximately microperiodic multifunction is close to a constant one, provided it is continuous at some point. As a consequence we obtain stability result for an approximately microperiodic single-valued function.

Keywords: Multifunction, microperiodic function, functional inequality, functional inclusion.
MSC(2010): Primary: 39B62; Secondary: 39B82, 26E25, 54C60.

1. **Introduction**

It is known that every element of the set $P = \{np + mq : \ n, m \in \mathbb{Z}\}$ is a period of a biperiodic real function $f : \mathbb{R} \to \mathbb{R}$ which is a function satisfying the equalities

$$f(x + p) = f(x) = f(x + q), \qquad x \in \mathbb{R},$$

with $p \neq q$. Moreover the above mentioned set P is dense provided $\frac{p}{q} \notin \mathbb{Q}$ (cf. [5, 19]). Biperiodic functions appear also as a solution of a system of two difference equations

$$\Delta_p(f) = 0 = \Delta_q(f),$$

where $\Delta_p(f)(x) = f(x + p) - f(x)$ is the difference operator (cf. [20]).

According to [19], we say a function f defined on a topological group $(X, \cdot)$ is microperiodic if there exists a dense set $P \subset X$ with

$$f(px) = f(x), \qquad x \in X, p \in P. \tag{1.1}$$

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It is known that a real microperiodic fuction f continuous at some point is constant.

Analogous results for functions satisfying the weaker condition

$$f(px) \geq f(x), \qquad x \in X, p \in P, \tag{1.2}$$

were proved by P. Montel [21] (see also [22], [18, p. 228]). Those results have been generalized and extended in several ways in [5,6,8,14–17,19] (in particular in a connection with some problem arising in a characterization of L^p norm).

J. Brzdęk has asked whether a multifunction in some sense microperiodic is constant. Motivated by this question, we introduce the notion of microperiodic set-valued maps (called multifunctions for brevity), replacing equality in (1.1) by an inclusion. In the third section, applying the above mentioned results for single-valued functions, we present sufficient regularity conditions which imply that such multifunctions are constant.

Section 4 contains some stability results for microperiodic multifunctions corresponding to the notion of Hyers-Ulam stability (see [1, 11, 23] and for the recent results and more references see e.g. [4,12,13]). Namely we show that, under some regularity conditions, an approximately microperiodic multifunction is close (in some sense) to a constant one.

For the convenience of the reader, in the following section some needful facts concerning two kinds of continuity of multifunctions are collected.

2. **Preliminaries and auxiliary results**

Denote by $P(X)$ the family of nonempty subsets of $X \neq \emptyset$. If X is a Hausdorff topological space (linear or metric space, when appropriate) $P_f(X)$, $P_c(X)$, $P_{fc}(X)$, $P_{bf}(X)$, P_{bc}, P_{bfc}, $P_k(X)$ the families of closed, convex, closed convex, bounded closed, bounded convex, bounded closed convex, and finally compact members of $P(X)$, respectively.

Given topological spaces X, Y we say that a multifunction $F : X \to 2^Y$ is upper semicontinuous at $x_0 \in X$ if for every open set $V \subset Y$ such that $F(x_0) \subset V$ there exists $U \in \mathcal{N}(x_0)$ (an open neighbourhood of x_0) such that $F(x) \subset V$ for every $x \in U$. F is lower semicontinuous at $x_0 \in X$ if for every open set $V \subset Y$ such that $F(x_0) \cap V \neq \emptyset$ there exists $U \in \mathcal{N}(x_0)$ such that $F(x) \cap V \neq \emptyset$ for every $x \in U$. We say that F is continuous at x_0 if it is both upper and lower semicontinuous at this point (see [2, 3, 10]). F is said to be semicontinuous on a set $A \subset X$ if it is semicontinuous at every point $x \in A$.

Here and subsequently, $\overline{\mathbb{R}}$ stands for the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$. A single-valued function $f : X \to \overline{\mathbb{R}}$ on a topological space X is lower semicontinuous at $x_0 \in X$ if for every $\varepsilon > 0$ there exists $U \in \mathcal{N}(x_0)$ such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U$. The function f is upper semicontinuous if $-f$ is lower semicontinuous (cf. Definition A.1.29 and Proposition A.1.30 in [10]).

We will abbreviate upper and lower semicontinuity to usc and lsc, respectively.

If Y is a topological vector space, then Y^* denotes the topological dual of Y, i.e., the space of all continuous linear functionals on Y. In this case, semicontinuity of a multifunction is related to semicontinuity of the support function $x \mapsto \sigma(y^*, F(x)) := \sup\{(y^*, y) : \; y \in F(x)\}$, $x \in X$, $y^* \in Y^*$. It is easy to proof the following local versions of Prop. I.2.29, I.2.35 form [10].

Lemma 2.1. *Let Y be a normed space furnished with the weak topology. If $F : X \to P(Y)$ is usc at x_0, then for every $y^* \in Y^*$ the support function is usc at x_0.*

Lemma 2.2. *Let Y be a locally convex topological space. If $F : X \to P_{fc}(Y)$ is lsc at x_0, then for every $y^* \in Y^*$ the support function is lsc at x_0.*

To define another concept of continuity for multifunctions, recall the Hausdorff pseudometric. If (Y, d) is a metric space, following [10, Def. I.1.1] (cf. also [9]), for every $A, B \subset Y$ define excess of A over B and the Hausdorff distance between A, B

$$e(A, B) := \sup\{d(a, B) : \; a \in A\},$$
$$h(A, B) := \max\{e(A, B), e(B, A)\},$$

where $d(a, B) := \inf\{d(a, b), b \in B\}$. It is easily seen that the excess has the following properties.

Remark 2.3. If $A, B, C \in P(X)$, then

$$A \subset B \Longrightarrow e(A, C) \leq e(B, C), \tag{2.1}$$
$$B \subset C \Longrightarrow e(A, B) \geq e(A, C). \tag{2.2}$$

The function $h : P(Y) \times P(Y) \to [0, +\infty]$ is a pseudometric on $P(Y)$ with the property

$$h(A, B) = 0 \Longrightarrow \overline{A} = \overline{B}, \qquad A, B \in P(Y).$$

Therefore $(P_{bf}(Y), h)$ h is a metric space. Furthermore we have the following formulas.

Lemma 2.4. *[10, Th. I.1.13, Rem. I.1.14] If Y is a normed space $A, B \in P_{bfc}(Y)$, then*

$$e(A, B) = \sup\{\sigma(y^*, A) - \sigma(y^*, B) : \; \|y^*\| \leq 1\},$$
$$h(A, B) = \sup\{|\sigma(y^*, A) - \sigma(y^*, B)| : \; \|y^*\| \leq 1\}.$$

Here we assume that X is a Hausdorff topological space, Y is a metric space. A multifunction $F : X \to P(Y)$ is said to be h-upper semicontinuous (h-usc for short) at $x_0 \in X$ if $x \mapsto e(F(x), F(x_0))$ is continuous at x_0, it is h-lower semicontinuous (h-lsc for short) at x_0 if $x \mapsto e(F(x_0), F(x))$ is continuous at

x_0. We say that F is h-continuous if it is h-upper semicontinuous and h-lower semicontinuous at x_0.

The notions of semicontinuity and h-semicontinuity are not equivalent in general. For the convenience of the reader we remind some theorems comparing the continuity concepts (cf. [10, Prop. I.2.61, I.2.66]) and recal some usefull properties.

Lemma 2.5. *If $F : X \to P(Y)$ is usc at $x_0 \in X$, then it is h-usc at x_0.*

Lemma 2.6. *If $F : X \to P(Y)$ is h-lsc at $x_0 \in X$, then it is lsc at x_0.*

In the case of compact-valued multifunctions the notions of semicontinuity and h-semicontinuity are equivalent (see [10, Prop. I.2.68 and I.2.69]). If F is single-valued, then the mentioned above concepts of continuity reduce to continuity.

If Y is a topological space we denote by $\overline{A}$ the closure of $A \subset Y$. Given a multifunction $F : X \to P(Y)$ we define the multifunction $\overline{F} : X \to P_f(Y)$ by $\overline{F}(x) := \overline{F(x)}$, $x \in X$.

3. Microperiodic multifunctions

Let $(X, \cdot)$ be a topological group and let $P \subset X$. Observe that a function $f : X \to \overline{\mathbb{R}}$ satisfying the condition

$$f(px) \leq f(x), \qquad x \in X, p \in P, \tag{m}$$

satisfies (1.2) with the set $P^{-1} = \{p^{-1} : \ p \in P\}$. Moreover, if the set P is dense in X, then P^{-1} is dense as well. Therefore without loss of generality we consider functions fulfilling the condition (m).

We first prove elementary properties of semicontinuous microperiodic functions.

Lemma 3.1. *Let $(X, \cdot)$ be a topological group, P dense subset of X. If $f : X \to \overline{\mathbb{R}}$ satisfying (m) is lsc (usc respectively) at some $x_0 \in X$, then*

$$f(x) \geq (\leq) f(x_0) \qquad x \in X.$$

Proof. Assume that f is lsc at $x_0 \in X$. Fix $x \in X$ and take any $\varepsilon > 0$. By the semicontinuity of f, there exists an open neighborhood U of x_0 such that $f(y) \geq f(x_0) - \varepsilon$ for every $y \in U$. Since P is dense, there exists $p \in P$ with $px \in U$. From the above and (m) we have

$$f(x) \geq f(px) \geq f(x_0) - \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain our assertion.

The proof for usc function f is similar or we an capply the previous case for the function $-f$, which is microperiodic and lsc. □

As a consequence of the above lemma we get the following generalization of Theorem 2 in [5] to the class of functions with values in the extended real line.

Proposition 3.2. *Let $(X, \cdot)$ be a topological group, P dense subset of X. If $f : X \to \overline{\mathbb{R}}$ satisfying (m) is continuous at some $x_0 \in X$, then it is constant.*

We introduce the notion of microperiodic multifunction, replacing in (1.1) the equality by an inclusion.

Definition 3.3. We say that a multifunction $F : X \to 2^Y$ is microperiodic if there exists a dense subset P of X such that

$$F(px) \subset F(x), \qquad x \in X, p \in P. \tag{M}$$

Observe that, similarly to single-valued functions, the inclusion in the above condition can be replaced by the opposite one, which holds with the dense set P^{-1}.

Let $A, B \subset Y$. Observe that a multifunction $F : \mathbb{R} \to 2^Y$ of the form

$$F(x) = \begin{cases} A & \text{for } x \in \mathbb{Q}, \\ B & \text{for } x \notin \mathbb{Q} \end{cases} \tag{3.1}$$

is microperiodic. The following examples present microperiodic multifunctions $F : \mathbb{R} \to P(\mathbb{R})$ which are not constant, although they are semicontinuous or even continuous at some point.

Example 3.4. The multifunction $F(x) = \begin{cases} [0,1] & \text{for } x \in \mathbb{Q}, \\ \{0\} & \text{for } x \notin \mathbb{Q} \end{cases}$ is microperiodic, usc on $\mathbb{Q}$ (h-usc on $\mathbb{Q}$ in view of Lemma 2.5), h-lsc on $\mathbb{R} \setminus \mathbb{Q}$ (lsc on $\mathbb{R} \setminus \mathbb{Q}$, according to Lemma 2.6).

Example 3.5. The multifunction $F(x) = \begin{cases} [0,1] & \text{for } x \in \mathbb{Q}, \\ (0,1) & \text{for } x \notin \mathbb{Q} \end{cases}$ is microperiodic, h-continuous on $\mathbb{R}$, since $h([0,1], (0,1)) = 0$, therefore it is lsc on $\mathbb{R}$. It is easy to show that it is usc only on $\mathbb{Q}$.

Example 3.6. The multifunction $F(x) = \begin{cases} (0,1) & \text{for } x \in \mathbb{Q}, \\ (0,1) \setminus \{\frac{1}{2}\} & \text{for } x \notin \mathbb{Q} \end{cases}$ is microperiodic, h-continuous on $\mathbb{R}$, since $h((0,1), (0,1) \setminus \{\frac{1}{2}\}) = 0$, therefore it is lsc on $\mathbb{R}$. It is easy to show that it is usc only on $\mathbb{Q}$.

Example 3.7. The multifunction $F(x) = \begin{cases} [0,1] & \text{for } x \in \mathbb{Q}, \\ \mathbb{Q} \cap [0,1] & \text{for } x \notin \mathbb{Q} \end{cases}$ is microperiodic, h-continuous on $\mathbb{R}$, since $h([0,1], [0,1] \cap \mathbb{Q}) = 0$, therefore it is lsc on $\mathbb{R}$. It is easy to show that it is usc only on $\mathbb{Q}$.

Remark 3.8. Same as in [5], it is easily seen that a multifunction F defined on a commutative group $(X, +)$ satisfying the inclusions

$$\begin{aligned} F(x + p_i) &\subset F(x), \qquad x \in X, i \in \{1, \dots, k\} \\ F(x + q_i) &\supset F(x), \qquad x \in X, i \in \{1, \dots, j\}, \end{aligned}$$

where $k, j \in \mathbb{N}$, $p_1, \dots p_k, q_1, \dots, q_j \in X$ fulfills (M) with the set

$$P := \{\sum_{i=1}^{k} n_i p_i - \sum_{i=1}^{j} m_i q_i : \ n_1, \dots n_k, m_1, \dots m_j \in \mathbb{N}\}.$$

F is microperiodic provided P is dense in the topological group X.

Particulary, a multifunction $F : \mathbb{R} \to P(\mathbb{R})$ such that

$$F(x+1) \subset F(x) \subset F(x+q), \qquad x \in X,$$

where $q \in (0, +\infty) \setminus \mathbb{Q}$ is microperiodic.

Theorem 3.9. *Let X be a topological group, Y a metric space, $F : X \to P(Y)$ a microperiodic multifunction. If F is h-continuous at some x_0, then the multifunction $\overline{F}$ is constant.*

Proof. Define $f(x) := e(F(x_0), F(x))$, $x \in X$. According to (M) and (2.2)

$$f(x) \leq f(px), \qquad x \in X, p \in P. \tag{3.2}$$

Since F is h-continuous at x_0, f is continuous at x_0. Therefore by Proposition [5], f is constant. Consequently

$$e(F(x_0), F(x)) = f(x) = f(x_0) = e(F(x_0), F(x_0)) = 0, \qquad x \in X.$$

Thus

$$F(x_0) \subset \overline{F}(x), \qquad x \in X. \tag{3.3}$$

On the other hand, from (2.1) the function $g(x) := e(F(x), F(x_0))$, $x \in X$ satisfies

$$g(px) \leq g(x), \qquad x \in X, p \in P. \tag{3.4}$$

Moreover, by h-continuity of F, g is continuous and therefore constant (see Proposition 3.2. It follows that

$$e(F(x), F(x_0)) = g(x) = g(x_0) = e(F(x_0), F(x_0)) = 0, \qquad x \in X,$$

which yields

$$F(x) \subset \overline{F}(x_0), \qquad x \in X. \tag{3.5}$$

Combining (3.3) and (3.5), we get for every $x \in X$

$$F(x) \subset \overline{F}(x_0) \subset \overline{F}(x),$$

hence

$$\overline{F}(x) \subset \overline{F}(x_0) \subset \overline{F}(x),$$

which completes the proof. □

According to Lemma 2.5 we have what follows.

Corollary 3.10. *Let X be a topological group, Y a metric space, $F : X \to P(Y)$ a microperiodic multifunction. If F is usc and h-lsc at some x_0, then the multifunction $\overline{F}$ is constant.*

The case of multifunctions with closed values is analogous to the case of single valued functions.

Corollary 3.11. *Let X be a topological group, Y a metric space and let multifunction $F : X \to P_f(Y)$ be microperiodic. If F is h-continuous at some x_0, then it is constant.*

Observe that if values of F are singletons, h-continuity is just continuity of a single-valued function. Therefore the above proposition is a generalization of the fact that a microperiodic function continuous at some point is constant. If F has compact values, then continuity is equivalent to h-continuity, and we get what follows.

Corollary 3.12. *Let X be a topological group, Y a metric space and let multifunction $F : X \to P_k(Y)$ be microperiodic. If F is continuous at some x_0, then it is constant.*

Theorem 3.13. *Assume that X is a topological group, Y a normed space, $F : X \to P_{fc}(Y)$ is a microperiodic multifunction. If F is lsc at x_0 and usc at x_0 (Y endowed with the weak topology), then it is constant.*

Proof. Fix $y^* \in Y^*$. By our assumptions and Lemmas 2.1 and 2.2, the support function $f_{y^*}(x) := \sigma(y^*, F(x))$, $x \in X$ is continuous at x_0. According to condition (M),

$$f_{y^*}(px) = \sigma(y^*, F(px)) \le \sigma(y^*, F(x)) = f_{y^*}(x), \qquad x \in X, p \in P.$$

Therefore on account of Proposition 3.2, f_{y^*} is constant. Consequently for every $x \in X$

$$\sigma(y^*, F(x)) = \sigma(y^*, F(x_0)), \qquad y^* \in Y^*.$$

By the separation theorem (cf. [2, Th. 2.4.2]) $F(x) = F(x_0)$, $x \in X$, since values of F are closed and convex. □

If multifunction F with convex values is continuous at some point, then $\overline{F}$ satisfies assumptions of the above theorem (see [10, Prop. 2.38, 2.40]), therefore we have the following corollary.

Corollary 3.14. *Assume that X is a topological group, Y a normed space, $F : X \to P_c(Y)$ is a microperiodic multifunction. If F is continuous at x_0, then $\overline{F}$ is constant.*

Examples 3.5 – 3.7, show that lower semicontinuity on the whole domain of a microperiodic multifunction do not imply it is constant, as well as upper semicontinuity on a dense set. Below we give sufficient conditions for a multifunction with open values.

Theorem 3.15. *Assume that X is a topological group, Y a topological space, $F : X \to P(Y)$ a microperiodic multifunction with open values. If F is usc on some open neighborhood of x_0, then it is constant.*

Proof. Let $U \in \mathcal{N}_{x_0}$ be an open neighborhood of x_0 such that F is usc on U. Since $F(x_0)$ is open, there exists $W \in \mathcal{N}_{x_0}$ such that

$$F(x) \subset F(x_0), \qquad x \in W.$$

We show that f is constant on $U \cap W$. Fix $x \in U \cap W$. Then F is usc at x and $F(x) \subset V := F(x)$. Therefore there exists $U_x \in \mathcal{N}_x$ satisfying

$$F(y) \subset V = F(x), \qquad y \in U_x.$$

Since P is dense, there exists $p \in P$ with $p^{-1}x_0 \in U_x$ and thus

$$F(x_0) = F(pp^{-1}x_0) \subset F(p^{-1}x_0) \subset F(x) \subset F(x_0).$$

Now take any $y \in X$. Then there exists $p, q \in P$ such that $p^{-1}y, qy \in U \cap W$ and

$$F(y) = F(pp^{-1}y) \subset F(p^{-1}y) = F(x_0) = F(qy) \subset F(y),$$

which completes the proof. □

4. Approximately microperiodic multifunctions

The notion of approximately microperiodic functions is connected to the issue of Hyers-Ulam type stability and its generalizations (see e.g. [4, 12, 13]). This section is motivated by the paper [7]. We show that, under some continuity assumptions, a multifunction satisfying the inclusion (M) approximately in some sense is close to a constant multifunction, which is a solution of this inclusion (as was shown in the previous section).

From now on we will need the following assumptions

$$\text{(A)} \quad \begin{cases} (X, \cdot) \text{ is a topological group, } P \subset X \text{ is a dense set,} \\ (Y, d) \text{ is a metric space, } \varepsilon : X \to [0, +\infty). \end{cases}$$

Theorem 4.1. *Under the assumptions* (A)*, let $F : X \to P(Y)$ satisfies*

$$e(F(px), F(x)) \le \varepsilon(x), \qquad x \in X, p \in P \tag{4.1}$$

$$\Big(e(F(x), F(px)) \le \varepsilon(x), \qquad x \in X, p \in P\Big). \tag{4.2}$$

If F is h-continuous at some x_0, then

$$e(F(x_0), F(x)) \le \varepsilon(x), \qquad x \in X$$

$$\Big(e(F(x), F(x_0)) \le \varepsilon(x), \qquad x \in X \text{ respectively}\Big).$$

Proof. Assume that (4.1) holds. Fix $n \in \mathbb{N}$. Since F is continuous at $x_0 \in X$, there exists an open neighbourhood U_n of x_0 such that

$$h(F(x), F(x_0)) < \frac{1}{n}, \qquad x \in U_n.$$

Take any $x \in X$. By the density of P, there exists $p_n \in P$, such that $p_n x \in U_n$. Therefore for every $n \in \mathbb{N}$

$$e(F(x_0), F(x)) \leq e(F(x_0), F(p_n x)) + e(F(p_n x), F(x)) < \frac{1}{n} + \varepsilon(x).$$

Letting $n \to \infty$ we obtain the first assertion. The proof of the second one is simlar. □

Below we give an example, which shows that the assumption on h-continuity of F is essential in the above theorem.

Example 4.2. Multifunction $F : \mathbb{R} \to P(\mathbb{R})$ of the form (3.1) with $A = \{1\}$, $B = \{0\}$ is microperiodic with $P = \mathbb{Q}$. It is not h-continuous at each $x_0 \in \mathbb{R}$, and satisfies

$$h(F(px), F(x)) = 0 < \frac{1}{2} =: \varepsilon(x), \qquad x \in X, p \in P.$$

On the other hand, $h(F(x), F(y)) > \frac{1}{2}$ for all $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$.

Note that applying [7, Th. 2], we immediately obtain its counterpart for multifunctions with bounded closed values and a constant function ε. By our Theorem 4.1, we obtain a more general result.

Corollary 4.3. *Under the assumptions* (A)*, let* $F : X \to P(Y)$ *satisfy*

$$h(F(px), F(x)) \leq \varepsilon(x), \qquad x \in X, p \in P. \tag{4.3}$$

If F *is* h*-continuous at some* x_0*, then*

$$h(F(x), F(x_0)) \leq \varepsilon(x), \qquad x \in X.$$

Applying the above corollary for the multifunction $F(x) = \{f(x)\}$, where $f : X \to Y$, we have a kind of a generalization of the result for single-valued functions.

Corollary 4.4. *Under the assumptions* (A)*, let* $f : X \to Y$ *satisfy the inequality*

$$d(f(px), f(x)) \leq \varepsilon(x), \qquad x \in X, p \in P.$$

If f *is continuous at some* x_0*, then*

$$d(f(x), f(x_0)) \leq \varepsilon(x), \qquad x \in X.$$

In order to relax the assumption (4.3) (replacing the Hausdorff metric by an excess), we prove the following theorem.

Theorem 4.5. *Assume* (A)*. Let ε be usc at some x_0 and let $F : X \to P(Y)$ satisfy the inequality* (4.1) *(or satisfies* (4.2)*). If F is h-continuous at x_0, then*

$$e(F(x), F(x_0)) \le \varepsilon(x_0), \qquad x \in X$$

$$\Big(e(F(x_0), F(x)) \le \varepsilon(x_0), \qquad x \in X \textit{ respectively.}\Big)$$

Proof. Assume that (4.1) holds and substitute x by $p^{-1}x$ in this condition. It follows that

$$e(F(x), F(p^{-1}x)) \le \varepsilon(p^{-1}x), \qquad x \in X, p \in P.$$

Fix $n \in \mathbb{N}$. Since F is continuous at $x_0 \in X$, there exists $U_n \in \mathcal{N}(x_0)$ with

$$h(F(x), F(x_0)) < \frac{1}{n}, \qquad x \in U_n.$$

By the upper semicontinuity of ε, there exists $V_n \in \mathcal{N}(x_0)$ such that

$$\varepsilon(x) \le \varepsilon(x_0) + \frac{1}{n}, \qquad x \in V_n.$$

Take any $x \in X$. As P^{-1} is dense, there exists $p_n \in P$ such that $p_n^{-1}x \in U_n \cap V_n$. Therefore for every $n \in \mathbb{N}$

$$\begin{aligned} e(F(x), F(x_0)) &\le e(F(x), F(p_n^{-1}x)) + e(F(p_n^{-1}x), F(x_0)) + < \varepsilon(p_n^{-1}x) + \frac{1}{n} \\ &\le \varepsilon(x_0) + \frac{1}{n} + \frac{1}{n}. \end{aligned}$$

Letting $n \to \infty$ we obtain our assertion. Similar proof works for the second one. $\square$

Combining Theorems 4.1 and 4.5 we obtain what follows.

Corollary 4.6. *Under the assumptions* (A)*, let ε be usc at some x_0 and let $F : X \to P(Y)$ satisfy one of the conditions* (4.1) *or* (4.2)*. If F is h-continuous at x_0, then*

$$h(F(x), F(x_0)) \le \max\{\varepsilon(x), \varepsilon(x_0)\}, \qquad x \in X.$$

We now apply the above result, to get a counterpart of Corollary 4.4 for single-valued functions with real values, which generalizes [7, Th. 1].

Corollary 4.7. *Under the assumptions* (A)*, let ε be usc at some x_0 and let $f : X \to \mathbb{R}$ satisfy one of the conditions*

$$\begin{aligned} f(px) - f(x) &\le \varepsilon(x), \qquad x \in X, p \in P, \\ f(x) - f(px) &\le \varepsilon(x), \qquad x \in X, p \in P. \end{aligned} \tag{4.4}$$

If f is continuous at x_0, then

$$| f(x) - f(x_0) | \le \max\{\varepsilon(x), \varepsilon(x_0)\}, \qquad x \in X.$$

Proof. Assume that (4.4) is fulfilled and define $F(x) := (-\infty, f(x)]$, $x \in X$. Then $| f(x) - f(x_0) | = h(F(x), F(x_0))$, $x \in X$. It is easy to check that F satisfies the assumptions of Corollary 4.6, which completes the proof. □

Note that taking subsets A, B, C of a normed space such that $A \subset B + C$ we have $e(A, B) \leq \|C\|$, where $\|C\| = \sup\{\|c\| : c \in C\}$. Therefore Corollary 4.6 may be formulated in the following way.

Corollary 4.8. *Let* (A) *holds. Assume that Y is a normed space, $G : X \to P(Y)$ is a multifunction such that $\|G\|$ is usc at some $x_0 \in X$. If $F : X \to P(Y)$ is h-continuous at x_0 and satisfies one of the conditions*

$$F(px) \subset F(x) + G(x), \qquad x \in X, p \in P, \tag{4.5}$$

$$F(x) \subset F(px) + G(x), \qquad x \in X, p \in P, \tag{4.6}$$

then

$$h(F(x), F(x_0)) \leq \max\{G(x), G(x_0)), \qquad x \in X.$$

As a consequence of the above we get some kind of a generalization of [7, Th. 2].

Corollary 4.9. *Assume that X is a topological group, $P \subset X$ a dense set, Y is a normed space. Let $C \subset Y$ and let $f : X \to Y$ satisfy one of the conditions*

$$f(px) \in f(x) + C, \qquad x \in X, p \in P,$$
$$f(x) \in f(px) + C, \qquad x \in X, p \in P,$$

If f is continuous at some x_0, then

$$\|f(x) - f(x_0)\| \leq \|C\|, \qquad x \in X.$$

In the case of Vietoris continuity, we appply Corollary 4.7 and properties of the support function for multifunctions with closed convex and bounded values.

Theorem 4.10. *Under the assumptions* (A)*, let ε be usc at some $x_0 \in X$ and let $F : X \to P_{bfc}(Y)$ satisfy* (4.1) *or* (4.2)*. If F is lsc at x_0 and usc at x_0 (Y endowed with the weak topology), then*

$$h(F(x), F(x_0)) \leq \max\{\varepsilon(x), \varepsilon(x_0)\}, \qquad x \in X.$$

Proof. Assume that (4.1) is fulfilled. By Lemma 2.4, for every $y^* \in Y^*$, $x \in X$, $p \in P$

$$\sigma(y^*, F(px)) - \sigma(y^*, F(x)) \leq \varepsilon(x).$$

Take $y^* \in Y^*$ with $\|y^*\| \leq 1$ and set $f(x) := \sigma(y^*, F(x)) \in \mathbb{R}$, $x \in X$. It follows that $f(px) - f(x) \leq \varepsilon(x)$ for every $x \in X$. By our assumptions and Lemmas 2.1 and 2.2, f is continuous at x_0. On account of Corollary 4.7

$$| f(x) - f(x_0) | \leq \max\{\varepsilon(x), \varepsilon(x_0)\}, \qquad x \in X,$$

and consequently for every $x \in X$ and $y^* \in Y^*$ with $\|y^*\| \le 1$ we have

$$| \sigma(y^*, F(x)) - \sigma(y^*, F(x_0)) | \le \max\{\varepsilon(x), \varepsilon(x_0)\}.$$

Lemma 2.4 completes the proof. □

Finally we have a counterpart of Corollary 4.8 for a multifunction continuous in the sense of Vietoris.

Corollary 4.11. *Let* (A) *holds. Assume that Y is a normed space, $G : X \to P(Y)$ is a multifunction such that $\|G\|$ is usc at some $x_0 \in X$. If $F : X \to P_{bc}(Y)$ is continuous at x_0 and satisfies* (4.5) *or* (4.6)*, then*

$$h(F(x), F(x_0)) \le \max\{\|G(x)\|, \|G(x_0)\|), \qquad x \in X.$$

Proof. Assume that (4.5) holds. Since F is continuous at x_0 so $\overline{F}$ is (see [10, Prop. 2.41]). Therefore $\overline{F}$ is usc at x_0 with respect to the weak topology in Y and its values belong to $P_{bfc}(Y)$. Moreover

$$e(\overline{F}(px), \overline{F}(x) = e(F(px), F(x)) \le \|G(x)\|, \qquad x \in X, p \in P.$$

Applying the above theorem, we have

$$h(F(x), F(x_0)) = h(\overline{F}(x), \overline{F}(x_0)) \le \max\{\|G(x)\|, \|G(x_0))\|, \qquad x \in X,$$

which finishes the proof. □

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