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ON LINEAR PRESERVERS OF SGUT-MAJORIZATION ON $\mathbf{M}_{n,m}$

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Abstract. Let $\mathbf{M}_{n,m}$ be the set of n-by-m matrices with entries in the field of real numbers. A matrix R in $\mathbf{M}_n = \mathbf{M}_{n,n}$ is a generalized row substochastic matrix (g-row substochastic, for short) if $Re \leq e$, where $e = (1, 1, \ldots, 1)^t$. For X, $Y \in \mathbf{M}_{n,m}$, X is said to be sgut-majorized by Y (denoted by $X \prec_{sgut} Y$) if there exists an n-by-n upper triangular g-row substochastic matrix R such that $X = RY$. This paper characterizes all linear preservers and strong linear preservers of $\prec_{sgut}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$ respectively.
Keywords: Linear preserver, strong linear preserver, g-row substochastic matrices, sgut-majorization.
MSC(2010): Primary: 15A03, 15A04; Secondary: 15A51.

1. Introduction

Vector majorization is a much studied concept in linear algebra and its applications. The reader can find that majorization has been connected with combinatorics, analytic inequalities, numerical analysis, matrix theory, probability and statistics in a book written by Marshall, Olkin, and Arnold [13]. Several generalization of this concept have also been introduced. For more information we refer the reder to [2–12] . The purpose of this paper is introducing and studying a new type of generalized majorization. For more information on the type of majorization and linear preservers of majorization see [1] and [14].

Let $\mathcal{V}$ be a linear space of matrices, T be a linear function on $\mathcal{V}$, and $\mathcal{R}$ be a relation on $\mathcal{V}$. The linear function T is said to preserve $\mathcal{R}$, if $\mathcal{R}(\mathcal{TX}, \mathcal{TY})$ whenever $\mathcal{R}(\mathcal{X}, \mathcal{Y})$. Also, T is said to strongly preserve $\mathcal{R}$, if

$$\mathcal{R}(\mathcal{TX}, \mathcal{TY}) \Leftrightarrow \mathcal{R}(\mathcal{X}, \mathcal{Y}).$$

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Throughout this paper, let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices, $\mathbb{R}^n$ be the set of all n-by-1 real column vectors, $\{e_1,\ldots,e_n\}$ be the standard basis of $\mathbb{R}^n$, $A(n_1,\ldots,n_l|m_1,\ldots,m_k)$ be the submatrix of A obtained from A by deleting rows $n_1,\ldots,n_l$ and columns $m_1,\ldots,m_k$, $A(n_1,\ldots,n_l)$ be the abbreviation of $A(n_1,\ldots,n_l|n_1,\ldots,n_l)$, $\mathbb{N}_k$ be the set $\{1,\ldots,k\}\subset\mathbb{N}$, $J_{n,m}$ be the $n\times m$ matrix with all of the entries equal to one, E be the n-by-n matrix with all of the entries of the last column equal to one and the other entries equal to zero, A^t be the transpose of a given matrix $A\in\mathbf{M}_{n,m}$, card(S) be the cardinal number of a set S, where S is a finite set, $[T]$ be the matrix representation of a linear function $T:\mathbb{R}^n\to\mathbb{R}^n$ with respect to the standard basis, $diag(a_1,\ldots,a_n)$ be the matrix $A=[a_{ij}]\in\mathbf{M}_n$ such that $a_{ii}=a_i$ for each $i=1,\ldots,n$ and $a_{ij}=0$ if $i\neq j$, r_i be the sum of entries on the ith row of $[T]$, and $\mathcal{A}(S)$ be the set $\{\sum_{i=1}^m\lambda_i a_i \mid m\in\mathbb{N},\ \sum_{i=1}^m\lambda_i\leq 1,\ a_i\in S,\ \forall i\in\mathbb{N}_m\}$, where $S\subseteq\mathbb{R}^n$, aff(S) be the set
$\{\sum_{i=1}^m\lambda_i a_i \mid m\in\mathbb{N},\ \sum_{i=1}^m\lambda_i=1,\ a_i\in S,\ \lambda_i\in\mathbb{R},\ \forall i\in\mathbb{N}_m\}$, where $S\subseteq\mathbb{R}^n$.

A real matrix R is called *g-row stochastic* provided that each its row sums is equal to one. For $X,Y\in\mathbf{M}_{n,m}$, X is said to be *gut-majorized* by Y, and write $X\prec_{gut}Y$, if there exists an n-by-n upper triangular g-row stochastic matrix R such that $X=RY$. In [4], the authors, obtained the structure of linear preservers and strong linear preservers of $\prec_{gut}$ on $\mathbb{R}^n$ and $\mathbf{M}_{n,m}$ respectively. In fact, they proved the following theorems:

Theorem 1.1. *Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear function. Then T preserves $\prec_{gut}$ if and only if one of the following assertions hold:*
(i) $Te_1=\cdots=Te_{n-1}=0$. In other words

$$[T]=\begin{pmatrix}0&\ldots&0&a_{1n}\\0&\ldots&0&a_{2n}\\\vdots&\vdots&\vdots&\vdots\\0&\ldots&0&a_{nn}\end{pmatrix}.$$

(ii) There exist $t\in\mathbb{N}_{n-1}$ and $1\leq i_1<\cdots<i_m\leq n-1$ such that $a_{i_1t},a_{i_2t+1},\ldots,a_{i_mn-1}\neq 0$,

$$[T]=\begin{pmatrix}0&*&&&&\\&a_{i_1t}&&&*&\\&&\ddots&&&\\&&&a_{i_2t+1}&&\\&&&&\ddots&\\&0&&&a_{i_mn-1}&\\&&&&0&*\end{pmatrix},$$

and $r_{i_k}\in\mathrm{aff}\{r_{i_k+1},\ldots,r_n\}$ *for all* $k\in\mathbb{N}_m$.

Let $\mathcal{R}_n^{gut}$ be the collection of all n-by-n upper triangular g-row stochastic matrices.

Theorem 1.2. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear function. Then T strongly preserves $\prec_{gut}$ if and only if $TX = AXR + EXS$ for some $R, S \in \boldsymbol{M}_m$ and invertible matrix $A \in \mathcal{R}_n^{gut}$, such that $R(R + S)$ is invertible.*

In this work, we focus on the upper triangular g-row substochastic matrices and introduce a new type of majorization.

Definition 1.3. A matrix $R \in \mathbf{M}_n$ is called *g-row substochastic* if all its row sums is less than or equal to one.

Let $\mathcal{RS}_n^{gut}$ be the collection of all n-by-n upper triangular g-row substochastic matrices.

Definition 1.4. Let $X, Y \in \mathbf{M}_{n,m}$. We say that X is *sgut-majorized* by Y (denoted by $X \prec_{sgut} Y$) if $X = RY$, for some $R \in \mathcal{RS}_n^{gut}$.

This paper is organized as follows. In section 2, we state a necessary and sufficient condition for $x \prec_{sgut} y$ and some properties of sgut-majorization on $\mathbb{R}^n$. Then we characterize all (strong) linear preservers of sgut-majorization on $\mathbb{R}^n$. The last section of this paper studies some facts of this concept that are necessary for studying the strong linear preservers of $\prec_{sgut}$ on $\mathbf{M}_{n,m}$. Also, the strong linear preservers of $\prec_{sgut}$ on $\mathbf{M}_{n,m}$ are obtained.

2. Sgut-majorization on $\mathbb{R}^n$

In this section we state some properties of sgut-majorization on $\mathbb{R}^n$. Also, we characterize all (strong) linear preservers of sgut-majorization on $\mathbb{R}^n$.
The following proposition can be easily obtained from the definition of sgut-majorization.

Proposition 2.1. *Let $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$. Then $x \prec_{sgut} y$ if and only if $x_i \in \mathcal{A}\{y_i, \ldots, y_n\}$, for all $i \in \mathbb{N}_n$.*

Now, we state some lemmas, which are necessary to prove the main results.

Lemma 2.2. *Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{sgut}$ and let $S : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the linear function with $[S] = [T](1, \ldots, k)$. Then S preserves $\prec_{sgut}$ on $\mathbb{R}^{n-k}$.*

Proof. Consider $x' = (x_{k+1}, \ldots, x_n)^t$, $y' = (y_{k+1}, \ldots, y_n)^t \in \mathbb{R}^{n-k}$ such that $x' \prec_{sgut} y'$. Proposition 2.1 ensures that $x := (0, \ldots, 0, x_{k+1}, \ldots, x_n)^t \prec_{sgut} y := (0, \ldots, 0, y_{k+1}, \ldots, y_n)^t$, where $x,\ y \in \mathbb{R}^n$, and so $Tx \prec_{sgut} Ty$. This implies that $Sx' \prec_{sgut} Sy'$. Therefore, S preserves $\prec_{sgut}$ on $\mathbb{R}^{n-k}$, as desired. □

Lemma 2.3. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\prec_{sgut}$, then $[T]$ is upper triangular.*

Proof. Assume that $[T] = [a_{ij}]$. If $n = 1$; Then $A = [a_{11}]$ and the result is trivial. We proceed by induction on n. Suppose that $n \geq 2$ and that the assertion has been established for all linear preservers of $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. Lemma 2.2 ensures that S preserves $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. According to the induction hypothesis $[S]$ is an $n-1$-by-$n-1$ upper triangular matrix. So it is enough to show that $a_{21} = \cdots = a_{n1} = 0$. As $e_1 \prec_{sgut} e_2$, we observe that $Te_1 \prec_{sgut} Te_2$ and hence $(a_{11}, \ldots, a_{n1})^t \prec_{sgut} (a_{12}, a_{22}, 0, \ldots, 0)^t$. This shows that $a_{31} = \cdots = a_{n1} = 0$. So it remains to prove that $a_{21} = 0$. Assume, if possible, that $a_{21} \neq 0$. By setting $x = e_1$ and $y = (\frac{-a_{22}}{a_{21}}, 1, 0, \ldots, 0)^t$, we observe that $x \prec_{sgut} y$, and then $Tx \prec_{sgut} Ty$. This ensures that $a_{21} = 0$, which is a contradiction. Hence $a_{21} = 0$ and the proof is complete. ☐

Lemma 2.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear preserver of $\prec_{sgut}$, and let $[T] = [a_{ij}]$. If there exist some $k, t \in \mathbb{N}_{n-1}$ such that $a_{kt} \neq 0$, and $a_{k+1t} = a_{k+2t} = \cdots = a_{nt} = 0$, then for each j $(t+1 \leq j \leq n)$ there is some l $(k+1 \leq l \leq n)$ such that $a_{lj} \neq 0$.*

Proof. Since T preserves $\prec_{sgut}$ if and only if αT preserves $\prec_{sgut}$, for all $\alpha \in \mathbb{R} \setminus \{0\}$, we can assume without loss of generality that $a_{kt} = 1$. Suppose that there exists some j $(t+1 \leq j \leq n)$ such that $a_{k+1j} = a_{k+2j} = \cdots = a_{nj} = 0$. Let $x = e_t$ and $y = -a_{kj}e_t + e_j$. Then $x \prec_{sgut} y$, but $Tx \not\prec_{sgut} Ty$. This contradiction shows that for each j $(t+1 \leq j \leq n)$ $a_{lj} \neq 0$, for some l $(k+1 \leq l \leq n)$. ☐

In the following theorem we characterize the structure of linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving sgut-majorization.

Theorem 2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function, and let $[T] = [a_{ij}]$. Then T preserves $\prec_{sgut}$ if and only if one of the following conditions hold:*
(a) $Te_1 = \cdots = Te_{n-1} = 0$. In other words

$$[T] = \begin{pmatrix} 0 & \ldots & 0 & a_{1n} \\ 0 & \ldots & 0 & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & a_{nn} \end{pmatrix}.$$

(b) *There exist* $t \in \mathbb{N}_{n-1}$ *and* $1 \leq i_1 < \cdots < i_m \leq n$ *such that* $a_{i_1 t}, a_{i_2 t+1}, \ldots, a_{i_m n} \neq 0$,

$$[T] = \begin{pmatrix} 0 & * & & & & & & \\ & a_{i_1 t} & & & * & & & \\ & & \ddots & & & & & \\ & & & a_{i_2 t+1} & & & & \\ & & & & \ddots & & & \\ & 0 & & & & a_{i_{m-1} n-1} & & \\ & & & & & & \ddots & \\ & & & & & & & a_{i_m n} \\ & & & & & & & * \end{pmatrix},$$

and one of the following statement happens.
(i) $\mathrm{card}(\mathrm{h_m}) \geq 2$, *where* $h_m = \{r_{i_{m-1}+1}, \ldots, r_n\}$.
(ii) *there exists* $k \in \mathbb{N}_{m-1}$ *such that* $\mathrm{card}(\mathrm{h_k}) \geq 2$, $r_{i_k} = r_{i_k+1} = \cdots = r_n$, *and for each* $i \geq i_k$, *and for each* $j \in \mathbb{N}_n$, $a_{ij} \geq 0$ *or* $a_{ij} \leq 0$, *where* $h_1 = \{r_1, r_2, \ldots, r_{i_1-1}, r_n\}$, *and* $h_j = \{r_{i_{j-1}+1}, \ldots, r_{i_j-1}, r_n\}$ *for each* j $(2 \leq j \leq m-1)$.
(iii) $r_1 = r_2 = \cdots = r_n$, *and for each* $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ *or* $a_{ij} \leq 0$.

Proof. First, assume that (a) or (b) holds. Let $x = (x_1, \ldots, x_n)^t, y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$ and let $x \prec_{sgut} y$. We should prove $Tx \prec_{sgut} Ty$. If (a) holds; It is easy to see that $Tx \prec_{sgut} Ty$. If (b) holds; Then $n \geq 2$. By induction on n we prove the statement. Let $n = 2$; Proof, which is easy, is omitted for the sake of brevity. Assume that $n \geq 3$ and that the assertion has been established for the case $n-1$. Let $x = (x_1, \ldots, x_n)^t, y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$ and let $x \prec_{sgut} y$. Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. Since $x' = (x_2, \ldots, x_n)^t, y' = (y_2, \ldots, y_n)^t \in \mathbb{R}^{n-1}$ and $x' \prec_{sgut} y'$, by the induction hypothesis, $Sx' \prec_{sgut} Sy'$. So for proving $Tx \prec_{sgut} Ty$ it suffices to show that $(Tx)_1 \in \mathcal{A}\{(Ty)_1, \ldots, (Ty)_n\}$. If $\mathrm{card}\{(Ty)_1, \ldots, (Ty)_n\} \geq 2$, then it holds. Otherwise, $(Ty)_1 = \ldots = (Ty)_n$.
If (i) occurs; There is some j $(i_{m-1} + 1 \leq j \leq n-1)$ such that $a_{jn} \neq a_{nn}$. As $(Ty)_j = (Ty)_n$, so $y_n = 0$. $(Ty)_{i_1} = \ldots = (Ty)_{i_{m-1}} = (Ty)_n$ and $a_{i_1 t}, \ldots, a_{i_{m-1} n-1} \neq 0$ show that $y_t = \cdots = y_n = 0$, and hence $x_t = \cdots = x_n = 0$. This means that $(Tx)_1 \in \mathcal{A}\{(Ty)_1, \ldots, (Ty)_n\}$.
As a similar fashion, the cases (ii) and (iii) can be proved.

Next, suppose that T preserves $\prec_{sgut}$, $[T] = [a_{ij}]$, and (a) dose not hold. We show that (b) holds. Use induction on n. For $n = 2$, the proof is easy. Now assume that $n \geq 3$ and the statement holds for all linear preservers of $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. From Lemma 2.3, we observe that $[T]$ is upper triangular. Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. Lemma 2.2 ensures that S preserves $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. Apply induction hypothesis for S. So the proof will

be divided into two steps.
Step 1. If $[S]$ satisfies (a); Lemma 2.4 ensures then that the first nonzero column of $[T]$ should be its $(n-1)st$ column. If $\text{card}(\text{h}_\text{m}) \geq 2$, then $(b)-(i)$ occurs. Otherwise, $r_2 = \cdots = r_n$. Without loss of generality, assume that $a_{1n-1} = 1$. We should prove $r_1 = r_n$, $a_{1n}, a_{nn} \geq 0$, and $a_{nn} \neq 0$. Lemma 2.4 ensures that $a_{nn} \neq 0$. If $r_1 \neq r_n$; Let $x_{n-1} \in \mathbb{R}$. Choose $x = x_{n-1}e_{n-1}$ and $y = (a_{nn} - a_{1n})e_{n-1} + e_n$. We observe that $x \prec_{sgut} y$, and thus $Tx \prec_{sgut} Ty$. This follows that $x_{n-1} \in \mathcal{A}\{a_{nn}\}$, a contradiction. So $r_1 = r_n$. If $a_{nn} < 0$; Since $e_n \prec_{sgut} (e_{n-1} + e_n)$, we obtain a contradiction. This contradiction implies that $a_{nn} > 0$. Since $e_{n-1} \prec_{sgut} (e_{n-1} + e_n)$, we conclude that $a_{1n} \geq 0$. Thus (iii) holds for $[T]$.
Step 2. If $[S]$ satisfies (b); Let the first nonzero column of $[S]$ be the t^{th} column of $[T]$. We have two cases.
Case 1. The first nonzero column of $[T]$ is its t^{th} column. We see that $i_1 > 1$. If for $[S]$ one of the forms of (i) or (ii) happens, then there is no thing to prove. Otherwise, (iii) occurs for $[S]$. That is, $r_2 = \cdots = r_n$ and for each i, j $(2 \leq i, j \leq n)$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$. If $r_1 \neq r_n$, then (ii) occurs for $[T]$ with $k = 1$. If not; Then $r_1 = r_n$. Without loss of generality assume that for each i, j $(2 \leq i, j \leq n)$ $a_{ij} \geq 0$. We should just prove $a_{1t}, \ldots, a_{1n} \geq 0$. Define $J_1 = \{j : t \leq j \leq n,\ a_{1j} \geq 0\}$ and $J_2 = \{j : t \leq j \leq n,\ a_{1j} < 0\}$. It is enough to show that $J_2 = \emptyset$. If $J_2 \neq \emptyset$, then $r_1 \geq 0$. If $J_1 = \emptyset$, we conclude that $r_1 < 0$, a contradiction. This contradiction shows that $J_1 \neq \emptyset$. Set $x = \sum_{j \in J_1} e_j$ and $y = \sum_{j=t}^{n} e_j$. We see that $x \prec_{sgut} y$, and then $Tx \prec_{sgut} Ty$. This shows that $\sum_{j \in J_1} a_{1j} \in \mathcal{A}\{r_1\}$. So $\sum_{j \in J_1} a_{1j} \leq r_1$, and then $\sum_{j \in J_1} a_{1j} \leq \sum_{j \in J_1} a_{1j} + \sum_{j \in J_2} a_{1j}$. This means that $\sum_{j \in J_2} a_{1j} \geq 0$. Contradiction. So $J_2 = \emptyset$. We see that (iii) holds for $[T]$.
With an argument almost identical to that of the above, the following theme can be proved.
Case 2. The first nonzero column of $[T]$ is not its t^{th} column. Lemma 2.4 ensures then that the first nonzero column of $[T]$ is its $t - 1^{th}$ column. □

Lemma 2.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear preserver of $\prec_{sgut}$, and let $[T] = [a_{ij}]$. Then $[T]$ is upper triangular, $\prod_{i=1}^{n} a_{ii} \neq 0$, $r_1 = r_2 = \cdots = r_n$, and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$.*

Proof. By Lemma 2.3, $[T]$ is an upper triangular matrix. Since $[T]$ is upper triangular and invertible, we observe that $\prod_{i=1}^{n} a_{ii} \neq 0$. Theorem 2.5 ensures that $r_1 = r_2 = \cdots = r_n$ and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$.. □

Lemma 2.7. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear function that strongly preserves sgut-majorization. Then T is invertible.*

Proof. Suppose that $T(A) = 0$, where $A \in \mathbf{M}_{n,m}$. Notice that since T is linear, we have $T(0) = 0 = T(A)$. Then it is obvious that $T(A) \prec_{sgut} T(0)$. Therefore,

$A \prec_{sgut} 0$, because T strongly preserves sgut-majorization. Then A = R0, for some $R \in \mathcal{RS}_n^{gut}$. So $A = 0$, and hence T is invertible. □

Theorem 2.8. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function. Then T strongly preserves $\prec_{sgut}$ if and only if $[T] = \alpha I_n$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

Proof. As the sufficiency part is clear, we prove the necessity part. Suppose that T strongly preserves $\prec_{sgut}$ and $[T] = [a_{ij}]$. Lemma 2.7 ensures that T is invertible. From Lemma 2.6, we have $[T]$ is an upper triangular matrix, $\prod_{i=1}^n a_{ii} \neq 0$, $r_1 = \cdots = r_n$, and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$. We prove the statement by induction. The result is trivial for $n = 1$. Assume that our claim has been proved for all strong linear preservers of $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. Let $S : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear function with $[S] = [T](1)$. Conclude from Lemma 2.2 that S preserves $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. We claim that S strongly preserves $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. Let $x' = (x_2, \ldots, x_n)^t$, $y' = (y_2, \ldots, y_n)^t \in \mathbb{R}^{n-1}$, and let $Sx' \prec_{sgut} Sy'$. Set $x = (0, x')^t$, $y = (0, y')^t \in \mathbb{R}^n$. Observe that $Tx = (\sum_{i=2}^n a_{1i}x_i, Sx')^t$ and $Ty = (\sum_{i=2}^n a_{1i}y_i, Sy')^t$. For proving $Tx \prec_{sgut} Ty$ it suffices to show that $(Tx)_1 \in \mathcal{A}\{(Ty)_1, \ldots, (Ty)_n\}$. If $\text{card}\{(Ty)_1, \ldots, (Ty)_n\} \geq 2$, then $(Tx)_1 \in \mathcal{A}\{(Ty)_1, \ldots, (Ty)_n\}$. Otherwise, $(Ty)_2 = \cdots = (Ty)_n$. Then $y_2 = \cdots = y_n$, because $\prod_{i=2}^n a_{ii} \neq 0$ and $r_2 = \cdots = r_n$. Since $(Ty)_1 = (Ty)_n$, if $y_n \neq 0$ occurs, then $a_{11} = 0$, which is a contradiction. So $y_2 = \cdots = y_n = 0$, and then $x_2 = \cdots = x_n = 0$. Thus $(Tx)_1 \in \mathcal{A}\{(Ty)_1, \ldots, (Ty)_n\}$. It is obvious that $Tx \prec_{sgut} Ty$, and then, since T strongly preserves $\prec_{sgut}$ on $\mathbb{R}^n$, $x \prec_{sgut} y$. This implies that $x' \prec_{sgut} y'$. Therefore, S strongly preserves $\prec_{sgut}$ on $\mathbb{R}^{n-1}$. The induction hypothesis ensures that there exists some $\alpha \in \mathbb{R} \setminus \{0\}$ such that $[S] = \alpha I_{n-1}$. If we prove that $a_{12} = \cdots = a_{1n} = 0$, since $r_1 = \cdots = r_n$, we deduce that $[T] = \alpha I_n$. By a simple calculation, one may show that

$$[T^{-1}] = \begin{pmatrix} \frac{1}{a_{11}} & \frac{-a_{12}}{a_{11}\alpha} & \frac{-a_{13}}{a_{11}\alpha} & \ldots & 0 & 0 & \frac{-a_{1n}}{a_{11}\alpha} \\ 0 & \frac{1}{\alpha} & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{\alpha} \end{pmatrix}.$$

Since T strongly preserves $\prec_{sgut}$, the operator T^{-1} is a linear preserver of $\prec_{sgut}$, and hence, from Theorem 2.5, all entries of $[T^{-1}]$ have the same sign. As all entries of $[T]$ have the same sign too, it follows that $a_{12} = \cdots = a_{1n} = 0$. □

3. Sgut-majorization on $\mathbf{M}_{n,m}$

In this section we discuss some properties of sgut-majorization on $\mathbf{M}_{n,m}$, and we find the structure of strong linear preservers of $\prec_{sgut}$ on $\mathbf{M}_{n,m}$. First, we state some lemmas.

Lemma 3.1. *The following statements about a given matrix $A \in \boldsymbol{M}_n$ are equivalent.*
(a) *For all $D \in \mathcal{RS}_n^{gut}$, $AD = DA$.*
(b) *For some $\alpha \in \mathbb{R}$, $A = \alpha I$.*
(c) *For all $D \in \mathcal{RS}_n^{gut}$ and all $x, y \in \mathbb{R}^n$, $(Dx + ADy) \prec_{sgut} (x + Ay)$.*

Proof. To verify that (a) implies (b), suppose that $D := diag(1, \frac{1}{2}, \ldots, \frac{1}{n})$. Since $D \in \mathcal{RS}_n^{gut}$, observe that $a_{ij} = 0$ for each $i, j \in \mathbb{N}_n$, $i \neq j$. Also, for $D = E \in \mathcal{RS}_n^{gut}$, this follows that $a_{11} = \cdots = a_{nn}$. Thus (b) holds. Clearly, (b) implies (c). For each $y \in \mathbb{R}^n$, let $x = -Ay$. Put x, y in the relation (c). It is easy to see that $DAy = ADy$. Then $AD = DA$, and so (c) implies (a). □

Remark 3.2. Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear function. For every $i, j \in \mathbb{N}_m$, consider the embedding $E^j : \mathbb{R}^n \to \mathbf{M}_{n,m}$ and the projection $E_i : \mathbf{M}_{n,m} \to \mathbb{R}^n$ which are defined by $E^j(x) = xe_j^t$ and $E_i(A) = Ae_i$ respectively. Put $T_i^j = E_i T E^j$, for all $i, j \in \mathbb{N}_m$. Then $TX = T[x_1 \mid \ldots \mid x_m] = [\sum_{j=1}^m T_1^j x_j \mid \ldots \mid \sum_{j=1}^m T_m^j x_j]$. Moreover, if T preserves $\prec_{sgut}$, then T_i^j preserves $\prec_{sgut}$ for all $i, j \in \mathbb{N}_m$ too.

Lemma 3.3. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ preserve $\prec_{sgut}$. If for any $i \in \mathbb{N}_m$ there exists $k \in \mathbb{N}_m$ such that T_i^k is invertible, then $\sum_{j=1}^m A_i^j x_j = A_i^k \sum_{j=1}^m \alpha_i^j x_j$, for some $\alpha_i^j \in \mathbb{R}$, where $A_i^j = [T_i^j]$.*

Proof. We may assume without loss of generality that $i, k = 1$ and $j = 2$. We prove that there exists $\alpha_1^2 \in \mathbb{R}$ such that $A_1^2 = \alpha_1^2 A_1^1$. Let $D \in \mathcal{RS}_n^{gut}$ and $x, y \in \mathbb{R}^n$. So $D[x|y|0|\ldots|0] \prec_{sgut} [x|y|0|\ldots|0]$. This implies that $T[Dx|Dy|0|\ldots|0] \prec_{sgut} T[x|y|0|\ldots|0]$, and then $[A_1^1 Dx + A_1^2 Dy \mid * \mid *] \prec_{sgut} [A_1^1 x + A_1^2 y \mid * \mid *]$. This ensures that $A_1^1 Dx + A_1^2 Dy \prec_{sgut} A_1^1 x + A_1^2 y$, and Lemma 3.1 ensures then that there exists $\alpha_1^2 \in \mathbb{R}$ such that $A_1^2 = \alpha_1^2 A_1^1$. □

Lemma 3.4. *Suppose that $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ is a strong linear preserver of $\prec_{sgut}$. Then for each $i \in \mathbb{N}_m$ there exists $j \in \mathbb{N}_m$ such that T_i^j is invertible.*

Proof. Define $I = \{i \in \mathbb{N}_m \mid T_i^j e_1 = 0, \ \forall j \in \mathbb{N}_m\}$. We prove that I is empty. We may assume that I is not empty. Assume without loss of generality that $I = \{1, 2, \ldots, k\}$, where $k \in \mathbb{N}_m$. If $k = m$, we choose $X = [e_1 \mid 0 \mid \ldots \mid 0] \in \mathbf{M}_{n,m}$. We see that $X \neq 0$, but $TX = 0$. It follows that T is not invertible, while T strongly preserves $\prec_{sgut}$. This contradiction determines $k < m$. Lemma 3.3 ensures for each i ($k+1 \leq i \leq m$) and $j \in \mathbb{N}_m$, there exist invertible matrices A_i

and $\alpha_i^j \in \mathbb{R}$ such that $\sum_{j=1}^m A_i^j x_j = A_i \sum_{j=1}^m \alpha_i^j x_j$. So there exist $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$, not all zero, such that $\gamma_1(\alpha_{k+1}^1, \ldots, \alpha_m^1)^t + \cdots + \gamma_m(\alpha_{k+1}^m, \ldots, \alpha_m^m)^t = 0$. Define $x_j = \gamma_j e_1$ for each $j \in \mathbb{N}_m$, and let $X = [x_1 \mid \ldots \mid x_m] \in \mathbf{M}_{n,m}$. It is obvious that $X \neq 0$ and $TX = 0$, which would be a contradiction. We conclude that for each $i \in \mathbb{N}_m$, there exists $j \in \mathbb{N}_m$ such that $T_i^j e_1 \neq 0$, and hence T_i^j is invertible. □

Theorem 3.5. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ strongly preserve $\prec_{sgut}$. Then $TX = AXR$ for some invertible matrices $R \in \boldsymbol{M}_m$ and $A \in \mathcal{R}_n^{gut}$.*

Proof. Theorem 2.8 ensures that the case $m = 1$. So assume that $m > 1$. Lemma 3.4 ensures then that, since T is invertible, for each $i \in \mathbb{N}_m$ there exists $j \in \mathbb{N}_m$ such that T_i^j is invertible. Now, by Lemma 3.3, there exist invertible matrices $A_1, \ldots, A_m \in \mathbf{M}_n$ and vectors $a_1, \ldots, a_m \in \mathbb{R}^m$ such that $TX = [A_1 X a_1 \mid \ldots \mid A_m X a_m]$.

We claim that $\dim(\text{span}\{a_1, \ldots, a_m\}) \geq 2$. If not; then $\{a_1, \ldots, a_m\} \subseteq \text{span}\{a\}$, for some $a \in \mathbb{R}^m$. Since $m > 1$, we can choose $0 \neq b \in (\text{span}\{a\})^{\perp}$. Define $X \in \mathbf{M}_{n,m}$ such that the first row is b^t and the other rows are zero. We see that $X \neq 0$ and $TX = 0$, which would be a contradiction, because of Lemma 2.7. Thus $\text{rank}\{a_1, \ldots, a_m\} \geq 2$.

Without loss of generality, assume that $\{a_1, a_2\}$ is a linearly independent set. Let $X \in \mathbf{M}_{n,m}$ and $D \in \mathcal{RS}_n^{gut}$. So $DX \prec_{sgut} X$, and hence $TDX \prec_{sgut} TX$. Thus $[A_1 DX a_1 \mid \ldots \mid A_m DX a_m] \prec_{sgut} [A_1 X a_1 \mid \ldots \mid A_m X a_m]$, and so $A_1 DX a_1 + A_2 DX a_2 \prec_{sgut} A_1 X a_1 + A_2 X a_2$, and then

$$DXa_1 + A_1^{-1}A_2 DXa_2 \prec_{sgut} Xa_1 + A_1^{-1}A_2 Xa_2, \text{for all } X \in \mathbf{M}_{n,m}, D \in \mathcal{RS}_n^{gut}. \quad (1)$$

Since $\{a_1, a_2\}$ is linearly independent, for every $x, y \in \mathbb{R}^n$, there exists $B_{x,y} \in \mathbf{M}_{n,m}$ such that $B_{x,y} a_1 = x$ and $B_{x,y} a_2 = y$. Set $X = B_{x,y}$ in (1). So we have $Dx + A_1^{-1}A_2 Dy \prec_{sgut} x + A_1^{-1}A_2 y$, for all $D \in \mathcal{RS}_n^{gut}, x, y \in \mathbb{R}^n$. Lemma 3.1 ensures that $A_1^{-1}A_2 = \alpha I$, for some $\alpha \in \mathbb{R}$, and hence $A_2 = \alpha A_1$. For every $i \geq 3$ if $a_i = 0$, choose $A_i = A_1$, and if $a_i \neq 0$, then $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is linearly independent. Then, in a similar fashion, one shows that $A_i = \gamma_i A_1$, for some $\gamma_i \in \mathbb{R}$, or $A_i = \lambda_i A_2$, for some $\lambda_i \in \mathbb{R}$. Consider $A = A_1$. So for every $i \geq 2$, $A_i = r_i A$, for some $r_i \in \mathbb{R}$ and hence $TX = [AXa_1 \mid AX(r_2 a_2) \mid \ldots \mid AX(r_m a_m)] = AXR$, where $R = [a_1 \mid r_2 a_2 \mid r_m a_m]$. As T and A are invertible, we conclude that R is invertible too. Since A is the matrix representation of an invertible linear preserver of $\prec_{sgut}$, then there is some nonzero multiple of it belongs to $\mathcal{R}_n^{gut}$. □

Lemma 3.6. *Suppose that $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ satisfies $TX = AXR$ for some invertible matrices $R \in \boldsymbol{M}_m$ and $A \in \mathcal{R}_n^{gut}$. Let $A = [a_{ij}]$. If T strongly preserves $\prec_{sgut}$, then $a_{11} = 1$ and $a_{12} = a_{13} = \cdots = a_{1n} = 0$.*

Proof. Without loss of generality assume that $R = I_m$. First, we prove that $a_{11} = 1$. Clearly, $a_{11} \neq 0$. The proof is first divided into two steps.

Step 1. If $a_{11} < 0$ or $1 < a_{11}$. If $a_{11} < 0$, choose α such that $\alpha < \frac{1}{a_{11}}$, and if $1 < a_{11}$, select α such that $\frac{1}{a_{11}} < \alpha \leq 1$. Define $X \in \mathbf{M}_{n,m}$ such that its first row is $(\alpha\ \alpha \ldots \alpha)$ and all its other rows are zero, $Y = J_{n,m}$, and $R \in \mathbf{M}_n$ such that its first row is $(\alpha\ \ 0 \ldots 0)$ and all its other rows are zero. We observe that $X = RY$. As $R \in \mathcal{R}_n^{gut}$, then $X \prec_{sgut} Y$. If $TX \prec_{sgut} TY$, then there exists some $H \in \mathcal{RS}_n^{gut}$ such that $TX = HTY$. This follows that $\alpha a_{11} \leq 1$, which is a contradiction. So $TX \not\prec_{sgut} TY$. Thus T does not strongly preserve $\prec_{sgut}$.
Step 2. If $0 < a_{11} < 1$. Set $X \in \mathbf{M}_{n,m}$ such that its first row is $(\frac{1}{a_{11}}\ \ \frac{1}{a_{11}} \ldots \frac{1}{a_{11}})$ and all its other rows are zero, and $Y = J_{n,m}$. We see that $TX = RTY$, where $R \in \mathbf{M}_n$ is the matrix that its first row is $(a_{11}\ \ 0 \ldots 0)$ and all its other rows are zero. Hence $TX \prec_{sgut} TY$. But if $X \prec_{sgut} Y$, then there exists some $H \in \mathcal{RS}_n^{gut}$ such that $X = HY$. This shows that $\frac{1}{a_{11}} \leq 1$, which would be a contradiction. So $X \not\prec_{sgut} Y$, and thus T does not strongly preserve $\prec_{sgut}$.
Therefore, $a_{11} = 1$.

Now, we claim that $a_{12} = a_{13} = \cdots = a_{1n} = 0$. Assume, if possible, that $a_{1j} \neq 0$, for some $2 \leq j \leq n$. Choose x_1 such that $0 \leq 1 - x_1 < a_{1j}$ if $a_{1j} > 0$, and $0 \leq 1 - x_1$ if $a_{1j} < 0$. Select x_j such that $\frac{1-x_1}{a_{1j}} < x_j < 1$ if $a_{1j} > 0$, and $x_j < \frac{1-x_1}{a_{1j}}$ if $a_{1j} < 0$. Define $X \in \mathbf{M}_{n,m}$ such that its first row is $(x_1 \ldots x_1)$, its j^{th} row is $(x_j \ldots x_j)$, and all its other rows are zero, $Y = J_{n,m}$, and $R \in \mathbf{M}_n$ such that its first row is $(x_1\ \ 0 \ldots 0)$, its j^{th} row is $(0 \ldots 0\ \ x_j\ \ 0 \ldots 0)$ where x_j is j^{th} entry, and all its other rows are zero. We have $X = RY$. As $R \in \mathcal{RS}_n^{gut}$, we deduce that $X \prec_{sgut} Y$. If $TX \prec_{sgut} TY$, then there exists some $H \in \mathcal{RS}_n^{gut}$ such that $TX = HTY$. This implies that $x_1 + a_{1j}x_j \leq 1$, which is a contradiction. Therefore, $a_{12} = a_{13} = \cdots = a_{1n} = 0$. $\square$

Lemma 3.7. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ satisfy $TX = AXR$ for some invertible matrices $R \in \boldsymbol{M}_m$ and $A \in \mathcal{R}_n^{gut}$. Suppose that T strongly preserves $\prec_{sgut}$. If $S : \boldsymbol{M}_{n-1,m} \to \boldsymbol{M}_{n-1,m}$ satisfies $SX = A(1)XR$, then S is a strong linear preserver of $\prec_{sgut}$ on $\boldsymbol{M}_{n-1,m}$.*

Proof. It can be assumed without loss of generality that $R = I_m$. Lemma 3.6 ensure that $A = \begin{pmatrix} 1 & 0 \\ 0 & A(1) \end{pmatrix}$. Let $X', Y' \in \mathbf{M}_{n-1,m}$ such that $X' \prec_{sgut} Y'$. Then there is some $R' \in \mathcal{RS}_{n-1}^{gut}$ such that $X' = R'Y'$. Set $X = \begin{pmatrix} 0 \\ X' \end{pmatrix} \in \mathbf{M}_{n,m}$ and $Y = \begin{pmatrix} 0 \\ Y' \end{pmatrix} \in \mathbf{M}_{n,m}$. We see that $X = RY$, where $R = \begin{pmatrix} 0 & 0 \\ 0 & R' \end{pmatrix} \in \mathcal{RS}_n^{gut}$, and then $X \prec_{sgut} Y$. Since T preserves $\prec_{sgut}$ on $\mathbf{M}_{n,m}$, we conclude that $TX \prec_{sgut} TY$. Hence there exists some $H \in \mathcal{RS}_n^{gut}$ such that $TX = HTY$. Partition $H = \begin{pmatrix} H_1 & H_2 \\ 0 & H_3 \end{pmatrix}$, where $H_3 \in \mathbf{M}_{n-1}$. This implies that $A(1)X' = H_3A(1)Y'$. That is, $SX' = H_3SY'$. As $H_3 \in \mathcal{RS}_{n-1}^{gut}$, we have $SX' \prec_{sgut} SY'$.

Now, suppose that $X', Y' \in \mathbf{M}_{n-1,m}$ such that $SX' \prec_{sgut} SY'$. So there is some $R' \in \mathcal{RS}^{gut}_{n-1}$ such that $SX' = R'SY'$. Define $X = \begin{pmatrix} 0 \\ X' \end{pmatrix} \in \mathbf{M}_{n,m}$ and $Y = \begin{pmatrix} 0 \\ Y' \end{pmatrix} \in \mathbf{M}_{n,m}$. Set $R = \begin{pmatrix} 0 & 0 \\ 0 & R' \end{pmatrix}$. We have $R \in \mathcal{RS}^{gut}_n$ and $TX = RTY$. Thus $TX \prec_{sgut} TY$, and so $X \prec_{sgut} Y$. Then there exists some $H \in \mathcal{RS}^{gut}_n$ such that $X = HY$. Partition $H = \begin{pmatrix} H_1 & H_2 \\ 0 & H_3 \end{pmatrix}$, where $H_3 \in \mathbf{M}_{n-1}$. This concludes that $X' = H_3Y'$, and then $X' \prec_{sgut} Y'$. Therefore, S strongly preserves $\prec_{sgut}$. □

In the next theorem the structure of linear functions $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ strongly preserving sgut-majorization will be characterized.

Theorem 3.8. *Let $T : \boldsymbol{M}_{n,m} \to \boldsymbol{M}_{n,m}$ be a linear function. Then T strongly preserves $\prec_{sgut}$ if and only if $TX = XR$ for some invertible matrix $R \in \boldsymbol{M}_m$.*

Proof. As the sufficiency of the condition is easy to see, we just prove the necessity of the condition. Assume that T strongly preserves $\prec_{sgut}$. Theorem 3.5 ensures that $TX = AXR$ for some invertible matrices $R \in \mathbf{M}_m$ and $A \in \mathcal{R}^{gut}_n$. Let $A = [a_{ij}]$. From Lemma 3.6, we see that $a_{11} = 1$ and $a_{12} = a_{13} = \ldots = a_{1n} = 0$. Now, we claim that $A = I_n$. We proceed by induction on n. There is nothing to prove for $n = 1$. Suppose that $n \geq 2$ and the assertion has been established for all strong linear preservers of $\prec_{sgut}$ on $\mathbf{M}_{n-1,m}$. Lemma 3.6 ensures that $A = \begin{pmatrix} 1 & 0 \\ 0 & A(1) \end{pmatrix}$. Let $S : \mathbf{M}_{n-1,m} \to \mathbf{M}_{n-1,m}$ be the linear function by $SX = A(1)XR$. Lemma 3.7 implies that S strongly preserves $\prec_{sgut}$. Since $A(1) \in \mathcal{R}^{gut}_{n-1}$ is an invertible matrix, the induction hypothesis insures that $A(1) = I_{n-1}$. Therefore, $A = I_n$. □

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