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EXAMPLES OF NON-QUASICOMMUTATIVE SEMIGROUPS DECOMPOSED INTO UNIONS OF GROUPS

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Abstract. Decomposability of an algebraic structure into a union of its sub-structures have been studied by many authors for groups, rings and non-group semigroups since 1926. A sub-class of non-group semigroups is the well known quasicommutative semigroups where it is known that a regular quasicommutative semigroup is decomposable into a union of groups. The converse of this result is a natural question. Obviously, if a semigroup S is decomposable into a union of groups then S is regular so, the aim of this paper is to give examples of non-quasicommutative semigroups which are decomposable into the disjoint unions of groups. Our examples are two infinite classes of finite semigroups.
Keywords: Quasicommutative semigroups, finitely presented semigroups, decomposition.
MSC(2010): Primary: 20M05; Secondary: 20M17, 11Y50.

1. **Introduction**

Decomposition of a group G into the union of three pairwise disjoint normal subgroups of G is a result of Scorza [8]. An interesting analogue of this result for rings is given recently by A. Lucchini [5]. In the decomposition of non-group non-commutative semigroups into the union of groups, specially when the semigroup is finite (because of their useful applications), one may consult Clifford [3] where the construction of inverse semigroups has been investigated. Since then, Mukherjee [6] in the investigation of quasicommutative semigroups proved that " a regular quasicommutative semigroup may be decomposed into a union of groups", (a semigroup S is called quasicommutative if for every two elements $x, y \in S$ there exists a positive integer $r \geq 1$ such that $xy = y^r x$). In this paper we discuss the converse of this result by giving examples of finite semigroups which are unions of groups but they are not quasicommutative.

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Let S_1 and S_2 be the semigroups defined by the presentations

$$\pi_1 = \langle a, b \mid a^{n+1} = a, b^3 = b, ba = a^{n-1}b\rangle,\ (n \geq 3)$$

and

$$\pi_2 = \langle a, b \mid a^{1+p^{\alpha}} = a, b^{1+p^{\beta}} = b, ab = ba^{1+p^{\alpha-\gamma}}\rangle,$$

respectively, where p is a prime, α, β and γ are integers such that $\alpha \geq 2\gamma$, $\beta \geq \gamma \geq 1$ and $\alpha + \beta > 3$.

Proposition 1.1. *of the noncommutative semigroups S_1 and S_2 is disjoint union of groups. Moreover, they are not quasicommutative.*

Proposition 1.2. *For every prime p and positive integers α and β with $\alpha+\beta > 3$, there exist $[\frac{\alpha}{2}]$ non-isomorphic semigroups of order $p^{\alpha} + p^{\beta} + p^{\alpha+\beta}$ which are unions of groups.*

Our notation is standard. We follow [3,4] for the preliminaries on semigroup theory. As in the references [2, 7] we recall the notion of a presentation $\pi = \langle X|R\rangle$ of a set of formal generators X and a set of relators R where $\langle X|R\rangle$ is defined appropriately for finitely generated groups, semigroups and monoids. For more information on group, semigroup and monoid presentations one may see [2, 7] and their related references. To avoid confusion we use the notation $Sg(\pi)$ and $Gp(\pi)$ for the semigroup and the group presented by the presentation π, respectively.

2. **Proofs of the main results**

Let $S_1 = Sg(\pi_1)$ and $S_2 = Sg(\pi_2)$. Note that, by the definition of Green $\mathcal{J}$-classes of a semigroup S, two elements x and y are in the same $\mathcal{J}$-class if $S^1xS^1 = S^1yS^1$ and the set $\{y|S^1xS^1 = S^1yS^1\}$ is the $\mathcal{J}$-class J_x of x. Consequently, for every $x, y, z \in S$, $J_{xzy} \subseteq J_z$.

Lemma 2.1. The semigroup S_1 partitions into disjoint union of three groups.

Proof. Considering the above comment on the Green $\mathcal{J}$-classes we first show that the subsets

$$A = \{a^t|1 \leq t \leq n\},\ B = \{b, b^2\}\ , C = \{a^tb^j|1 \leq t \leq n, 1 \leq j \leq 2\},$$

are the Green $\mathcal{J}$-classes of the semigroup S_1.

For the subset A, $J_{xzy} \subseteq J_z$ yields

$$J_a = J_{a^{n+1}} \subseteq J_{a^n} \subseteq J_{a^{n-1}} \subseteq \cdots \subseteq J_{a^2} \subseteq J_a.$$

So, $J_a = J_{a^2} = \cdots = J_{a^n}$. Hence, $A = \{a, a^2, \cdots, a^n\}$ is a Green $\mathcal{J}$-class of S_1. There are similar proofs to show that the subsets B and C are Green $\mathcal{J}$-classes.

The classes A and B are cyclic groups of orders n and 2, respectively, (for, a^n and b^2 are the idempotent elements of S_1). However, $C \simeq D_{2n}$, the dihedral group of order $2n$, because letting $e = a^nb^2$, $x = ab$ and $y = a^{n-1}b^2$ gives rise to the relations $x^2 = e$, $y^n = e$ and $(xy)^2 = e$. Since $\mid C \mid = 2n$, then $C \simeq D_{2n}$. $\square$

The semigroup S_2 studied by Arjomandfar et al. [1] for its finiteness property in 2012, shows that S_2 is of order $p^{\alpha+\beta} + p^{\alpha} + p^{\beta}$. Considering the relation $ab = ba^{1+p^{\alpha-\gamma}}$ we get:

Lemma 2.2. The semigroup S_2 partitions into disjoint union of three groups.

Proof. An easy hand calculation shows that the elements $a^{p^{\alpha}}$, $b^{p^{\beta}}$ and $a^{p^{\alpha}}b^{p^{\beta}}$ are the idempotent elements of S_2. This yields in turn the following Green $\mathcal{J}$-classes for S_2:

$$\begin{aligned} A' &= \{a^i \mid 1 \le i \le p^{\alpha}\} \\ B' &= \{b^i \mid 1 \le i \le p^{\beta}\} \\ C' &= \{b^j a^i \mid 1 \le i \le p^{\alpha},\ 1 \le j \le p^{\beta}\}. \end{aligned}$$

Obviously, A' and B' are cyclic groups of orders p^{α} and p^{β}, where, $a^{p^{\alpha}}$ and $b^{p^{\beta}}$ are their identity elements, respectively. However, C' is a non-abelian group and has a presentation isomorphic to

$$G(\gamma) = \langle x, y \mid x^{p^{\alpha}} = 1, y^{p^{\beta}} = 1, xy = yx^{1+p^{\alpha-\gamma}} \rangle,$$

under the predefined conditions of α, β and γ. To prove $C' \simeq G(\gamma)$, we see that for fixed values of α and β there are $[\frac{\alpha}{2}]$ different values for γ where all of these groups are of order $p^{\alpha+\beta}$ but they are pairwise non-isomorphic (one may check by considering the derived subgroups for different values of γ.) This investigating the structure of $G(\gamma)$ leads us to consider $\gamma = 1$ and to prove that $C' \simeq \langle x, y \mid x^{p^{\alpha}} = 1, y^{p^{\beta}} = 1, xy = yx^{1+p^{\alpha-1}} \rangle$ it is sufficient to let $e = b^{p^{\beta}}a^{p^{\alpha}}$, $x = b^{p^{\beta}}a$ and $y = ba^{p^{\alpha}}$. So, the semigroup S_2 is disjoint union of three groups. $\square$

Proof of Proposition 1.1. By considering the above lemmas it is sufficient to examine the quasicommutativity of the semigroups S_1 and S_2.

The semigroup S_1 is not quasicommutative, for, if $x = ab^2$ and $y = b$ then there is no positive integer r such that $x.y = y^r.x$. Since $b^3 = b$ then the possible values for r are $r = 1$ or $r = 2$. However, in both cases we get contradictions as follows.

If $r = 1$ then

$$ab^2.b = b.ab^2 = a^{n-1}b.b^2 = a^{n-1}b$$

which yields the contradiction $ab = a^{n-1}b$, and if $r = 2$ then $ab^2.b = b^2.ab^2$ yields

$$ab = b.a^{n-1}.b^3 = b.a^{n-1}.b.$$

On the other hand, by using the relators $ba = a^{n-1}b$ and $a^{n+1} = a$ of S_1 we may get that $b.a^{n-1} = ab$, by almost a routine calculation. So, we get the contradiction $ab = ab.b = ab^2$, in this case. This proves that the semigroup S_1 is not quasicommutative.

The semigroup S_2 consists of three types of elements as indicated above. This semigroup is not quasicommutative, for, by taking the elements $x = ba$ and $y = a$ we are able to show that there is no positive integer r that satisfies $xy = y^r x$. Indeed, the possible values of r are $r = 1, 2, \ldots, p^\alpha$ (because of the relation $a^{1+p^\alpha} = a$) and by substituting for x and y in $xy = y^r x$ we get.

$$ba^2 = a^r ba = a^{r-1} ba^{2+p^{\alpha-\gamma}} = a^{r-2} ba^{3+2p^{\alpha-\gamma}} = \cdots = ba^{r+1+rp^{\alpha-\gamma}}.$$

Obviously, $ba^2 = ba^{r+1+rp^{\alpha-\gamma}}$ is a contradiction for every value of r. This completes the proof of Proposition 1.1 . $\square$

Proof of Proposition 1.2. At the end of the proof of Lemma 2.2 where we studied the semigroup S_2, we showed that for all values of $\gamma = 1, 2, \ldots, [\frac{\alpha}{2}]$, the groups C' are non-isomorphic, for a given prime p and the positive integers α and β where, $\alpha + \beta > 3$. So, all of the semigroups presented by π_2 are examples of non-isomorphic semigroups of order $p^\alpha + p^\beta + p^{\alpha+\beta}$. $\square$

3. Conclusion

Computing the characters of non-group non-commutative semigroups is one of the valuable applications of the decomposition of such semigroups in unions of finite groups. So far, the main attempts in this area of calculation concern the commutative semigroups, while the characters of our examples of non-commutative non-quasicommutative semigroups are now calculable via the characters of the groups which are the union components of the semigroups.

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