Title:

Pseudo Ricci symmetric real hypersurfaces of a complex projective space

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PSEUDO RICCI SYMMETRIC REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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ABSTRACT. Pseudo Ricci symmetric real hypersurfaces of a complex projective space are classified and it is proved that there are no pseudo Ricci symmetric real hypersurfaces of the complex projective space $\mathbb{CP}^n$ for which the vector field $\xi$ from the almost contact metric structure $(\phi, \xi, \eta, g)$ is a principal curvature vector field.

Keywords: real hypersurface, complex projective space, pseudo Ricci symmetric.


1. Introduction

Let $\mathbb{CP}^n$, $n \geq 2$ be an $n$-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4 and let $M$ be a real hypersurface of $\mathbb{CP}^n$. Then $M$ has an almost contact metric structure $(\phi, \xi, \eta, g)$ induced from the Kähler structure of $\mathbb{CP}^n$. The study of the geometry of real hypersurfaces in the space forms has been an object of significant interest. Important past results have been, for example, the classification of homogeneous real hypersurfaces in $\mathbb{CP}^n$ by Takagi [34], the characterization of real hypersurfaces with constant principal curvatures by Kimura [17] and the studies of real hypersurfaces in $\mathbb{CP}^n$ with special Ricci tensor by Hamada [9], by Cecil and Ryan [3] and by Ki [14]. Many differential geometers have studied real hypersurfaces of a complex projective space such as Bejancu and Deshmukh [2], Cho and Ki [6], Deshmukh [7], Hamada [10,11], Ikuta [13], Kimura [15,16,18], Kimura and Maeda [19], Maeda [20,21], Maeda [22], Matsuyama [23–25], Okumura [26], Perez et. al [29,30], [31,32], Takagi [35,36], Wang [37] and others.

It is well known that there does not exist a real hypersurface $M$ of $\mathbb{CP}^n$...
satisfying the condition that the second fundamental tensor $A$ of $M$ is parallel. Again, in [10], Hamada used the condition that the second fundamental tensor $A$ is recurrent, i.e. there exists an 1-form $\alpha$ such that $\nabla A = A \otimes \alpha$. And Hamada [10] proved that there are no real hypersurfaces of a complex projective space with parallel second fundamental tensor. In this connection, Hui and Matsuyama [12] studied real hypersurfaces of a complex projective space with pseudo parallel second fundamental tensor. Again many differential geometers studied real hypersurfaces of complex projective space satisfying some condition of Ricci tensor. In [14] Ki proved that there are no real hypersurfaces of a complex projective space with parallel Ricci tensor. Again, in [11], Hamada studied the real hypersurfaces of a complex projective space with recurrent Ricci tensor and proved that there are no real hypersurfaces with recurrent Ricci tensor of $CP^n$ under the condition that $\xi$ is a principal curvature vector.

A Riemannian space is said to be Ricci symmetric if its Ricci tensor $S$ of type $(0,2)$ satisfies $\nabla S = 0$, where $\nabla$ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent space [28], Ricci semisymmetric space [33], pseudo Ricci symmetric space by Deszcz [8] and pseudo Ricci symmetric space by Chaki [4].

A non-flat Riemannian space $(M^n, g)$ is said to be pseudo Ricci symmetric [4] if its Ricci tensor $S$ of type (0,2) is not identically zero and satisfies the condition

$$\langle \nabla_X S \rangle(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(X, Z) + \alpha(Z)S(Y, X),$$

for any vector field $X, Y, Z$, where $\alpha$ is a nowhere vanishing 1-form and $\nabla$ denotes the operator of covariant differentiation with respect to the metric tensor $g$. The pseudo Ricci symmetric spaces have been also studied by Arslan et. al [1], Chaki and Saha [5], Özen [27] and many others.

The relation (1.1) can be written as

$$\langle \nabla_X Q \rangle Y = 2\alpha(X)QY + \alpha(Y)QX + S(Y, X)\rho,$$

where $\rho$ is the vector field associated to the 1-form $\alpha$ such that $\alpha(X) = g(X, \rho)$ and $Q$ is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all $X, Y$.

Motivated by the above studies the present paper deals with the study of pseudo Ricci symmetric real hypersurfaces of a complex projective space. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 is devoted to the study of pseudo Ricci symmetric real hypersurfaces of a pseudo Ricci symmetric complex projective space and it is proved that there are no pseudo Ricci symmetric real hypersurfaces of the complex projective space $CP^n$ for which the vector field $\xi$ from the almost contact metric structure $(\phi, \xi, \eta, g)$ is a principal curvature vector field.
2. Preliminaries

Let $M$ be a real hypersurface of the complex projective space $CP^n$ equipped with the Fubini-Study Riemannian metric $G$. We denote the induced metric on $M$ by $g$, and let $N$ be a unit normal vector field to $M$. Then the Kähler structure $J$ on $CP^n$ induces an almost contact metric structure $(\phi, \eta, g)$ on $M$, i.e., in this setting we have a tensor field $\phi$ of type $(1,1)$ on $M$ characterized by $g(\phi X, Y) = G(JX, Y)$, and a vector field $\xi$, an 1-form $\eta$ on $M$ characterized by $g(\xi, X) = \eta(X) = G(JX, N)$, and with these data, the following relations hold:

\begin{equation}
\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = g(\xi, \xi) = 1 \quad \text{and} \quad \phi\xi = 0.
\end{equation}

The Riemannian connections $\nabla$ in $CP^n$ and $\nabla$ in $M$ are related by

\begin{equation}
\nabla_X Y = \nabla_X Y + g(AX, Y)N,
\end{equation}

\begin{equation}
\nabla_X N = -AX
\end{equation}

for arbitrary vector fields $X$ and $Y$ on $M$, where $A$ is the second fundamental tensor of $M$ in $CP^n$. Let $TM$ be the tangent bundle of $M$. An eigenvector $X$ of the second fundamental tensor $A$ is called a principal curvature vector. Also an eigenvalue $\lambda$ of $A$ is called a principal curvature.

Now it follows from (2.2) that

\begin{equation}
(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi,
\end{equation}

\begin{equation}
\nabla_X \xi = \phi AX.
\end{equation}

Let $\bar{R}$ and $R$ be the curvature tensors of $CP^n$ and $M$ respectively. From the expression of the curvature tensor of $CP^n$, we see that the curvature tensor, Codazzi equation and the Ricci tensor of type $(1,1)$ are given by

\begin{equation}
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY;
\end{equation}

\begin{equation}
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi,
\end{equation}

\begin{equation}
QX = (2n + 1)X - 3\eta(X)\xi + hAX - A^2 X,
\end{equation}

where $h = \text{trace} A$.

Again we have

\begin{equation}
(\nabla_X Q)Y = -3g(\phi AX, Y)\xi - 3\eta(Y)\phi AX + (Xh)AY + h(\nabla_X A)Y - A(\nabla_X A)Y - (\nabla_X A)AY.
\end{equation}

Also we recall the following:

**Lemma 2.1.** If $\xi$ is a principal curvature vector, then the corresponding principal curvature $\lambda$ is locally constant.
Lemma 2.2. Assume that \( \xi \) is a principal curvature vector and the corresponding principal curvature is \( a \). If \( AX = \lambda X \) for \( X \perp \xi \), then we have \( A\bar{\omega}X = \bar{\lambda} \), where \( \bar{\lambda} = \frac{(a\lambda + 2)}{(2\lambda - a)} \).

Theorem 2.3. Let \( M \) be a connected real hypersurface of \( CP^n \), \( n \geq 3 \), whose Ricci tensor \( S \) satisfies \( S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \) for some smooth functions \( a \) and \( b \) on \( M \). Then \( M \) is locally congruent to one of the following:

(i) a geodesic hyper-sphere,
(ii) a tube of radius \( r \) over a totally geodesic \( CP^k \), \( 1 \leq k \leq n - 2 \), where \( 0 < r < \frac{n}{2} \) and \( \cot^2 r = \frac{k}{n-1} \),
(iii) a tube of radius \( r \) over a complex quadric \( Q_{n-1} \), where \( 0 < r < \frac{n}{4} \) and \( \cot^2 2r = n - 2 \).

Theorem 2.4. Let \( M \) be a homogeneous real hypersurface of \( CP^n \). Then \( M \) is a tube of radius \( r \) over one of the following Kähler submanifolds:

(\( A_1 \)) hyperplane \( CP^{n-1} \), where \( 0 < r < \frac{n}{2} \),
(\( A_2 \)) totally geodesic \( CP^k \), \( 1 \leq k \leq n - 2 \), where \( 0 < r < \frac{n}{2} \),
(\( B \)) complex quadric \( Q_{n-1} \), where \( 0 < r < \frac{n}{2} \),
(\( C \)) \( CP^1 \times CP^{\frac{n-1}{2}} \), where \( 0 < r < \frac{n}{4} \) and \( n \geq 5 \) is odd,
(\( D \)) complex Grassman \( cG_{2,5} \), where \( 0 < r < \frac{n}{4} \) and \( n = 9 \),
(\( E \)) Hermitian symmetric space \( SO(10)/U(5) \), where \( 0 < r < \frac{n}{4} \) and \( n = 15 \).

Theorem 2.5. Let \( M \) be a real hypersurface of \( CP^n \). Then \( M \) has constant principal curvatures and \( \xi \) is a principal curvature vector if and only if \( M \) is locally congruent to a homogeneous real hypersurface.

Theorem 2.6. There are no real hypersurfaces with parallel Ricci tensor of a complex space form \( cM^n, c \neq 0 \).

3. Pseudo Ricci symmetric Real hypersurfaces of a complex projective space \( CP^n \)

In this section, we have studied pseudo Ricci symmetric real hypersurfaces of a complex projective space \( CP^n \) and prove the following:

Lemma 3.1. Let \( M \) be a connected pseudo Ricci symmetric real hypersurface of a complex projective space \( CP^n \). If all eigenvalues of the Ricci operator \( Q \) are constant then the Ricci tensor \( S \) of \( M \) is parallel.

Proof. Let \( QX = \lambda X, QY = \mu Y \) and \( QZ = \nu Z \). Then we have

\[
g((\nabla_X Q)Y, Z) = (\mu - \nu)g(\nabla_X Y, Z). \tag{3.1}
\]

Again we have from (1.2) that

\[
g((\nabla_X Q)Y, Z) = 2\alpha(X)\mu g(Y, Z) + \alpha(Y)\nu g(X, Z) + \alpha(\mu)g(X, Y).
\]
If $\lambda \neq \mu$, $\lambda \neq \nu$ and $\mu \neq \nu$ then $g(\nabla_X Q)Y, Z = 0$. In the treatment of the case $\mu = \nu$, it follows from (3.1) that $g(\nabla_X Q)Y, Z = 0$.

Assume that $\mu \neq \nu$, $\lambda = \mu$. Then $g(\nabla_Z X, Y) = 0$.

On the other hand, we have

$$g(\nabla_Z X, Y) = 2\alpha(Z)\mu g(X, Y) + \left[ \alpha(X)S(Z, Y) + \alpha(Y)S(X, Z) \right].$$

Thus we obtain $\alpha(Z)\mu g(X, Y) = 0$. Hence $g(\nabla_X Q)Y, Z = 0$. Consequently, the Ricci tensor $S$ of $M$ is parallel.

From (2.8) and since $\xi$ is principal, the principal curvature vector will also be eigenvectors of $S$. Thus Ricci tensor of a homogeneous real hypersurface has constant eigenvalues. Again the hypersurface listed in Theorem 2.2 do not have parallel Ricci tensor. Thus from Lemma 3.1 and Theorem 2.3, we obtain

Proposition 3.2. A homogeneous real hypersurface of $CP^n$ can not be pseudo Ricci symmetric.

Since all the surfaces in (i) - (iii) of Theorem 2.1 are homogeneous, so by using Theorem 2.1, we have

Corollary 3.3. A real hypersurface of $CP^n$, $n \geq 3$ whose Ricci tensor $S$ satisfies $S(X, Y) = ag(X, Y) + b\eta(Y)\eta(Y)$ for some smooth functions $a$ and $b$ on $M$, can not be pseudo Ricci symmetric.

Now we prove the following:

Theorem 3.4. There are no pseudo Ricci symmetric real hypersurfaces in the complex projective space $CP^n$, for which $\xi$ is a principal curvature vector field.

Proof. Assume that $M$ is a real hypersurface in complex projective space $CP^n$ (with almost contact metric structure $(\phi, \xi, \eta, g)$), which is pseudo Ricci symmetric and for which $\xi$ is a principal curvature vector field. Then by virtue of (2.8) it follows from (1.2) that

$$g(\nabla_X Q)Y, Z = 2\alpha(X)[(2n + 1)g(Y, Z) - 3\eta(Y)\eta(Z)]$$
$$+ h\gamma(AY, Z) - g(A^2 Y, Z]$$
$$+ \alpha(Y)[(2n + 1)g(X, Z) - 3\eta(X)\eta(Z)]$$
$$+ h\gamma(AX, Z) - g(A^2 X, Z]$$
$$+ \alpha(Z)[(2n + 1)g(X, Y) - 3\eta(X)\eta(Y)]$$
$$+ h\gamma(AX, Y) - g(A^2 X, Y].$$
Using (2.9) in (3.2), we get
\[ 2\alpha(X)[(2n+1)g(Y, Z) - 3\eta(Y)\eta(Z) + hg(AY, Z) - g(A^2Y, Z)] + \alpha(Y)[(2n+1)g(X, Z) - 3\eta(X)\eta(Z) + hg(AX, Z) - g(A^2X, Z)] + \alpha(Z)[(2n+1)g(X, Y) - 3\eta(X)\eta(Y) + hg(AX, Y) - g(A^2X, Y)] + 3\eta(Z)g(\phi AX, Y) + 3\eta(Y)g(\phi AX, Z) - (Xh)g(AY, Z) - hg((\nabla_X A)Y, Z) + g(A(\nabla_X A)Y, Z) + g((\nabla_X A)AY, Z) = 0 \]
for any tangent vectors \( X, Y, Z \).

Putting \( Y = \xi \) and \( Z = \phi X \) in (3.3), we get
\[ 2\alpha(X)[\eta(\phi X) + 3\eta(\phi X)] + \alpha(\eta(X)\eta(X) - (Xh)g(AX, \phi X) - \eta(AX\phi X) + g(AX, \phi X) + g((\nabla_X A)\phi X) = 0. \]

Let us assume \( \xi = a \xi \). Then by Lemma 2.1, \( a \) is constant and hence we get
\[ (\nabla_X A)\xi = a\phi AX - A\phi AX. \]

Using (3.5) in (3.4), we obtain
\[ \lambda[\bar{\lambda}^2 - \{h + \alpha(\xi)\}\bar{\lambda} - (a^2 - ha + 3)] = 0. \]

Again from (3.7), we get
\[ \lambda\bar{\lambda} = \frac{\lambda + \bar{\lambda}}{2} a + 1. \]

If \( \lambda = \bar{\lambda} \) then (3.9) yields
\[ \lambda^2 = a\lambda + 1. \]

If 0 occurs as a principal curvature (for a principal vector orthogonal to \( \xi \)), then it follows from (3.9) that all principal curvatures are constant.

Next assuming that 0 is not a principal curvature (again we consider only
directions orthogonal to $\xi$, the relation (3.8) shows that there are at most two
distinct principal curvatures. If $\lambda$ and $\bar{\lambda}$ are distinct then we have
$$\lambda + \bar{\lambda} = h + \alpha(\xi) \quad \text{and} \quad \lambda\bar{\lambda} = -(a^2 - ha + 3),$$
which yields
$$-(a^2 - ha + 3) = \frac{\{h + \alpha(\xi)\}a}{2} + 1,$$
i.e.
$$a^2 = \frac{\{h - \alpha(\xi)\}a}{2} + 4 = 0.$$
Thus the coefficients in (3.8) are constants and hence so are $\lambda$ and $\bar{\lambda}$. The final
possibility is that all principal curvatures (with principal vectors orthogonal to $\xi$) satisfy (3.10) and are again constant.

Hence by Theorem 2.3, it follows that $M$ is locally homogeneous and there-fore the Ricci operator of $M$ has constant eigenvalues. Since $M$ is also pseudo
Ricci symmetric, so by Lemma 3.1, it follows that the Ricci tensor of $M$ is parallel.

On the other hand, the homogeneous real hypersurfaces of complex projective
space $CP^n$ have been classified by Takagi [34] (Theorem 2.2), and it turns
out that none of them have parallel Ricci tensor. Therefore we arrive at a
contradiction, and thus, no such hypersurface $M$ exists. □

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