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k-POWER CENTRALIZING AND k-POWER SKEW-CENTRALIZING MAPS ON TRIANGULAR RINGS

X. F. QI

(Communicated by Bamdad Yahaghi)

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring and $\mathcal{Z}(\mathcal{U})$ its center. Assume that $f : \mathcal{U} \to \mathcal{U}$ is a map satisfying $f(x + y) - f(x) - f(y) \in \mathcal{Z}(\mathcal{U})$ for all $x,\ y \in \mathcal{U}$ and k is a positive integer. It is shown that, under some mild conditions, the following statements are equivalent: (1) $[f(x), x^k] \in \mathcal{Z}(\mathcal{U})$ for all $x \in \mathcal{U}$; (2) $[f(x), x^k] = 0$ for all $x \in \mathcal{U}$; (3) $[f(x), x] = 0$ for all $x \in \mathcal{U}$; (4) there exist a central element $z \in \mathcal{Z}(\mathcal{U})$ and an additive modulo $\mathcal{Z}(\mathcal{U})$ map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $f(x) = zx + h(x)$ for all $x \in \mathcal{U}$. It is also shown that there is no nonzero additive k-skew-centralizing maps on triangular rings.

Keywords: Triangular rings, centralizing maps, k-skew-centralizing maps, nest algebras.
MSC(2010): Primary: 16W10; Secondary: 47B47, 46L10.

1. Introduction

Let $\mathcal{R}$ be an associative ring with the center $\mathcal{Z}(\mathcal{R})$. Recall that a map $f : \mathcal{R} \to \mathcal{R}$ is commuting if $[f(x), x] = 0$ for all $x \in \mathcal{R}$ and is centralizing if $[f(x), x] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Analogously, f is called skew-commuting (skew-centralizing) if $f(x)x + xf(x) = 0$ ($f(x)x + xf(x) \in \mathcal{Z}(\mathcal{R})$) for all $x \in \mathcal{R}$. Here, we denote the Jordan product of x and y by $[x, y]_{-1}$, that is, $[x, y]_{-1} = xy + yx$.

The study of (skew-)commuting and (skew-)centralizing maps was initiated by Divinsky [11], where he proved that a simple artinian ring is commutative if it has a commuting automorphism different from the identity map. Brešar in [2] proved that, if F is an additive commuting map from a von Neumann algebra $\mathcal{M}$ into itself, then there exist $Z \in \mathcal{Z}(\mathcal{M})$ and an additive map $h : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ such that $F(A) = ZA + h(A)$ for all $A \in \mathcal{M}$. Later, Brešar [4] gave the same characterization of additive commuting maps on prime rings. Cheung

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in [9] discussed linear commuting maps on triangular algebras and shown that, under some mild conditions, such maps also have the form $x \mapsto zx + h(x)$, where z is a central element and h is a central valued linear map. For skew-commuting (skew-centralizing) maps, Brešar [3] proved that, if $\mathcal{R}$ is a prime ring of characteristic not 2 and $f : \mathcal{R} \to \mathcal{R}$ is an additive map with f skew-commuting on some ideal $\mathcal{I}$ of $\mathcal{R}$, then $f(x) = 0$ for all $x \in \mathcal{I}$. Bell and Lucier [1] investigated additive skew-commuting or skew-centralizing maps on subsets of certain rings.

More generally, for a positive integer k, a map $f : \mathcal{R} \to \mathcal{R}$ is k-power commuting (k-power skew-commuting) if $[f(x), x^k] = 0$ ($[f(x), x^k]_{-1} = 0$) for all $x \in \mathcal{R}$ and is k-power centralizing (k-power skew-centralizing) if $[f(x), x^k] \in \mathcal{Z}(\mathcal{R})$ ($[f(x), x^k]_{-1} \in \mathcal{Z}(\mathcal{R})$) for all $x \in \mathcal{R}$. Obviously, f is (skew-)commuting ((skew-)centralizing) if $k = 1$. Brešar and Hvala [7] proved that every additive 2-power commuting map on a prime ring with characteristic not 2 is commuting. Recently, Inceboz, Koşan and Lee [13] generalized the above result to semiprime rings and proved that every additive 2-power commuting (respectively, centralizing) map $f : \mathcal{R} \to \mathcal{U}$ is commuting, where $\mathcal{R}$ is a semiprime ring and $\mathcal{U}$ is its maximal right ring of quotients. About other related results on various rings and algebras, see [6, 8] and the references therein.

We remark that another kind of k-commuting maps was introduced in [17], there, an additive map f on a ring is called k-commuting if $[f(x), x]_k = 0$ for all x, where $[x, y]_k = [[x, y]_{k-1}, y]$ with $[x, y]_0 = x$ and $[x, y]_1 = xy - yx$. From definitions, these two kinds of maps are not obviously equivalent, and so the conditions on discussed rings (algebras) and the conclusions are also different. There are some papers to discuss k-commuting maps on rings and algebras, for example, see [12, 14–17] and the references therein.

The purpose of the present paper is to consider the problem of characterizing the general k-power centralizing and k-power skew-centralizing maps on triangular rings $\mathcal{U}$. As corollaries, characterizations of all k-power commuting and k-power skew-commuting maps on $\mathcal{U}$ are obtained.

Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module, that is, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $a\mathcal{M} = \mathcal{M}b = \{0\}$ imply $a = 0$ and $b = 0$. The ring

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \{\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}$$

under the usual matrix operations is called a triangular ring. Note that triangular rings are neither prime nor semiprime. Denote by $\mathcal{Z}(\mathcal{U})$ the center of $\mathcal{U}$. By [9], it is known that

$$\mathcal{Z}(\mathcal{U}) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathcal{U} : am = mb \text{ for all } m \in \mathcal{M}\}. \tag{1.1}$$

This means that $a \in \mathcal{Z}(\mathcal{A})$ and $b \in \mathcal{Z}(\mathcal{B})$ if $am = mb$ holds for all $m \in \mathcal{M}$. Define two natural projections $\pi_A : \mathcal{U} \to \mathcal{A}$ and $\pi_B : \mathcal{U} \to \mathcal{B}$ respectively by

$$\pi_A(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = a \quad \text{and} \quad \pi_B(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = b.$$

It is shown from [9, Proposition 3] that $\pi_A(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{A})$, $\pi_B(\mathcal{Z}(\mathcal{U})) \subseteq \mathcal{Z}(\mathcal{B})$ and there exists a unique ring isomorphism $\tau : \pi_A(\mathcal{Z}(\mathcal{U})) \to \pi_B(\mathcal{Z}(\mathcal{U}))$ such that

$$am_1 = m_1\tau(a) \quad \text{and} \quad \tau^{-1}(b)m_2 = m_2 b \tag{1.2}$$

for all $a \in \pi_A(\mathcal{Z}(\mathcal{U}))$, $b \in \pi_B(\mathcal{Z}(\mathcal{U}))$ and all $m_1, m_2 \in \mathcal{M}$.

Recall that a map ϕ from a ring $\mathcal{T}$ into itself is additive modulo $\mathcal{Z}(\mathcal{T})$ (or equivalently, almost additive) if ϕ satisfies $\phi(x+y) - \phi(x) - \phi(y) \in \mathcal{Z}(\mathcal{T})$ for all $x, y \in \mathcal{T}$ ([5]).

Assume that $f : \mathcal{U} \to \mathcal{U}$ is an additive modulo $\mathcal{Z}(\mathcal{U})$ map and k is a positive integer. In this paper, we show that, under some mild conditions about $\mathcal{U}$, the following several statements are equivalent: (1) f is k-power centralizing; (2) f is k-power commuting; (3) f is commuting; (4) there exist a central element $z \in \mathcal{Z}(\mathcal{U})$ and an additive modulo $\mathcal{Z}(\mathcal{U})$ map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $f(x) = zx + h(x)$ for all $x \in \mathcal{U}$ (Theorem 2.1). As corollaries and applications, a characterization of additive k-power centralizing maps on nest algebras is obtained (Corollaries 2.2-2.3). In addition, it is also shown that there exists no nonzero additive k-power skew-centralizing map on $\mathcal{U}$ (Theorem 2.4).

2. Main results and corollaries

In this section, we will give our main results and several corollaries.

For k-power commuting and k-power centralizing maps, we have

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be an $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let k be a positive integer and $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ the triangular ring with characteristic not 2 and k. Assume that $f : \mathcal{U} \to \mathcal{U}$ is an additive module $\mathcal{Z}(\mathcal{U})$ map. If $\mathcal{U}$ satisfies $\pi_A(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_B(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$ and there exists $m_0 \in \mathcal{M}$ such that $\mathcal{Z}(\mathcal{U}) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : \ a \in \mathcal{Z}(\mathcal{A}),\ b \in \mathcal{Z}(\mathcal{B}),\ am_0 = m_0 b\}$, then the following four statements are equivalent.*

(1) f is commuting.

(2) f is k-power commuting.

(3) f is k-power centralizing.

(4) There exist a central element $z \in \mathcal{Z}(\mathcal{U})$ and an additive module $\mathcal{Z}(\mathcal{U})$ map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $f(x) = zx + h(x)$ for all $x \in \mathcal{U}$.

Particularly, if f is additive, we immediately obtain the following corollary.

Corollary 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be an $(\mathcal{A},\mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Let k be a positive integer and $\mathcal{U} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ the triangular ring with characteristic not 2 and k. Assume that $f : \mathcal{U} \to \mathcal{U}$ is an additive map. If $\mathcal{U}$ satisfies $\pi_A(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_B(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$ and there exists $m_0 \in \mathcal{M}$ such that $\mathcal{Z}(\mathcal{U}) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : \ a \in \mathcal{Z}(\mathcal{A}),\ b \in \mathcal{Z}(\mathcal{B}),\ am_0 = m_0 b\}$, then the following four statements are equivalent.*

(1) f is commuting.

(2) f is k-power commuting.

(3) f is k-power centralizing.

(4) There exist a central element $z \in \mathcal{Z}(\mathcal{U})$ and an additive map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $f(x) = zx + h(x)$ for all $x \in \mathcal{U}$.

As an application of Theorem 2.1 to the nest algebra case, we have

Corollary 2.3. *Let $\mathcal{N}$ be a nest on a Banach space X over the real or complex field $\mathbb{F}$ and let $\mathrm{Alg}\mathcal{N}$ be the associated nest algebra. Assume that k is a positive integer and $\Phi : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is an additive map. If there exists a non-trivial element in $\mathcal{N}$ that is complemented in X, then Φ is k-power centralizing if and only if $\Phi(A) = \lambda A + h(A)I$ for all $A \in \mathrm{Alg}\mathcal{N}$, where $\lambda \in \mathbb{F}$ and $h : \mathrm{Alg}\mathcal{N} \to \mathbb{F}$ is an additive functional.*

Proof. By the assumption on the nest, there is a non-trivial element $N_1 \in \mathcal{N}$ such that N_1 is complemented in X. Thus, there exists an idempotent operator E with $\mathrm{ran}(E) = N_1 \in \mathcal{N}$. It is clear that $E \in \mathrm{Alg}\mathcal{N}$. Decompose X into the direct sum $X = \mathrm{ran}(E) \oplus \ker E$. Then with respect to this decomposition, we have $E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Let $\mathcal{N}_E = \{N \cap N_1 : \forall N \in \mathcal{N}\}$ and $\mathcal{N}_{I-E} = \{N \cap \ker E : \forall N \in \mathcal{N}\}$. Then $\mathcal{N}_E$ and $\mathcal{N}_{I-E}$ are nests on Banach spaces N_1 and $\ker E$, respectively. Thus $E(\mathrm{Alg}\mathcal{N})E|_{N_1} = \mathrm{Alg}(\mathcal{N}_E)$, $(I-E)(\mathrm{Alg}\mathcal{N})(I-E)|_{\ker E} = \mathrm{Alg}(\mathcal{N}_{I-E})$ and

$$\mathrm{Alg}\mathcal{N} = \{\begin{pmatrix} C & W \\ 0 & D \end{pmatrix} : \quad C \in \mathrm{Alg}(\mathcal{N}_E), \\ W \in \mathcal{B}(\ker E, \mathrm{ran}(E)),\ D \in \mathrm{Alg}(\mathcal{N}_{I-E})\}.$$

It is easy to prove that $\mathcal{B}(\ker E, \mathrm{ran}(E))$ is a faithful left $\mathrm{Alg}(\mathcal{N}_E)$-module and a faithful right $\mathrm{Alg}(\mathcal{N}_{I-E})$-module. So, $\mathrm{Alg}\mathcal{N}$ is a triangular ring. Note that the center of any nest algebra is $\mathbb{F}I$. Thus, $\mathrm{Alg}\mathcal{N}$ satisfies all assumptions about $\mathcal{U}$ in Theorem 2.1. Therefore, by Theorem 2.1, the corollary holds. □

For k-power skew-commuting and k-power skew-centralizing maps, we have the following theorem.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings, and $\mathcal{M}$ be an $(\mathcal{A},\mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Assume* *that $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is the triangular ring and satisfies $\pi_A(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{A})$, $\pi_B(\mathcal{Z}(\mathcal{U})) = \mathcal{Z}(\mathcal{B})$ and there exists $m_0 \in \mathcal{M}$ such that $\mathcal{Z}(\mathcal{U}) = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in \mathcal{Z}(\mathcal{A}), b \in \mathcal{Z}(\mathcal{B}), am_0 = m_0 b\}$. Assume that k is a positive integer and $f : \mathcal{U} \to \mathcal{U}$ is an additive module $\mathcal{Z}(\mathcal{U})$ map. If f is k-power skew-centralizing, then $f(0) \in \mathcal{Z}(\mathcal{U})$ and $f(x) = f(0)$ for all $x \in \mathcal{U}$. Particularly, if f is additive, then $f \equiv 0$.*

By Theorem 2.4, there exists no nonzero additive k-power skew-commuting maps on triangular rings.

3. Proofs of main theorems

In this section, we will give proofs of our main results, Theorems 2.1 and 2.4.

The following lemma is needed.

Lemma 3.1. *Let $n \geq 1$ be a positive integer and let $\mathcal{R}$ be a unital ring with characteristic not n. Assume that $a_0 \in \mathcal{R}$. Then the following two statements hold.*

(1) If $[a_0, a^n] = 0$ for all $a \in \mathcal{R}$, then $a_0 \in \mathcal{Z}(\mathcal{R})$.

(2) If the characteristic of $\mathcal{R}$ is not 2 and $a_0 a^n + a^n a_0 = 0$ for all $a \in \mathcal{R}$, then $a_0 = 0$.

Proof. Denote by e the unit of $\mathcal{R}$. If $n = 1$, it is obvious that the lemma holds. In the sequel, we always assume $n > 1$.

(1) Since $[a_0, a^n] = 0$ for all $a \in \mathcal{R}$, replacing a by $a + e$ in the equation, we have $[a_0, (a + e)^n] = 0$, that is,

$$C_n^1[a_0, a^{n-1}] + C_n^2[a_0, a^{n-2}] + \cdots + C_n^{n-1}[a_0, a] = 0.$$

Next, replacing e by $2e, 3e, \ldots, (n-1)e$ in turn in the equation $[a_0, a^n] = 0$, and expressing the resulting system of $n - 1$ homogeneous equations in the variables $C_n^i[a_0, a^{n-i}],\ i = 1, 2, \ldots, n-1$. We see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{pmatrix}.$$

Since the determinant of the matrix is different from zero, the system has only a zero solution. It follows that $C_n^{n-1}[a_0, a] = 0$, and so $[a_0, a] = 0$ for all $a \in \mathcal{R}$.

(2) Since $a_0 a^n + a^n a_0 = 0$ for all $a \in \mathcal{R}$, replacing a by $a + e$ in the equation gets $a_0(a + e)^n + (a + e)^n a_0 = 0$, that is,

$$C_n^1(a_0 a^{n-1} + a^{n-1} a_0) + C_n^2(a_0 a^{n-2} + a^{n-2} a_0) + \ldots + C_n^{n-1}(a_0 a + a a_0) = 0.$$

Regarding $C_n^i(a_0a^{n-i}+a^{n-i}a_0)$ $(i=1,2,\dots,n-1)$ as the variables, by using the same argument as that of (1), one can obtain $a_0a+aa_0=0$ for all $a\in\mathcal{R}$. This implies $a_0=0$ by taking $a=e$.

The proof is completed. □

In the rest of the paper, denote by e_1 and e_2 the units of $\mathcal{A}$ and $\mathcal{B}$, respectively. Assume that $f:\mathcal{U}\to\mathcal{U}$ is an almost additive map, that is,

$$f(x+y)-f(x)-f(y)\in\mathcal{Z}(\mathcal{U})\ \text{ holds for all }\ x,y\in\mathcal{U}.$$

For any $x=\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in\mathcal{U}$, write

$$f(x)=\begin{pmatrix} f_A(a)+f_M(m)+f_B(b) & h_A(a)+h_M(m)+h_B(b) \\ 0 & g_A(a)+g_M(m)+g_B(b)\end{pmatrix}, \tag{3.1}$$

where $f_A:\mathcal{A}\to\mathcal{A}$, $f_M:\mathcal{M}\to\mathcal{A}$, $f_B:\mathcal{B}\to\mathcal{A}$, $h_A:\mathcal{A}\to\mathcal{M}$, $h_M:\mathcal{M}\to\mathcal{M}$, $h_B:\mathcal{B}\to\mathcal{M}$, $g_A:\mathcal{A}\to\mathcal{B}$, $g_M:\mathcal{M}\to\mathcal{B}$ and $g_B:\mathcal{B}\to\mathcal{B}$ are maps. By the almost additivity of f, it is clear that $f(0)\in\mathcal{Z}(\mathcal{U})$. Define $f'(x)=f(x)-f(0)$ for all $x\in\mathcal{U}$. It is easily checked that $f':\mathcal{U}\to\mathcal{U}$ is also an almost additive k-power centralizing map (respectively, k-power skew-centralizing map) satisfying $f'(0)=0$. So, without loss of generality, we always assume $f(0)=0$ in the sequel.

Thus, by Eq.(3.1), we have

$$h_A(0)+h_M(0)+h_B(0)=0, \tag{3.2}$$

$$f_A(0)+f_M(0)+f_B(0)=0\ \text{ and }\ g_A(0)+g_M(0)+g_B(0)=0. \tag{3.3}$$

For any $m_1,m_2\in\mathcal{M}$, by the almost additivity of f and Eqs.(3.2)-(3.3), we have

$$\begin{aligned} & f(\begin{pmatrix} 0 & m_1+m_2 \\ 0 & 0\end{pmatrix})-f(\begin{pmatrix} 0 & m_1 \\ 0 & 0\end{pmatrix})-f(\begin{pmatrix} 0 & m_2 \\ 0 & 0\end{pmatrix}) \\ = & \begin{pmatrix} f_A(0)+f_M(m_1+m_2)+f_B(0) & h_A(0)+h_M(m_1+m_2)+h_B(0) \\ 0 & g_A(0)+g_M(m_1+m_2)+g_B(0)\end{pmatrix} \\ & -\begin{pmatrix} f_A(0)+f_M(m_1)+f_B(0) & h_A(0)+h_M(m_1)+h_B(0) \\ 0 & g_A(0)+g_M(m_1)+g_B(0)\end{pmatrix} \\ & -\begin{pmatrix} f_A(0)+f_M(m_2)+f_B(0) & h_A(0)+h_M(m_2)+h_B(0) \\ 0 & g_A(0)+g_M(m_2)+g_B(0)\end{pmatrix} \\ = & \begin{pmatrix} f_1(m_1,m_2) & f_2(m_1,m_2) \\ 0 & f_3(m_1,m_2)\end{pmatrix}\in\mathcal{Z}(\mathcal{U}), \end{aligned}$$

where

$$\begin{aligned} f_1(m_1,m_2) &= f_M(m_1+m_2)-f_M(m_1)-f_M(m_2)+f_M(0), \\ f_2(m_1,m_2) &= h_M(m_1+m_2)-h_M(m_1)-h_M(m_2)+h_M(0), \\ f_3(m_1,m_2) &= g_M(m_1+m_2)-g_M(m_1)-g_M(m_2)+g_M(0). \end{aligned}$$

It follows from Eq.(1.1) that

$$(3.4)\qquad \begin{cases} f_2(m_1,m_2) = h_M(m_1+m_2) - h_M(m_1) - h_M(m_2) + h_M(0) = 0, \\ f_1(m_1,m_2) = f_M(m_1+m_2) - f_M(m_1) - f_M(m_2) + f_M(0) \in \mathcal{Z}(\mathcal{A}), \\ f_3(m_1,m_2) = g_M(m_1+m_2) - g_M(m_1) - g_M(m_2) + g_M(0) \in \mathcal{Z}(\mathcal{B}), \\ f_1(m_1,m_2)m = mf_3(m_1,m_2) \ \text{ for all } \ m \in \mathcal{M}. \end{cases}$$

Similarly, one can prove that

$$(3.5)\qquad \begin{cases} h_A(a_1+a_2) - h_A(a_1) - h_A(a_2) + h_A(0) = 0, \\ (f_A(a_1+a_2) - f_A(a_1) - f_A(a_2) + f_A(0))m \\ = m(g_A(a_1+a_2) - g_A(a_1) - g_A(a_2) + g_A(0)) \end{cases}$$

hold for all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$; and

$$(3.6)\qquad \begin{cases} h_B(b_1+b_2) - h_B(b_1) - h_B(b_2) + h_B(0) = 0, \\ (f_B(b_1+b_2) - f_B(b_1) - f_B(a_2) + f_B(0))m \\ = m(g_B(b_1+b_2) - g_B(b_1) - g_B(b_2) + g_B(0)) \end{cases}$$

hold for all $b_1, b_2 \in \mathcal{B}$ and $m \in \mathcal{M}$.

In addition, note that $x^k = \begin{pmatrix} a^k & \sum_{i=0}^{k-1} a^i m b^{k-1-i} \\ 0 & b^k \end{pmatrix}$. A direct calculation obtains

$$(3.7)\qquad \begin{aligned} & [f(x), x^k] \\ = & \begin{pmatrix} [f_A(a) + f_M(m) + f_B(b), a^k] & \phi(x) \\ 0 & [g_A(a) + g_M(m) + g_B(b), b^k] \end{pmatrix} \\ \in & \ \mathcal{Z}(\mathcal{U}) \end{aligned}$$

and

$$(3.8)\qquad \begin{aligned} & f(x)x^k + x^k f(x) = [f(x), x^k]_{-1} \\ = & \begin{pmatrix} [f_A(a) + f_M(m) + f_B(b), a^k]_{-1} & \psi(x) \\ 0 & [g_A(a) + g_M(m) + g_B(b), b^k]_{-1} \end{pmatrix} \\ \in & \ \mathcal{Z}(\mathcal{U}), \end{aligned}$$

where

$$\begin{aligned} \phi(x) = \ & (h_A(a) + h_M(m) + h_B(b))b^k - a^k(h_A(a) + h_M(m) + h_B(b)) \\ & + \textstyle\sum_{i=0}^{k-1} (f_A(a) + f_M(m) + f_B(b)) a^i m b^{k-1-i} \\ & - \textstyle\sum_{i=0}^{k-1} a^i m b^{k-1-i} (g_A(a) + g_M(m) + g_B(b)) \end{aligned}$$

and

$$\begin{aligned} \psi(x) = \ & (h_A(a) + h_M(m) + h_B(b))b^k + a^k(h_A(a) + h_M(m) + h_B(b)) \\ & + \textstyle\sum_{i=0}^{k-1} (f_A(a) + f_M(m) + f_B(b)) a^i m b^{k-1-i} \\ & + \textstyle\sum_{i=0}^{k-1} a^i m b^{k-1-i} (g_A(a) + g_M(m) + g_B(b)). \end{aligned}$$

Now, we are in the position to give proofs of Theorem 2.1 and Theorem 2.4.

Proof of Theorem 2.1. It is clear that "(1) $\Rightarrow$ (2) $\Rightarrow$ (3)" and "(4) $\Rightarrow$ (1)".

In the following, assume that f is a k-power centralizing map. We will prove "(3) $\Rightarrow$ (4)" by the following several steps.

Step 1. For any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have:
(i) $h_A(a) = h_A(0)$ and $h_B(b) = h_B(0)$;
(ii) $[f_A(a) - f_A(0), a^k] = 0$ and $[g_B(b) - g_B(0), b^k] = 0$.
Take any $a \in \mathcal{A}$, any $b \in \mathcal{B}$ and $m = 0$ in Eq.(3.7). Then

$$\begin{pmatrix} [f_A(a) + f_M(0) + f_B(b), a^k] & \begin{array}{l}(h_A(a) + h_M(0) + h_B(b))b^k \\ -a^k(h_A(a) + h_M(0) + h_B(b))\end{array} \\ 0 & [g_A(a) + g_M(0) + g_B(b), b^k] \end{pmatrix} \in \mathcal{Z}(\mathcal{U}).$$

By Eq.(1.1), this implies that

$$(3.9) \qquad (h_A(a) + h_M(0) + h_B(b))b^k = a^k(h_A(a) + h_M(0) + h_B(b))$$

and

$$(3.10) \qquad [f_A(a) + f_M(0) + f_B(b), a^k]m' = m'[g_A(a) + g_M(0) + g_B(b), b^k]$$

hold for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m' \in \mathcal{M}$. Particularly, letting $b = 0$ and $a = 0$ in Eqs.(3.9)-(3.10), respectively, we obtain that

$$(3.11) \qquad a^k(h_A(a) + h_M(0) + h_B(0)) = 0, \quad (h_A(0) + h_M(0) + h_B(b))b^k = 0$$

and

$$(3.12) \qquad [f_A(a) + f_M(0) + f_B(0), a^k] = 0, \quad [g_A(0) + g_M(0) + g_B(b), b^k] = 0$$

hold for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. So, combining Eq.(3.2) and Eq.(3.11) yields

$$(3.13) \qquad a^k(h_A(a) - h_A(0)) = 0 \text{ and } (h_B(b) - h_B(0))b^k = 0,$$

which imply

$$(3.14) \qquad h_A(e_1) = h_A(0) \text{ and } h_B(e_2) = h_B(0).$$

Next, comparing Eq.(3.2), Eq.(3.9) and Eq.(3.13), we achieve $(h_A(a) - h_A(0))b^k = a^k(h_B(b) - h_B(0))$ for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$. By respectively taking $b = e_2$ and $a = e_1$ in the equation, and noting that Eq.(3.14), we obtain

$$h_A(a) - h_A(0) = a^k(h_B(e_2) - h_B(0)) = 0$$

and

$$h_B(b) - h_B(0) = (h_A(e_1) - h_A(0))b^k = 0,$$

that is, $h_A(a) = h_A(0)$ and $h_B(b) = h_B(0)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. (i) is true.
Now, combining Eq.(3.2) and (3.12), we obtain that $[f_A(a) - f_A(0), a^k] = 0$ and $[g_B(b) - g_B(0), b^k] = 0$ hold for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and so (ii) holds.

Step 2. For any $m \in \mathcal{M}$, we have:
(i) $h_M(m) - h_M(0) = a_0m - mb_0 = mb_0' - a_0'm$, where $a_0 = f_A(e_1) - f_A(0)$, $b_0 = g_A(e_1) - g_A(0)$, $a_0' = f_B(e_2) - f_B(0)$ and $b_0' = g_B(e_2) - g_B(0)$;
(ii) $\begin{pmatrix} f_M(m) - f_M(0) & 0 \\ 0 & g_M(m) - g_M(0) \end{pmatrix} \in \mathcal{Z}(\mathcal{U})$;
(iii) $a^k(h_M(m) - h_M(0)) = (f_A(a) - f_A(0))a^{k-1}m - a^{k-1}m(g_A(a) - g_A(0))$ for all $a \in \mathcal{A}$;

(iv) $(h_M(m)-h_M(0))b^k = mb^{k-1}(g_B(b)-g_B(0))-(f_B(b)-f_B(0))mb^{k-1}$ for all $b\in\mathcal{B}$.

Firstly, letting $b=0$ in Eq.(3.7), one gets that

$$\begin{pmatrix} [f_A(a)+f_M(m)+f_B(0),a^k] & \begin{array}{l} -a^k(h_A(a)+h_M(m)+h_B(0)) \\ +(f_A(a)+f_M(m)+f_B(0))a^{k-1}m \\ -a^{k-1}m(g_A(a)+g_M(m)+g_B(0)) \end{array} \\ 0 & 0 \end{pmatrix}$$

is in $\mathcal{Z}(\mathcal{U})$, which implies

$$[f_A(a)+f_M(m)+f_B(0),a^k]=0 \tag{3.15}$$

and

$$\begin{array}{rl} & a^k(h_A(a)+h_M(m)+h_B(0)) \\ = & (f_A(a)+f_M(m)+f_B(0))a^{k-1}m \\ & -a^{k-1}m(g_A(a)+g_M(m)+g_B(0)). \end{array} \tag{3.16}$$

Comparing Eq.(3.3), Eq.(3.15) and Step 1(ii), one has

$$[f_M(m)-f_M(0),a^k]=0$$

for all $a\in\mathcal{A}$ and $m\in\mathcal{M}$. It follows from Lemma 3.1 that

$$f_M(m)-f_M(0)\in\mathcal{Z}(\mathcal{A}). \tag{3.17}$$

Next, letting $a=0$ in Eq.(3.7), one can similarly obtain

$$[g_A(0)+g_M(m)+g_B(b),b^k]=0 \tag{3.18}$$

and

$$\begin{array}{rl} & (h_A(0)+h_M(m)+h_B(b))b^k \\ = & mb^{k-1}(g_A(0)+g_M(m)+g_B(b)) \\ & -(f_A(0)+f_M(m)+f_B(b))mb^{k-1}. \end{array} \tag{3.19}$$

Comparing Eq.(3.3), Eq.(3.18) and Step 1(ii), one gets

$$[g_M(m)-g_M(0),b^k]=0$$

for all $b\in\mathcal{M}$ and $m\in\mathcal{M}$. It follows from Lemma 3.1 that

$$g_M(m)-g_M(0)\in\mathcal{Z}(\mathcal{B}). \tag{3.20}$$

Thirdly, letting $a=e_1$ in Eq.(3.16), by Step 1(i) and Eq.(3.3), one has

$$\begin{array}{rl} & h_M(m)-h_M(0) \\ = & (f_A(e_1)+f_M(m)+f_B(0))m-m(g_A(e_1)+g_M(m)+g_B(0)) \\ = & (f_A(e_1)-f_A(0)+f_M(m)-f_M(0))m \\ & -m(g_A(e_1)-g_A(0)+g_M(m)-g_M(0)). \end{array} \tag{3.21}$$

Now replacing m by $2m$ in Eq.(3.21), by using Eq.(3.4), one obtains

$$\begin{array}{rl} h_M(m)-h_M(0)= & (f_A(e_1)-f_A(0)+2f_M(m)-2f_M(0))m \\ & -m(g_A(e_1)-g_A(0)+2g_M(m)-2g_M(0)), \end{array}$$

which and Eq.(3.21) imply

$$(f_M(m) - f_M(0))m = m(g_M(m) - g_M(0)). \tag{3.22}$$

Thus, Eq.(3.21) reduces to

$$h_M(m) - h_M(0) = (f_A(e_1) - f_A(0))m - m(g_A(e_1) - g_A(0)).$$

On the other hand, by the arbitrariness of m in Eq.(3.22), we get

$$(f_M(m_0) - f_M(0))m_0 = m_0(g_M(m_0) - g_M(0)) \tag{3.23}$$

and

$$\begin{aligned} &(f_M(m + m_0) - f_M(0))(m + m_0) \\ = \ &(m + m_0)(g_M(m + m_0) - g_M(0)). \end{aligned} \tag{3.24}$$

Note that Eqs.(3.17) and (3.20) imply

$$f_M(m_0) - f_M(0) \in \mathcal{Z}(\mathcal{A}) \qquad \text{and} \qquad g_M(m_0) - g_M(0) \in \mathcal{Z}(\mathcal{B}).$$

By the assumption about $\mathcal{Z}(\mathcal{U})$, these and Eq.(3.23) yield

$$\begin{pmatrix} f_M(m_0) - f_M(0) & 0 \\ 0 & g_M(m_0) - g_M(0) \end{pmatrix} \in \mathcal{Z}(\mathcal{U}),$$

and so $(f_M(m_0) - f_M(0))m = m(g_M(m_0) - g_M(0))$ holds for all $m \in \mathcal{M}$. Hence, by using Eq.(3.4) to Eq.(3.24), a simple calculation yields $(f_M(m) - f_M(0))m_0 = m_0(g_M(m) - g_M(0))$. It follows from the assumption about $\mathcal{Z}(\mathcal{U})$, Eq.(3.17) and Eq.(3.20) that

$$\begin{pmatrix} f_M(m) - f_M(0) & 0 \\ 0 & g_M(m) - g_M(0) \end{pmatrix} \in \mathcal{Z}(\mathcal{U}).$$

That is, (ii) holds.

Finally, by Eqs.(3.2)-(3.3) and (ii), Eq.(3.16) can be reduced to $a^k(h_M(m) - h_M(0)) = (f_A(a) - f_A(0))a^{k-1}m - a^{k-1}m(g_A(a) - g_A(0))$, that is, (iii) holds.

Now, for Eq.(3.19), by a similar argument to that of Eq.(3.16), one can easily see that (iv) and the other relation in (i) are true.

Step 3. $a_0 \in \mathcal{Z}(\mathcal{A})$ and $b_0 \in \mathcal{Z}(\mathcal{B})$.

For any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, by Eq.(3.3) and Step 1(ii), Eq.(3.10) reduces to

$$[f_B(b) - f_B(0), a^k]m = m[g_A(a) - g_A(0), b^k] \text{ for all } m \in \mathcal{M}.$$

Letting $b = e_2$ in the above equation, one gets $[f_B(e_2) - f_B(0), a^k]m = [a_0', a^k]m = 0$, which implies $[a_0', a^k] = 0$ for all $a \in \mathcal{A}$ since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module. It follows from Lemma 3.1 that $a_0' \in \mathcal{Z}(\mathcal{A}) = \pi_A(\mathcal{Z}(\mathcal{U}))$. Also note that $mb_0' - a_0'm = a_0m - mb_0$ by Step 2(i). So

$$a_0m = mb_0' - a_0'm + mb_0 = m(b_0' + b_0 - \tau(a_0')) \text{ for all } m \in \mathcal{M}.$$

It follows that $a_0 \in \mathcal{Z}(\mathcal{A})$. Symmetrically, one can check $b_0 \in \mathcal{Z}(\mathcal{B})$.

Step 4. For any $a \in \mathcal{A}$, $b \in \mathcal{B}$ and any $m \in \mathcal{M}$, we have

$$(f_A(a) - f_A(0) - a_0a + \tau^{-1}(b_0)a)m = m(g_A(a) - g_A(0))$$

and

$$(f_B(b)-f_B(0))m = m(g_B(b)-g_B(0)+b_0b-\tau(a_0)b).$$

Here, we only give the proof of the first equation. The proof of the other equation is similar and we omit it here.

Note that it is shown from Step 2(i) and (iii) that

$$a^k(a_0m-mb_0) = (f_A(a)-f_A(0))a^{k-1}m - a^{k-1}m(g_A(a)-g_A(0))$$

holds for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. By Eq.(1.2) and Step 3, the above equation can be rewritten as

$$(f_A(a)-f_A(0)-a_0a+\tau^{-1}(b_0)a)a^{k-1}m - a^{k-1}m(g_A(a)-g_A(0)) = 0. \tag{3.25}$$

Replacing a by $a+e_1$ in Eq.(3.25), by Eq.(3.5) and Eq.(1.2), one achieves

$$(f_A(a)-f_A(0)-a_0a+\tau^{-1}(b_0)a)(a+e_1)^{k-1}m-(a+e_1)^{k-1}m(g_A(a)-g_A(0)) = 0.$$

For the convenience, write $f_A(a)-f_A(0)-a_0a+\tau^{-1}(b_0)a = F(a)$ and $G(a) = g_A(a)-g_A(0)$. Then the above equation becomes

$$\begin{aligned} & F(a)(a+e_1)^{k-1}m-(a+e_1)^{k-1}mG(a) \\ = & \textstyle\sum_{i=1}^{k-1} C_{k-1}^i(F(a)a^{k-1-i}m - a^{k-1-i}mG(a)) = 0. \end{aligned}$$

Similarly, replacing a by $a+2e_1, a+3e_1, \ldots, a+(k-1)e_1$ in turn in Eq.(3.25), and expressing the resulting system of $k-1$ homogeneous equations in the variables $C_{k-1}^i(F(a)a^{k-1-i}m - a^{k-1-i}mG(a))$, $i = 1, \ldots, k-1$. Still, the coefficient matrix of the system is nonsingular and the system has only a zero solution. It follows that

$$C_{k-1}^i(F(a)a^{k-1-i}m - a^{k-1-i}mG(a)) = 0, \ i = 1, \ldots, k-1;$$

particularly, $F(a)m-mG(a) = 0$. That is, $(f_A(a)-f_A(0)-a_0a+\tau^{-1}(b_0)a)m = m(g_A(a)-g_A(0))$.

Step 5. There exist a central element $z \in \mathcal{Z}(\mathcal{U})$ and an additive module $\mathcal{Z}(\mathcal{U})$ map $h : \mathcal{U} \to \mathcal{Z}(\mathcal{U})$ such that $f(x) = zx + h(x)$ for all $x \in \mathcal{U}$. Therefore, (4) holds.

So far, we have proved that, for any $x=\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}\in\mathcal{U}$, it holds that

$$\begin{aligned}
&f(x)=f(x)-f(0)\\
=&\begin{pmatrix} f_A(a)-f_A(0)+f_B(b)-f_B(0) & a_0m-mb_0\\ 0 & g_A(a)-g_A(0)+g_B(b)-g_B(0)\end{pmatrix}\\
&+\begin{pmatrix} f_M(m)-f_M(0) & 0\\ 0 & g_M(m)-g_M(0)\end{pmatrix}\\
=&\begin{pmatrix} a_0-\tau^{-1}(b_0) & 0\\ 0 & \tau(a_0)-b_0\end{pmatrix}\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}\\
&+\begin{pmatrix} f_M(m)-f_M(0) & 0\\ 0 & g_M(m)-g_M(0)\end{pmatrix}\\
&+\begin{pmatrix} \begin{array}{c} f_A(a)-f_A(0)+f_B(b)\\ -f_B(0)-a_0a+\tau^{-1}(b_0)a\end{array} & 0\\ 0 & \begin{array}{c} g_A(a)-g_A(0)+g_B(b)\\ -g_B(0)+b_0b-\tau(a_0)b\end{array}\end{pmatrix}.
\end{aligned}$$

Let $z=\begin{pmatrix} a_0-\tau^{-1}(b_0) & 0\\ 0 & \tau(a_0)-b_0\end{pmatrix}$ and

$$\begin{aligned}
h(x)=\;&\begin{pmatrix} f_M(m)-f_M(0) & 0\\ 0 & g_M(m)-g_M(0)\end{pmatrix}\\
&+\begin{pmatrix} f_A(a)-f_A(0)-a_0a+\tau^{-1}(b_0)a & 0\\ 0 & g_A(a)-g_A(0)\end{pmatrix}\\
&+\begin{pmatrix} f_B(b)-f_B(0) & 0\\ 0 & g_B(b)-g_B(0)+b_0b-\tau(a_0)b\end{pmatrix}\\
=\;&\begin{pmatrix} f_M(m)-f_M(0) & 0\\ 0 & g_M(m)-g_M(0)\end{pmatrix}+h_1(x)+h_2(x).
\end{aligned}$$

For z, by the definition of τ, we have $(a_0-\tau^{-1}(b_0))m=m(\tau(a_0)-b_0)$ for all $m\in\mathcal{M}$. It follows from Eq.(1.1) that $z\in\mathcal{Z}(\mathcal{U})$. In addition, by Step 4 and Eq.(1.1), it is obvious that $h_i(x)\in\mathcal{Z}(\mathcal{U})$ $(i=1,2)$. Moreover, since f is additive module $\mathcal{Z}(\mathcal{U})$, it is easy to check that h is also additive module $\mathcal{Z}(\mathcal{U})$.

The proof of the theorem is finished. $\square$

Proof of Theorem 2.4. Assume that f is k-power skew-centralizing additive module $\mathcal{Z}(\mathcal{U})$ map. So Eq.(3.8) holds. We will prove the theorem by checking the following several steps.

Step 1. For any $a\in\mathcal{A}$ and $b\in\mathcal{B}$, we have $h_A(a)=h_A(0)$ and $h_B(b)=h_B(0)$.

Take any $a\in\mathcal{A}$, any $b\in\mathcal{B}$ and $m=0$ in Eq.(3.8). Then

$$\begin{aligned}
&\begin{pmatrix} [f_A(a)+f_M(0)+f_B(b),a^k]_{-1} & \begin{array}{c}(h_A(a)+h_M(0)+h_B(b))b^k\\ +a^k(h_A(a)+h_M(0)+h_B(b))\end{array}\\ 0 & [g_A(a)+g_M(0)+g_B(b),b^k]_{-1}\end{pmatrix}\\
\in\;&\mathcal{Z}(\mathcal{U}).
\end{aligned}$$

By Eq.(1.1), this implies

$$(h_A(a) + h_M(0) + h_B(b))b^k + a^k(h_A(a) + h_M(0) + h_B(b)) = 0$$

and

$$[f_A(a) + f_M(0) + f_B(b), a^k]_{-1}m' = m'[g_A(a) + g_M(0) + g_B(b), b^k]_{-1} \tag{3.26}$$

for all $m' \in \mathcal{M}$. Now, by a similar argument to that of Step 1(i) in the proof of Theorem 2.1, one can check that $h_A(a) = h_A(0)$ and $h_B(b) = h_B(0)$ hold for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Step 2. For any $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}$, the following statements hold:
(i) $[f_A(a) - f_A(0), a^k]_{-1} = 0$ and $[g_B(b) - g_B(0), b^k]_{-1} = 0$;
(ii) $f_M(m) - f_M(0) = 0$ and $g_M(m) - g_M(0) = 0$;
(iii) $h_M(m) - h_M(0) = a_1m - mb_1 = a'_1m - mb'_1$, where $a_1 = f_A(0) - f_A(e_1)$, $b_1 = g_A(e_1) - g_A(0)$, $a'_1 = f_B(e_2) - f_B(0)$ and $b'_1 = g_B(0) - g_B(e_2)$;
(iv) $a^k(h_M(m) - h_M(0)) = -(f_A(a) - f_A(0))a^{k-1}m - a^{k-1}m(g_A(a) - g_A(0))$;
(v) $(h_M(m) - h_M(0))b^k = -mb^{k-1}(g_B(b) - g_B(0)) - (f_B(b) - f_B(0))mb^{k-1}$.

By taking $b = 0$ in Eq.(3.8), one gets that

$$\begin{pmatrix} [f_A(a) + f_M(m) + f_B(0), a^k]_{-1} & \begin{matrix} a^k(h_A(a) + h_M(m) + h_B(0)) \\ +(f_A(a) + f_M(m) + f_B(0))a^{k-1}m \\ +a^{k-1}m(g_A(a) + g_M(m) + g_B(0)) \end{matrix} \\ 0 & 0 \end{pmatrix}$$

belongs to $\mathcal{Z}(\mathcal{U})$, and so

$$[f_A(a) + f_M(m) + f_B(0), a^k]_{-1} = 0 \tag{3.27}$$

and

$$\begin{aligned} &a^k(h_A(a) + h_M(m) + h_B(0)) \\ &+(f_A(a) + f_M(m) + f_B(0))a^{k-1}m \\ &+a^{k-1}m(g_A(a) + g_M(m) + g_B(0)) = 0. \end{aligned} \tag{3.28}$$

Particularly, taking $m = 0$ in Eq.(3.27) yields $[f_A(a) - f_A(0), a^k]_{-1} = 0$ for all $a \in \mathcal{A}$ by Eq.(3.3). Thus, by using this relation and Eq.(3.3), Eq.(3.27) reduces to $[f_M(m) - f_M(0), a^k]_{-1} = 0$ for all $a \in \mathcal{A}$ and $m \in \mathcal{M}$. It follows from Lemma 3.1 that $f_M(m) - f_M(0) = 0$.

Similarly, by taking $a = 0$ in Eq.(3.8), one can obtain

$$[g_A(0) + g_M(m) + g_B(b), b^k]_{-1} = 0 \tag{3.29}$$

and

$$\begin{aligned} &(h_A(0) + h_M(m) + h_B(b))b^k \\ = \; &-mb^{k-1}(g_A(0) + g_M(m) + g_B(b)) \\ &-(f_A(0) + f_M(m) + f_B(b))mb^{k-1}. \end{aligned} \tag{3.30}$$

Moreover, Eq.(3.29), Eq.(3.3) and Lemma 3.1 imply $[g_B(b) - g_B(0), b^k]_{-1} = 0$ for each $b \in \mathcal{B}$ and $g_M(m) - g_M(0) = 0$ for all $m \in \mathcal{M}$. Hence (i) and (ii) are true.

Now, by using Eq.(3.3), Step 1 and (ii), Eq.(3.28) and Eq.(3.30) respectively imply that (iv) and (v) hold; by taking $a = e_1$ in Eq.(3.28) and $b = e_2$ in Eq.(3.30), it follows from (ii) that (iii) is also true.

Step 3. $a_1 = b_1 = a_1' = b_1' = 0$, that is, $f_A(e_1) - f_A(0) = 0$, $g_A(e_1) - g_A(0) = 0$, $f_B(e_2) - f_B(0) = 0$ and $g_B(e_2) - g_B(0) = 0$. Therefore, $h_M(m) - h_M(0) = 0$ for all $m \in \mathcal{M}$.

For any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, by Eq.(3.3) and Step 2, Eq.(3.26) reduces to

$$[f_B(b) - f_B(0), a^k]_{-1} m = m[g_A(a) - g_A(0), b^k]_{-1} \text{ for all } m \in \mathcal{M}.$$

By taking $b = e_2$ and $a = a_1$ in the above equation, noting that the characteristic of $\mathcal{U}$ is not 2, one gets $(f_B(e_2) - f_B(0))m = m(g_A(e_1) - g_A(0))$, that is,

$$a_1' m = m b_1 \quad \text{for every} \quad m \in \mathcal{M}. \tag{3.31}$$

On the other hand, letting $a = e_1$ and $b = e_2$ in Step 2(i), we have $f_A(e_1) - f_A(0) = 0$ and $g_B(e_2) - g_B(0) = 0$, that is, $a_1 = b_1' = 0$. Thus, Step 2(iii) implies $-mb_1 = a_1' m$ for all $m \in \mathcal{M}$, which and Eq.(3.31) force to $a_1' = b_1 = 0$ since $\mathcal{M}$ is a faithful left $\mathcal{A}$-module and right $\mathcal{B}$-module. By Step 2, the step holds.

Step 4. For any and $b \in \mathcal{B}$, we have $g_A(a) = g_A(0)$ and $f_B(b) = f_B(0)$.

For any $a \in \mathcal{A}$ and any $b \in \mathcal{B}$, since $[f_A(a) + f_M(0) + f_B(b), a^k]_{-1} m' = m'[g_A(a) + g_M(0) + g_B(b), b^k]_{-1}$ for all $m' \in \mathcal{M}$ (that is, Eq.(3.26)), replacing a by e_1 and by Step 2(i), Step 3 and Eq.(3.2), we have

$$2(f_B(b) - f_B(0))m' = m'[g_B(b) - g_B(0), b^k]_{-1} = 0 \text{ for all } m' \in \mathcal{M},$$

which implies $f_B(b) - f_B(0) = 0$ as $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module with characteristic not 2.

Similarly, replacing b by e_2 in Eq.(3.26), one can prove that $g_A(a) = g_A(0)$ holds for all $a \in \mathcal{A}$.

Step 5. For any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have $f_A(a) - f_A(0) = 0$ and $g_B(b) - g_B(0) = 0$.

Combining Step 2(iv)-(v) and Steps 3-4, one can obtain that

$$(f_A(a) - f_A(0))a^{k-1} = 0 \quad \text{and} \quad b^{k-1}(g_B(b) - g_B(0)) = 0 \tag{3.32}$$

hold for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $m \in \mathcal{M}$. Note that, by Eq.(3.5) and Step 4, it is true that

$$\begin{aligned} & (f_A(a_1 + a_2) - f_A(a_1) - f_A(a_2) + f_A(0))m \\ = \quad & m(g_A(a_1 + a_2) - g_A(a_1) - g_A(a_2) + g_A(0)) = 0 \end{aligned}$$

holds for all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$. This implies

$$f_A(a_1 + a_2) = f_A(a_1) + f_A(a_2) - f_A(0) \text{ for all } a_1, a_2 \in \mathcal{A}. \tag{3.33}$$

Replacing a by $a+e_1$ in Eq.(3.32) and by using Eq.(3.33), one gets

$$(f_A(a)-f_A(0))(a+e_1)^{k-1}=\sum_{i=1}^{k-1}C_{k-1}^i(f_A(a)-f_A(0))a^{k-1-i}=0.$$

Next, replacing a by $a+2e_1, a+3e_1, \ldots, a+(k-1)e_1$ in turn in Eq.(3.32), and expressing the resulting system of $k-1$ homogeneous equations in the variables $C_{k-1}^i(f_A(a)-f_A(0))a^{k-1-i},\ i=1,\ldots,k-1$. Still, the system has only a zero solution. It follows that $f_A(a)-f_A(0)=0$ holds for all $a\in\mathcal{A}$.

Similarly, one can show that $g_B(b)-g_B(0)=0$ holds for all $b\in\mathcal{B}$. The step is true.

Now, by Steps 1-5, it is easily seen that $f(x)=0=f(0)$ holds for $x\in\mathcal{U}$, completing the proof of the theorem.

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