TOTAL PERFECT CODES, OO-IRREDUNDANT AND TOTAL SUBDIVISION IN GRAPHS

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Abstract. Let \( G = (V(G), E(G)) \) be a graph, \( \gamma_t(G) \). Let \( ooir(G) \) be the total domination and OO-irredundance number of \( G \), respectively. A total dominating set \( S \) of \( G \) is called a total perfect code if every vertex in \( V(G) \) is adjacent to exactly one vertex of \( S \). In this paper, we show that if \( G \) has a total perfect code, then \( \gamma_t(G) = ooir(G) \). As a consequence, we determine the value of \( ooir(G) \) for some classes of graphs. Finally, we prove some new bounds for the total subdivision number.

Keywords: Total domination number, OO- irredundance number, total subdivision number.

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1. Introduction

Let \( G = (V(G), E(G)) \) be a simple graph and \( u \in X \subseteq V(G) \). The open neighborhood of \( u \) is denoted by \( N(u) \) and, the close neighborhood of \( u \) is \( N[u] = N(u) \cup \{u\} \). The open (closed) neighborhood of \( X \) is \( N(X) = \bigcup_{u \in X} N(u) \) \( (N[X] = \bigcup_{u \in X} N[u]) \). A subset \( S \subseteq V(G) \) is a dominating set of \( G \) if every vertex of \( V(G)/S \) is adjacent to some vertices of \( S \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of all dominating sets of \( G \). A dominating set of cardinality \( \gamma(G) \) is called a \( \gamma(G) \)-set. If every vertex of \( V(G)/S \) is adjacent to exactly one vertex of \( S \) and \( S \) is also an independent set, then \( S \) is called a perfect dominating set [5]. Domination is one of the major and well studied areas in graph theory. For more details on this concept see [5]. Among many types of dominating sets, total dominating sets have been investigated extensively [8]. A subset \( S \) of vertices in a graph \( G \) is a total dominating set if each vertex of \( V(G) \) is adjacent to some vertices of \( S \). The total domination number is the minimum cardinality of all minimal total dominating sets of \( G \) and is...
denoted by $\gamma_t(G)$. The total dominating set of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. If every vertex in $V(G)$ is adjacent to exactly one vertex of $S$, then $S$ is called a total perfect code or an efficient open dominating set [3]. Not necessarily all graphs have a perfect or total perfect code. For example, the cycle $C_5$ has neither a perfect code nor a total perfect code, $C_6$ has a perfect code but not any total perfect code and $C_4$ has total perfect code but not any perfect code. In this paper we focus on total perfect codes. At the beginning we state some basic results which are used in our proofs.

**Theorem 1.1.** [3] If a graph $G$ has a total perfect code $S$, then $|S| = \gamma_t(G)$ and all total perfect codes have the same cardinality.

**Theorem 1.2.** [3] For a graph $G$ the following are equivalent:

(a) $C = \{v_1, v_2, \ldots, v_k\}$ is a total perfect code of $G$.

(b) $\{N(v_1), N(v_2), \ldots, N(v_k)\}$ is a partition of $V(G)$.

(c) $|V(G)| = \sum_{v \in C} \deg(v)$

We note that not necessarily all $\gamma_t(G)$-sets are perfect. For example, in Figure 1, squared vertices is the total perfect code of $P_6 \square P_6$ and black vertices form the total dominating set of minimum size which is not perfect.

Determining whether an arbitrary graph $G$ has a total perfect code is an NP-complete problem [5] but the existence of total perfect codes for some classes of graphs has been examined. A total perfect code allows a graph to enjoy several properties. For example, as an easy observation it can lead to a better bound for the matching number. We recall that a matching in a graph is a set of pairwise nonadjacent edges. Number of edges in the maximum matching of graph $G$ is the matching number $\alpha'(G)$ of $G$.

It is not hard to see that for every graph $G$ with no isolated vertex, $\gamma(G) \leq \alpha'(G)$. Since $\gamma(G) \leq \gamma_t(G)$, Henning naturally asked in his survey [7] if the
inequality \( \gamma_t(G) \leq \alpha'(G) \) holds for every graph \( G \) with sufficiently large minimum degree? This inequality was proved for the family of claw-free graphs with minimum degree at least three and for the family of \( k \)-regular graphs when \( k \geq 3 \) (see [7]). We will show that this is true for all graphs which have total perfect code.

**Observation:** If \( G \) is a graph with \( \delta(G) \geq 2 \) that has a total perfect code, then \( \gamma_t(G) \leq \alpha'(G) \).

**Proof.** Let \( C = \{v_1, v_2, \ldots, v_k\} \) be a total perfect code of \( G \). We may assume that \( v_iv_{i+1} \in E(G) \), for all odd \( i \) with \( 1 \leq i \leq k - 1 \). By Theorem 1.1, \( N(v_1) \cup \cdots \cup N(v_k) \) is a partition of \( V(G) \). Since \( \delta(G) \geq 2 \), for each \( i, 1 \leq i \leq k \), there is \( u_i \in N(v_i) \setminus C \). The edges \( u_iv_i \) for \( 1 \leq i \leq k \) form a matching for the graph \( G \). Therefore, \( \gamma_t(G) \leq \alpha'(G) \). \( \square \)

In the next section we will establish the exact values of OO-irredundance number for some classes of graphs. In Section 3 we will give some bounds for total subdivision domination number of some graphs.

2. Total perfect codes and OO-irredundance

The set \( X \subseteq V(G) \) is OO-irredundance if and only if for each \( v \in X \), \( N(v) - N(X \setminus \{v\}) \neq \emptyset \). For convenience we let \( PN(v) = N(v) \setminus N(X \setminus \{v\}) \). The minimum cardinality among all maximal OO-irredundant set denoted by \( ooir(G) \) and is called OO-irredundance number of the graph \( G \). To the best of our knowledge \( ooir \) has only been determined for paths and cycles [1]. The main theorem of this section leads to determination of the exact value of OO-irredundance number for several classes of graphs.

The following proposition is an easy consequence of definitions.

**Proposition 2.1.** Any minimal total dominating set of a graph \( G \) is also a maximal OO-irredundant set. Therefore, \( ooir(G) \leq \gamma_t(G) \).

**Proof.** Assume that \( S \) is a minimal total dominating set of \( G \). Then, for each \( u \in S \) there is \( v \in V(G) \) such that \( N(v) \cap N(S) = \{u\} \). Therefore, \( N(v) \setminus N(S \setminus \{v\}) \neq \emptyset \) and \( S \) is an OO-irredundant set. On the other hand, for each \( x \in V \setminus S \), \( S \cup \{x\} \) is not an OO-irredundant set since otherwise there is \( v \in V \) that is not adjacent to any vertex in \( S \). \( \square \)

The following lemma is our main result of this section.

**Lemma 2.2.** Let \( G = (V, E) \) be a simple graph, \( C = \{v_1, v_2, \ldots, v_k\} \) total perfect code and \( S \subseteq V \) a maximal OO-irredundant set of \( G \). Then, \( |S| \geq |C| \).

**Proof.** Without loss of generality we can assume that \( v_1 \notin S \). We claim that we can find a vertex \( x(v_1) \in (V(G) \setminus C) \cap S \). Since \( S \) is a maximal OO-irredundant set, \( S \cup \{v_1\} \) is not OO-irredundant. This means that there is at least one vertex
in $S \cup \{v_1\}$ that has no private neighborhood. To proceed with the proof, we consider the following two cases:

Case 1) There is $y \in S$ such that $N(y) \setminus N(S \setminus \{y\}) \subseteq N(v_1)$. This means that all private neighbors of $y$ belong to $N(v_1)$. Accordingly, we define $x(v_1) = y$. Furthermore, we define $X_1(v_1) \subseteq S$ to be the set of all vertices $x(v_1)$ (The index 1 for $X$ indicates the Case 1.)

Case 2) $N(v_1) \subseteq N(S)$. Note that this implies $N(v_1) \setminus N(S \cup \{v_1\}) = \emptyset$. In this case we define $x(v_1)$ to be a vertex in $S$ which has a common neighbor with $v_1$. Likewise, we define the set $X_2(v_1)$ consisting of vertices in $S$ of this type.

Note that in both cases $X_i(v_1) \cap C = \emptyset$ ($i \in \{1, 2\}$). Now it remains to prove that if $v_i, v_j \notin S$ then $x(v_i) \neq x(v_j)$. Since $X_2(v_i) \cap X_2(v_j) = \emptyset$ without loss of generality we can assume $X_2(v_i) = X_2(v_j) = \emptyset$ and $X_1(v_i) = X_1(v_j) = \{y\}$. This means that $y$ has at least one private neighbor in $N(v_i)$ and one is in $N(v_j)$. Now consider $S \cup \{v_1\}$. Since it is not an OO-irredundant set, there is $z \in S \cup \{v_1\}$ such that $N(z) \setminus N((S \cup \{v_1\}) \setminus z) \neq \emptyset$. We know that $z \neq v_i$ since otherwise $z \in X_2(v_i)$. Also $z \neq y$ contradicts the assumption that $X_1(v_i) = \{y\}$. By symmetry this is true for $v_j$. We conclude that $S \setminus \{y\} \cup \{v_i, v_j\}$ is an OO-irredundant set for a subset of $G$ properly containing $S$, and this contradiction with maximality of $S$. Hence either $X_2(v_i) = X_2(v_j) \neq \emptyset$ or $|X_1(v_i) \Delta X_1(v_j)| \geq 2$. \hfill \Box

The previous lemma yields that if the graph $G$ has a total perfect code, then $\gamma_t(G) \leq ooir(G)$. Therefore, by Proposition 2.1 we have the following theorem.

**Theorem 2.3.** If a graph $G$ has a total perfect code, then $ooir(G) = \gamma_t(G)$.

The tree $T$ in Figure 2 shows that the difference between $ooir(G)$ and $\gamma_t(G)$ can be arbitrarily large. Note that $T$ is of order $2k + 2$. The vertex of degree $k + 1$ is called the root of $T$. The root and one of its neighbors is a maximal OO-irredundant set of minimum size, hence $ooir(T) = 2$. While, we need to pick up all vertices of degree 2 and the root in the minimum total dominating set of $T$, we have $\gamma_t(T) = k + 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree.png}
\caption{A tree with large total domination number and small OO-irredundance number}
\end{figure}
The existence of total perfect codes for some classes of graphs has been investigated in the literatures. For example, $P_n$ has a total perfect code if and only if $n \not\equiv 1 \pmod{4}$ and $C_n$ has a total perfect code if and only if $n \equiv 0 \pmod{4}$ ([3]). Also we have $\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod{4} \\ \frac{n}{2} + 1 & \text{otherwise} \end{cases}$ [5]. This means we can easily compute the value of $\gamma_t$ for infinitely many cycles and paths. This has been a challenge in [1]. Grid graphs $P_n \square P_m$ which have total perfect code are characterized in [9, 12]. In particular, it is proved in [9] that for two positive integers $m$ and $n$, $n \geq m \geq 2$, $P_n \square P_m$ has a total perfect code if and only if $m$ is even and $n = m + c(m + 1)$ or $n = m + 2 + c(m + 1)$ or $n = m - 2 + c(m + 1)$, for a nonnegative integer $c$. All trees which have a total perfect code are found in [3]. So using our result, we can find the value of $\gamma_t$ for these graphs.

In the following we consider the existence of total perfect codes for some other graphs.

**Proposition 2.4.** The hypercube $Q_n$ with $2^n$ vertices has a total perfect code if and only if $n = 2^m$ for some integer $m$.

*Proof.* First assume that $C = \{v_1, \ldots, v_k\}$ is a total perfect code of $Q_n$. Then, by Theorem 1.1, $V(Q_n) = N(v_1) \cup \cdots \cup N(v_k)$. Since for each $1 \leq i \leq k$, $|N(v_i)| = n$, thus $2^n = |V(Q_n)| = k \times n$. So $n = 2^m$ for an integer $m$. Also $Q_{2^m} = Q_{2^{m-1}} \square P_2$, where $\square$ shows the Cartesian product. But it is proved that $Q_{2^{m-1}}$ has perfect codes [10]. Let $C$ be a perfect code of $Q_{2^{m-1}}$, then $V(Q_{2^{m-1}})[C] \square P_2$ is a total perfect code of $Q_{2^n}$. \(\square\)

**Proposition 2.5.** If both $m$ and $n$ are integer multiplies of 4, then $C_n \square C_m$ has a total perfect code.

*Proof.* We know that $C_n$ ($n \equiv 0 \pmod{4}$) has a total perfect code of size $\frac{n}{2}$. So we can assume $P$ and $\bar{P}$ are two disjoint total perfect codes for $C_n$ such that $P \cup \bar{P} = V(C_n)$. Let $C_{n_1}^1, C_{n_2}^2, \ldots, C_{n}^m$ be $m$ copies of $C_n$ in $C_n \square C_m$. For convenience we denote the vertex set of $P$ in $j$th copy of $C_n$ by $P_j$ ($1 \leq j \leq m$). We define two sequences $\{a_i\}_{i=2}^m$ and $\{b_i\}_{i=2}^m$ as follows: $a_{i+1} = a_i + 4$, $a_1 = 1$ and $b_{j+1} = b_j + 4$, $b_1 = 3$. Also we define the set $T = \{\bigcup_{i=1}^m P_{a_i}, \bigcup_{j=1}^m P_{b_j}\}$. The condition $m \equiv 0 \pmod{4}$, guarantees that $T$ is a total perfect code of $C_n \square C_m$. \(\square\)

Therefore, there are plenty of graphs for which we can compute their OO-irredundance numbers rather easily by Theorem 2.3, while it is too computational to determine them directly.
3. Total perfect code and total subdivision number

We say that an edge $uv \in E$ is subdivided if the edge $uv$ is deleted, but a new vertex $x$ is added along with two new edges $ux$ and $xv$. We only permit an edge to be subdivided once. The domination subdivision number $\text{sd}_γ(G)$ of a graph $G$ is the minimum number of edges that must be subdivided in order to increase the domination number. We also define the total domination subdivision number $(\text{sd}_{tγ}(G))$ of $G$ in a same way. Since the total domination number of a single edge does not change when it is subdivided, in the study of total domination subdivision numbers we must assume that the graph is of order $n ≥ 3$. It is known that these two parameters can be arbitrarily large [4,6]. It is also difficult to find good upper bounds for them. The best known upper bound for the total domination subdivision number of a connected graph $G$ of order $n ≥ 3$ is $\left\lfloor \frac{2n}{3} \right\rfloor$.

The following theorems shows that if a graph $G$ has a total perfect code, then we can find a new upper bound for the total subdivision number of $G$.

**Theorem 3.1.** Let $G$ be a graph with $\delta(G) ≥ 2$ and let $C = \{v_1, v_2, \ldots, v_k\}$ be a total perfect code of $G$. If there is an edge $v_iv_j \in E(C)$ such that $N(v_i) \cup N(v_j)/\{v_i, v_j\}$ is not an independent set in $G$, then $\text{sd}_{tγ}(G) ≤ 2$.

**Proof.** We show that there are two edges in the graph $G$ such that the total domination number of $G$ will increase if we subdivide them. Assume $v_iv_j \in E(C)$ and that $N(v_i) \cup N(v_j)/\{v_i, v_j\}$ is not an independent set. Let $G'$ be the graph obtained by subdividing two edges $v_iv_j$ and $ab \in E(G(N(v_i) \cup N(v_j))/\{v_i, v_j\})$ with two vertices $x$ and $y$, respectively. Also let $S'$ be a $\gamma_t(G')$-set. We consider the following two cases:

1. $|S' \cap \{x, y\}| ∈ \{1, 2\}$. It is not hard to see that vertices $x$ and $y$ can not totally dominate $N(v_i) \cup N(v_j)$ and we need at least two more vertices from $N(v_i) \cup N(v_j)$. So $|S' \cap \{N(v_i) \cup N(v_j)\}| ≥ 3$. Also we know that $V(G) = N(v_1) \cup \cdots \cup N(v_k)$. So to totally dominate other vertices of $V(G')$ we need at least one vertex of each set. This means we need at least $k + 1$ vertices in $S'$ and hence we have $\gamma_t(G') > \gamma_t(G)$.

2. $|S' \cap \{x, y\}| = 0$. First note that a unique vertex of $N(v_i) \cup N(v_j)$ can not dominate both $x$ and $y$, also $|S' \cap \{v_i, v_j\}| ≥ 1$. Without loss of generality we can assume $v_i \in S'$. Now to totally dominate $v_i$ we need $|S' \cap N(v_j)| ≥ 1$. But none of the already chosen vertices can dominate $v_j$. So we need at least one other vertex of $S' \cap N(v_j)$. This means $|S' \cap \{N(v_i) \cup N(v_j)\}| ≥ 3$. So, $\gamma_t(G') > \gamma_t(G)$.

□

Haynes et al. [6] proved that for any grid graph $P_n \square P_m$, $1 ≤ \text{sd}_{tγ}(P_n \square P_m) ≤ 4$. Soltankhah [11] improved this bound and proved that $\text{sd}_{tγ}(P_n \square P_m) ≤ 3$ for $m, n ≥ 3$. Since total perfect codes of grid graphs satisfy the conditions of
Theorem 3.1, we conclude that, $sd_{\gamma_t}(P_n \square P_m) \leq 2$ for any grid graphs which has a total perfect code.

The Conditions in Theorem 3.1 are in fact necessary. In the graph of Figure 3, black vertices form a total perfect code of the graph and it is easy to check that subdividing any two edges of the graph do not increase the total domination number. However, our next theorem shows that the total subdivision number of this graph is equal to 3.

**Theorem 3.2.** Let $G$ be a connected graph with $\delta(G) \geq 2$ and the set $C = \{v_1, v_2, \ldots, v_k\}$ be a total perfect code of $G$. If for each $v_i v_j \in E(G)$ such that $v_i, v_j \in C$, $N(v_i) \cup N(v_j)/\{v_i, v_j\}$ is an independent set in $G$, then $sd_{\gamma_t}(G) \leq 3$.

**Proof.** According to Theorem 1.1, $V(G) = N(v_1) \cup \ldots \cup N(v_k)$ is a partition of $V(G)$. We choose two integers $i \neq j \in \{1, \ldots, k\}$ such that $v_i v_j \in E(G)$. Let $x \in N(v_i)$ and $y \in N(v_j)$.

We can show that the total domination number increases by subdividing three edges $x v_i, v_i v_j$ and $v_j y$ with three vertices $a$, $b$ and $c$, respectively. Let $G'$ be the resulting graph and let $S'$ be a $\gamma_t(G')$-set. First assume that, $|S' \cap \{a, b, c\}| = 3$. To totally dominate vertices in $\bigcup_{k=1, k \neq i, j}^k N(v_i)$ we need at least $k-2$ vertices in $S'_i$, so we conclude $\gamma_t(G') > \gamma_t(G)$. Also if $|S' \cap \{a, b, c\}| = \{1, 2\}$, we need at least two other vertices of $N(v_i) \cup N(v_j)$ to totally dominate the set $\{a, b, c, v_i, v_j\}$. Thus again we have $\gamma_t(G') > \gamma_t(G)$. Finally, if $|S' \cap \{a, b, c\}| = 0$ then $|S' \cap \{v_i, v_j\}| \geq 1$. Let $v_i \in S'$, to totally dominate $v_i$ there must be a vertex $t \in N(v_i) \cap S'$ but neither $t$ nor $v_i$ can dominate $v_j$. Therefore, $S'$ contains at least one other vertex of $N(v_j)$. Thus $\gamma_t(G') > \gamma_t(G)$. \[\square\]

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