ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 3, pp. 507–519

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 3, pp. 507–519
Online ISSN: 1735-8515

THE THEORY OF MATRIX-VALUED MULTIRESOLUTION ANALYSIS FRAMES

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(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. A generalization of matrix-valued multiresolution analysis (MMRA) to matrix-valued frames, and the constructions of matrix-valued frames are considered and characterized. A matrix-valued frame multiresolution analysis is defined in this paper. We provide necessary and sufficient conditions for constructing matrix-valued frames and Riesz bases of translates, and give the calculation method of matrix-valued dual Riesz basis. These conclusions are useful in providing theoretical basis for constructing matrix-valued frames and Riesz basis.

Keywords: matrix-valued wavelet, frame, wavelets, matrix-valued dual Riesz basis.
MSC(2010): Primary: 42A38; Secondary: 42C40.

1. Introduction

The concept of matrix-valued wavelets was introduced [14, 15] by utilizing the theory of paraunitary matrix filterbanks. Matrix-valued signals are often encountered in applications, such as video images, multispectral images, and color images. We see in [3, 14] that multiwavelets [6, 8] can be generated from the component functions in matrix-valued wavelets and prefiltering does not necessary for matrix-valued wavelet transforms. The matrix-based method also is used for the construction of a non-trivial symmetric quaternion wavelet with compact support [7]. Therefore, studies on matrix-valued wavelets are useful in the theory of multiwavelets and in the representations of signals. Matrix-valued wavelets have drawn much attention [1, 3–5, 7, 9, 11–15, 18]. The construction of orthonormal matrix-valued filters for multi-resolution analysis of matrix-valued time-series is studied [13]. The concept of biorthogonal matrix-valued wavelets bases [5,12] and orthogonal matrix-valued wavelet packets [4] are proposed. The Riesz basis functions of matrix-valued wavelet series expansion are obtained

Article electronically published on June 29, 2016.
Received: 11 October 2014, Accepted: 26 January 2015.
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from wavelets of the matrix-valued Lemari-Meyer type and the power-spectral density of the process [18].

Theory of scalar-valued multiresolution analysis frames [2, 10, 17] have become a hot field in time-frequency analysis recently. Their applications almost involve all areas in signal processing and communication theory. Many application fields have benefited from the use of wavelet frames. This theory of scalar-valued multiresolution analysis frames is extended to vector-valued signal space recently. The paper [11] is to give the construction of vector-valued multivariate wavelet frame packets associated with arbitrary dilation matrix. Minimum-energy vector-valued wavelet tight frames are introduced in [9]. However, the present research results about theory of matrix-valued multiresolution analysis frames do not have according to our knowledge. That is our motivation for writing this paper.

The main purpose of this paper is to give the theory of matrix-valued multiresolution analysis frames. This paper is organized as follows. In Section 2, we describe some concepts and notations for matrix-valued function spaces. We will introduce a version of matrix-valued multiresolution analysis to frame. In Section 3, a necessary and sufficient condition is provided for constructing matrix-valued frames of translates. We derive the calculation method of matrix-valued dual Riesz basis. Example of matrix-valued frames according to our theory is constructed.

2. **Preliminary**

We proceed with a brief introduction on the notion of the matrix-valued signal space. For a more detailed description see [14, 15]. Let

$$\mathbf{C}^{N\times N} = \{A : A \text{ is an } N\times N \text{ matrix with entries in the complex plane } \mathbf{C}\},$$

and

$$L^2(a,b;\mathbf{C}^{N\times N}) = \left\{\mathbf{f}\,(\mathrm{t}) = (f_{k,l}\,(t))_{N\times N} : f_{k,l}\,(t)\in L^2\,(a,b)\,, 1\le k,l\le N\right\}.$$

The space $L^2(a,b;\mathbf{C}^{N\times N})$ is called a matrix-valued signal space. When $a = -\infty$ and b $= +\infty$, $L^2(a,b;\mathbf{C}^{N\times N})$ is also denoted by $L^2(\mathbf{R};\mathbf{C}^{N\times N})$.

For the matrix-valued function $\mathbf{f}\in L^2(\mathbf{R};\mathbf{C}^{N\times N})$, its integration $\int \mathbf{f}(t)dt$ is defined as

$$\int \mathbf{f}(t)dt = \begin{pmatrix} \int f_{11}\,(t)\,dt & \int f_{12}\,(t)\,dt & \cdots & \int f_{1N}\,(t)\,dt \\ \int f_{21}\,(t)\,dt & \int f_{22}\,(t)\,dt & \cdots & \int f_{2N}\,(t)\,dt \\ & & \cdots & \\ \int f_{N1}\,(t)\,dt & \int f_{N2}\,(t)\,dt & \cdots & \int f_{NN}\,(t)\,dt \end{pmatrix}$$

if

$$\mathbf{f}(t) = \begin{pmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1N}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2N}(t) \\ & & \cdots & \\ f_{N1}(t) & f_{N2}(t) & \cdots & f_{NN}(t) \end{pmatrix}$$

and the differentiation is defined accordingly.

For two matrix-valued functions $\mathbf{f}$, $\mathbf{g} \in L^2(\mathbf{R}; \mathbf{C}^{N\times N})$, $\langle \mathbf{f}, \mathbf{g} \rangle$ denotes the integration of the matrix product $\mathbf{f}(t)\,\mathbf{g}^*(t)$

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbf{R}} \mathbf{f}(t)\,\mathbf{g}^*(t)\,dt \tag{2.1}$$

where $*$ means the transpose and the complex conjugate. For convenience, we still call the operation $\langle , \rangle$ in Eq. (2.1) inner product.

For each $t \in \mathbf{R}$ and $\mathbf{f} \in L^2(\mathbf{R}; \mathbf{C}^{N\times N})$, $\|\mathbf{f}\|$ denotes a norm for the matrix $\mathbf{f}(t)$ as

$$\|\mathbf{f}\| = \left(\sum_{k,l=1}^{N} \int_{\mathbf{R}} |f_{kl}(t)|^2\,dt \right)^{1/2}. \tag{2.2}$$

That is $\|\mathbf{f}\|^2 = \text{trace}\,\langle \mathbf{f}, \mathbf{f} \rangle$.

The Fourier transform of the matrix-valued function $\mathbf{f}$ is defined by

$$\hat{\mathbf{f}}(\omega) = \int_{\mathbf{R}} \mathbf{f}(t)\,e^{-it\omega}dt.$$

Then, the inverse Fourier transform is

$$\mathbf{f}(t) = \int_{\mathbf{R}} \hat{\mathbf{f}}(\omega)\,e^{it\omega}d\omega.$$

Let

$$\chi_D(t) = \begin{cases} \mathbf{I}, & t \in D \\ \mathbf{0}, & \text{otherwise} \end{cases}$$

where $\mathbf{0}$ is the zero matrix and $\mathbf{I}$ is an identity matrix.

Let

$$l^2(\mathbf{Z}) = \left\{ \mathbf{C} = \{C_k\}_{k\in\mathbf{Z}} : \sum_{k\in\mathbf{Z}} \|C_k\|^2_{N\times N} < +\infty \right\}.$$

$$\|\mathbf{C}\|^2_{l^2(\mathbf{Z})} = \sum_{k\in\mathbf{Z}} \|C_k\|^2_{N\times N}$$

where the constant matrix $C_k = \begin{pmatrix} C_{11}^k & C_{12}^k & \cdots & C_{1N}^k \\ C_{21}^k & C_{22}^k & \cdots & C_{2N}^k \\ & & \cdots & \\ C_{N1}^k & C_{N2}^k & \cdots & C_{NN}^k \end{pmatrix}$, $\|C_k\|_{N\times N}^2 = \sum_{i,j=1}^{N} \left|C_{i,j}^k\right|^2$.

A sequence $\mathbf{\Phi}_k \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$, $k \in \mathbf{Z}$ is called an orthonormal set in $L^2(\mathbf{R};\mathbf{C}^{N\times N})$ if

$$\langle \mathbf{\Phi}_k, \mathbf{\Phi}_l \rangle = \delta\,(k-l)\,\mathbf{I}_N$$

where $\delta\,(j) = 1$ when j =0 and $\delta\,(j) = 0$ when $j \neq 0$, and $\mathbf{I}_N$ is the $N \times N$ identity matrix. A sequence $\mathbf{\Phi}_k \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$, $k \in \mathbf{Z}$, is called an orthonormal basis for $L^2(\mathbf{R};\mathbf{C}^{N\times N})$ if it satisfies $\langle \mathbf{\Phi}_k, \mathbf{\Phi}_l \rangle = \delta\,(k-l)\,\mathbf{I}_N$, and moreover for any $\mathbf{f} \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$ there exists a sequence of $N \times N$ constant matrices $\mathbf{F}_k$ such that

$$\mathbf{f}\,(t) = \sum_{k\in\mathbf{Z}} \mathbf{F}_k \mathbf{\Phi}_k\,(t), \quad \text{for} \quad t \in \mathbf{R}$$

where the multiplication $\mathbf{F}_k\mathbf{\Phi}_k\,(t)$ for each fixed t is the $N \times N$ matrix multiplication, and the convergence for the infinite summation is in the sense of the norm $\|\cdot\|$ defined by Eq.(2.2) for the matrix-valued signal space.

A matrix–valued function $\mathbf{F}\,(\omega)$ is called unitary if

$$\mathbf{F}\,(\omega)\,\mathbf{F}^*\,(\omega) = \mathbf{I}_N.$$

An **MMRA** (matrix-valued multiresolution analysis) of the signal space $L^2(\mathbf{R};\mathbf{C}^{N\times N})$ is a nested sequence of closed subspaces $\mathbf{V}_j$, $j \in \mathbf{Z}$ of $L^2(\mathbf{R};C^{N\times N})$ such that

(i) $\mathbf{V}_j \subset \mathbf{V}_{j+1}$, $j \in \mathbf{Z}$.

(ii) $\cup_{j\in\mathbf{Z}}\mathbf{V}_j$ is dense in $L^2(\mathbf{R};\mathbf{C}^{N\times N})$ and

$$\cap_{j\in\mathbf{Z}}\mathbf{V}_j = \{\mathbf{0}\},$$

where $\mathbf{0}$ is the zero matrix.

(iii) $\mathbf{f}\,(t) \in \mathbf{V}_j$ if and only if $\mathbf{f}\,(2t) \in \mathbf{V}_{j+1}$, $j \in \mathbf{Z}$.

(iv) There is a $\mathbf{\Phi} \in \mathbf{V}_0$ such that its integer translations $\mathbf{\Phi}_k\,(t) = \mathbf{\Phi}\,(t-k)$, $k \in \mathbf{Z}$, form a frame for $\mathbf{V}_0$, i.e., two constants A and B exist, $0 < A \leq B < +\infty$, such that

$$A\,\|\mathbf{f}\|^2 \leq \sum_{k\in\mathbf{z}} \|\langle \mathbf{f}, \mathbf{\Phi}_k \rangle\|^2 \leq B\,\|\mathbf{f}\|^2\,, \forall \mathbf{f} \in L^2(\mathbf{R};\mathbf{C}^{N\times N}),$$

where A, B are frame bounds. A frame $\{\mathbf{\Phi}_k\}$ is tight if $A = B$. A frame $\{\mathbf{\Phi}_k\}$ is exact if it ceases to be a frame when any one of its elements is removed.

In this case we call $\mathbf{\Phi}\,(t)$ a *matrix-valued scaling function* for the **MMRA** $\{\mathbf{V}_j\}$. Since $\mathbf{\Phi}\,(t) \in \mathbf{V}_0 \subset \mathbf{V}_1$, by the above definition there exist constant

$N \times N$ matrices $H_k, k \in \mathbf{Z}$ (or $H'_k, k \in \mathbf{Z}$) such that

$$\mathbf{\Phi}(t) = 2\sum_k \mathbf{\Phi}(2t-k)\, H_k,$$

or $\mathbf{\Phi}(t) = 2\sum_k H'_k \mathbf{\Phi}(2t-k)$.

Let

$$\mathbf{H}(\omega) = \sum_k H_k e^{-ik\omega},$$

or $\mathbf{H}'(\omega) = \sum_k H'_k e^{-ik\omega}$.

Then

$$\hat{\mathbf{\Phi}}(\omega) = \hat{\mathbf{\Phi}}\left(\frac{\omega}{2}\right)\mathbf{H}\left(\frac{\omega}{2}\right) = \hat{\mathbf{\Phi}}(0)\cdots\mathbf{H}\left(\frac{\omega}{4}\right)\mathbf{H}\left(\frac{\omega}{2}\right),$$

or

$$\hat{\mathbf{\Phi}}(\omega) = \mathbf{H}'\left(\frac{\omega}{2}\right)\hat{\mathbf{\Phi}}\left(\frac{\omega}{2}\right) = \mathbf{H}'\left(\frac{\omega}{2}\right)\mathbf{H}'\left(\frac{\omega}{4}\right)\cdots\hat{\mathbf{\Phi}}(0).$$

Similarly, one can also define a frame for a subspace of $L^2(\mathbf{R};\mathbf{C}^{N\times N})$. If $\{\mathbf{\Phi}_n\}$ is a frame, it has a dual frame$\left\{\tilde{\mathbf{\Phi}}_n\right\}$ such that

$$\left\langle \mathbf{\Phi}_m, \tilde{\mathbf{\Phi}}_n \right\rangle = \delta(m-n)\,\mathbf{I}_{\mathrm{N}},$$

and any matrix–valued function $\mathbf{f}(t) \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$ can be expanded as

$$\mathbf{f}(t) = \sum_k \left\langle \mathbf{f}(t), \tilde{\mathbf{\Phi}}_k(t)\right\rangle \mathbf{\Phi}_k(t) = \sum_k \left\langle \mathbf{f}(t), \mathbf{\Phi}_k(t)\right\rangle \tilde{\mathbf{\Phi}}_k(t).$$

3. Main results

For $\mathbf{\Phi} \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$, we consider the 2π-periodic and symmetric matrix–valued function,

$$\mathbf{E}_{\mathbf{\Phi}}(\omega) = \sum_k \hat{\mathbf{\Phi}}(\omega+2\pi k)\,\hat{\mathbf{\Phi}}^*(\omega+2\pi k).$$

It is clear that $\mathbf{E}_{\mathbf{\Phi}}(\omega)$ is self-adjoint and positive semidefinite matrix, and thus has real nonnegative eigenvalues $\lambda_k(\omega) \geq 0$, $k = 1, 2, \ldots N$.

Theorem 3.1. *Let* $\mathbf{V}_0 \equiv \overline{span}\{\mathbf{\Phi}_k(t) : \mathbf{\Phi}_k(t) = \mathbf{\Phi}(t-k), k \in \mathbf{Z}\}$ *for some* $\mathbf{\Phi} \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$*, and assume* $\mathbf{E}_{\mathbf{\Phi}}(\omega) \in L^2(0, 2\pi;\mathbf{C}^{N\times N})$*. The sequence* $\{\mathbf{\Phi}_k\}$ *is a frame for* $\mathbf{V}_0$ *with frame bounds A and B if and only if there are positive constants A and B such that, for all matrix trigonometric polynomials*

$$\theta(\omega) = \sum_k C_k e^{-i\omega k}$$

defined on $[0, 2\pi]$, the inequalities

$$A \cdot \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega \leq \tag{3.1}$$

$$\text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega \leq B \cdot \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega < +\infty$$

are hold.

Proof. Define $\mathbf{f} = \sum_k C_k \mathbf{\Phi}(t-k)$, by Parseval's Identity and the shift property of the Fourier transform we have

$$\|\mathbf{f}\|^2 = \text{trace} \frac{1}{2\pi} \left\langle \sum_k C_k e^{-i\omega k} \hat{\mathbf{\Phi}}(\omega), \sum_{k'} C_{k'} e^{-i\omega k'} \hat{\mathbf{\Phi}}(\omega) \right\rangle \tag{3.2}$$

$$= \text{trace} \frac{1}{2\pi} \left\langle \theta(\omega)\hat{\mathbf{\Phi}}(\omega), \theta(\omega)\hat{\mathbf{\Phi}}(\omega) \right\rangle$$

$$= \text{trace} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \theta(\omega)\hat{\mathbf{\Phi}}(\omega)\hat{\mathbf{\Phi}}^*(\omega)\theta^*(\omega)d\omega$$

$$= \text{trace} \frac{1}{2\pi} \int_0^{2\pi} \theta(\omega) \sum_k \hat{\mathbf{\Phi}}(\omega + 2\pi k)\hat{\mathbf{\Phi}}^*(\omega + 2\pi k)\theta^*(\omega)d\omega$$

$$= \text{trace} \frac{1}{2\pi} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega.$$

Again by Parseval's Identity we have

$$\sum_{k=-K}^{K} \|\langle \mathbf{f}, \mathbf{\Phi}_k \rangle\|^2 = \sum_{k--K}^{K} \text{trace} \langle \mathbf{f}, \mathbf{\Phi}_k \rangle \langle \mathbf{f}, \mathbf{\Phi}_k \rangle^* \tag{3.3}$$

$$= \text{trace} \sum_m C_m \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{\Phi}}(\omega)\hat{\mathbf{\Phi}}^*(\omega)e^{-i\omega m}.$$

$$\left(\sum_n \int_{-\infty}^{+\infty} \hat{\mathbf{\Phi}}(\omega')\hat{\mathbf{\Phi}}^*(\omega')e^{i\omega' n} \frac{1}{2\pi} \sum_{k=-K}^{K} e^{ik(\omega-\omega')} d\omega' C_n^* \right) d\omega$$

where $K \in \mathbf{Z}$. We define the Dirichlet kernel

$$D_K(\omega) = \frac{1}{2\pi} \sum_{k=-K}^{K} e^{ik\omega}.$$

Hence, (3.3) can be rewritten as

$$\sum_{k=-K}^{K} \|\langle \mathbf{f}, \mathbf{\Phi}_k \rangle\|^2$$

$$= \text{trace} \sum_m C_m \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{\Phi}}(\omega) \hat{\mathbf{\Phi}}^*(\omega) e^{-i\omega m} \cdot$$

$$\left(\sum_n \int_{-\infty}^{+\infty} \hat{\mathbf{\Phi}}(\omega') \hat{\mathbf{\Phi}}^*(\omega') e^{i\omega' n} D_K(\omega - \omega') d\omega' C_n^* \right) d\omega$$

$$= \text{trace} \sum_m C_m \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathbf{\Phi}}(\omega) \hat{\mathbf{\Phi}}^*(\omega) e^{-i\omega m} \cdot$$

$$\left(\sum_n \int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega') e^{i\omega' n} D_K(\omega - \omega') d\omega' C_n^* \right) d\omega$$

$$= \text{trace} \sum_m \sum_n C_m \frac{1}{2\pi} \int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega) e^{-i\omega m} \cdot$$

$$\left(\int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega') e^{i\omega' n} D_K(\omega - \omega') d\omega' \right) C_n^* d\omega = \text{trace} \sum_m \sum_n C_m \frac{1}{2\pi} \cdot$$

$$\left[\int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega) e^{-i\omega m} \left(\int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega') e^{i\omega' n} D_K(\omega - \omega') d\omega' \right) d\omega \right] C_n^*$$

where depending on the periodicity of $D_K(\omega)$. Using properties of the Dirichlet kernel to all components in the matrix and using the dominated convergence theorem, we have

$$\lim_{K \to +\infty} \int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega) e^{-i\omega m} \left(\int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega') e^{i\omega' n} D_K(\omega - \omega') d\omega' \right) d\omega$$

$$= \int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega) \mathbf{E}_{\mathbf{\Phi}}(\omega) e^{-i\omega(m-n)} d\omega.$$

Hence

$$\lim_{K \to +\infty} \sum_{k=-K}^{K} \|\langle \mathbf{f}, \mathbf{\Phi}_k \rangle\|^2 \tag{3.4}$$

$$= \text{trace}\sum_m\sum_n C_m \frac{1}{2\pi}\int_0^{2\pi} \mathbf{E}_{\mathbf{\Phi}}(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)e^{-i\omega(m-n)}d\omega C_n^*$$

$$= \text{trace}\frac{1}{2\pi}\int_0^{2\pi}\sum_m C_m e^{-i\omega m}\mathbf{E}_{\mathbf{\Phi}}(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\left(\sum_n C_n e^{-i\omega n}\right)^* d\omega$$

$$= \text{trace}\frac{1}{2\pi}\int_0^{2\pi}\theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega$$

(3.1) is a consequence of combining (3.2) and (3.4). □

We give necessary and sufficient conditions for matrix-valued functions to form a frame.

Theorem 3.2. *Let* $\mathbf{V}_0 \equiv \overline{span}\left\{\mathbf{\Phi}_k(t) : \mathbf{\Phi}_k(t) = \mathbf{\Phi}(t-k), k \in \mathbf{Z}\right\}$ *be a closed subspace of* $L^2(\mathbf{R};\mathbf{C}^{N\times N})$ *for some* $\mathbf{\Phi} \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$. *Assume the real non-negative eigenvalues of* $\mathbf{E}_{\mathbf{\Phi}}(\omega)$ *are* $\lambda_k(\omega)$, $k = 1, 2, \cdots N$, *and let* $\alpha(\omega) = \min_k \lambda_k(\omega)$, *and* $\beta(\omega) = \max_k \lambda_k(\omega)$ *on* $G_{\mathbf{\Phi}}$. *The sequence* $\{\mathbf{\Phi}_k\}$ *is a frame for* $\mathbf{V}_0$ *with frame bounds* A *and* B *if and only if there are positive constants* A *and* B *such that*

$$A \le \alpha(\omega) \le \beta(\omega) \le B, \text{ a.e. on } G_{\mathbf{\Phi}} \tag{3.5}$$

where $G_{\mathbf{\Phi}} = \{\omega : \mathbf{E}_{\mathbf{\Phi}}(\omega) \neq \mathbf{0}, \omega \in [0, 2\pi]\}$, *and* $\mathbf{0}$ *is zero matrix. In this case,* A *and* B *are frame bounds for* $\{\mathbf{\Phi}_k(t)\}$.

Sufficiency: Assume (3.5) holds, and let

$$\theta(\omega) = \sum_k C_k e^{-i\omega k}$$

be a matrix trigonometric polynomial on $[0, 2\pi]$. We shall prove the inequalities (3.1).

The $\mathbf{E}_{\mathbf{\Phi}}(\omega)$ is self-adjoint and a positive definite matrix on $G_{\mathbf{\Phi}}$, and thus has real eigenvalues $\lambda_k(\omega) > 0$, $k = 1, 2, \ldots, N$ on $G_{\mathbf{\Phi}}$, and has unitary matrix $\mathbf{U}$ such that

$$\mathbf{E}_{\mathbf{\Phi}}(\omega) = \mathbf{U}\mathbf{\Lambda}(\omega)\mathbf{U}^*, \text{ a.e. on } G_{\mathbf{\Phi}}$$

where $\mathbf{\Lambda}(\omega) = \begin{pmatrix} \lambda_1(\omega) & 0 & \cdots & 0 \\ 0 & \lambda_2(\omega) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_N(\omega) \end{pmatrix}$.

For convenience of description, suppose

$$\mathbf{X}(\omega) \equiv \theta(\omega)\mathbf{U} = \begin{pmatrix} x_{11}(\omega) & x_{12}(\omega) & \cdots & x_{1N}(\omega) \\ x_{21}(\omega) & x_{22}(\omega) & \cdots & x_{2N}(\omega) \\ \cdots & \cdots & \cdots & \cdots \\ x_{N1}(\omega) & x_{N2}(\omega) & \cdots & x_{NN}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{x}_1(\omega) & \mathbf{x}_2(\omega) & \cdots & \mathbf{x}_N(\omega) \end{pmatrix}$$

where $\mathbf{x}_j(\omega) = \begin{pmatrix} x_{1j}(\omega) & x_{2j}(\omega) & \cdots & x_{Nj}(\omega) \end{pmatrix}^T$, $j = 1, 2, \ldots, N$, here T means the matrix transpose.

$$A \cdot \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega = A \cdot \text{trace} \int_{G_{\mathbf{\Phi}}} \mathbf{X}(\omega)\mathbf{\Lambda}(\omega)\mathbf{X}^*(\omega)d\omega$$

$$= A \cdot \{ \int_{G_{\mathbf{\Phi}}} \lambda_1(\omega)\mathbf{x}_1(\omega)\mathbf{x}_1^*(\omega)d\omega + \int_{G_{\mathbf{\Phi}}} \lambda_2(\omega)\mathbf{x}_2(\omega)\mathbf{x}_2^*(\omega)d\omega + \cdots + \int_{G_{\mathbf{\Phi}}} \lambda_N(\omega)\mathbf{x}_N(\omega)\mathbf{x}_N^*(\omega)d\omega \}$$

$$\leq \{ \int_{G_{\mathbf{\Phi}}} \lambda_1^2(\omega)\mathbf{x}_1(\omega)\mathbf{x}_1^*(\omega)d\omega + \int_{G_{\mathbf{\Phi}}} \lambda_2^2(\omega)\mathbf{x}_2(\omega)\mathbf{x}_2^*(\omega)d\omega + \cdots + \int_{G_{\mathbf{\Phi}}} \lambda_N^2(\omega)\mathbf{x}_N(\omega)\mathbf{x}_N^*(\omega)d\omega \}$$

$$= \text{trace} \int_{G_{\mathbf{\Phi}}} \mathbf{X}(\omega)\mathbf{\Lambda}(\omega)\mathbf{\Lambda}(\omega)\mathbf{X}^*(\omega)d\omega = \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega$$

$$\leq B \cdot \{ \int_{G_{\mathbf{\Phi}}} \lambda_1(\omega)\mathbf{x}_1(\omega)\mathbf{x}_1^*(\omega)d\omega + \int_{G_{\mathbf{\Phi}}} \lambda_2(\omega)\mathbf{x}_2(\omega)\mathbf{x}_2^*(\omega)d\omega + \cdots + \int_{G_{\mathbf{\Phi}}} \lambda_N(\omega)\mathbf{x}_N(\omega)\mathbf{x}_N^*(\omega)d\omega \}$$

$$= B \cdot \text{trace} \int_{G_{\mathbf{\Phi}}} \mathbf{X}(\omega)\mathbf{\Lambda}(\omega)\mathbf{X}^*(\omega)d\omega = B \cdot \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega,$$

where the two inequalities follow from (3.5). So (3.1) holds with

$$A = \min\{\alpha(\omega) : \omega \in G_{\mathbf{\Phi}}\} \text{ and } B = \max\{\beta(\omega) : \omega \in G_{\mathbf{\Phi}}\}.$$

Necessity: Assume $A > \alpha(\omega)$, for $\omega \in F \subseteq G_{\mathbf{\Phi}}$, where $|F| > 0$. $|F|$ denotes the Lebesque measure of F. Without loss of generality, suppose $A >$

$\lambda_1(\omega)$, for $\omega \in F \subseteq G_{\mathbf{\Phi}}$. Let

$$\theta(\omega) \equiv \mathbf{X}(\omega)\mathbf{U}^* = \begin{cases} \begin{pmatrix} e^{-i\omega} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mathbf{U}^*, \omega \in F \\ \mathbf{0},\ \omega \notin F \end{cases}$$

hence

$$A \cdot \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega = A \cdot \text{trace} \int_{G_{\mathbf{\Phi}}} \mathbf{X}(\omega)\mathbf{\Lambda}(\omega)\mathbf{X}^*(\omega)d\omega$$

$$= A \cdot \left\{ \int_{G_{\mathbf{\Phi}}} \lambda_1(\omega)\mathbf{x}_1(\omega)\mathbf{x}_1^*(\omega)\,d\omega \right\} > \left\{ \int_{G_{\mathbf{\Phi}}} \lambda_1^2(\omega)\mathbf{x}_1(\omega)\mathbf{x}_1^*(\omega)\,d\omega \right\}$$

$$= \text{trace} \int_{G_{\mathbf{\Phi}}} \mathbf{X}(\omega)\mathbf{\Lambda}(\omega)\mathbf{\Lambda}(\omega)\mathbf{X}^*(\omega)d\omega = \text{trace} \int_0^{2\pi} \theta(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\mathbf{E}_{\mathbf{\Phi}}(\omega)\theta^*(\omega)d\omega.$$

We obtain a contradiction to the first inequality of (3.1) which is valid for all matrix trigonometric polynomials $\theta(\omega)$. When we assume $B < \beta(\omega)$, for $\omega \in F \subseteq G_{\mathbf{\Phi}}$ where $|F| > 0$, a similar contradiction arise to the second inequality of (3.1). □

The sequence $\{\mathbf{\Phi}_k\}$ is a Riesz basis if and only if the sequence $\{\mathbf{\Phi}_k\}$ is an exact frame [16]. The next result shows some necessary and sufficient conditions for matrix-valued functions to form Riesz bases.

Theorem 3.3. *Let* $\mathbf{V}_0 \equiv \overline{span}\{\mathbf{\Phi}_k(t) : \mathbf{\Phi}_k(t) = \mathbf{\Phi}(t-k), k \in \mathbf{Z}\}$ *be a closed subspace of* $L^2(\mathbf{R}; \mathbf{C}^{N\times N})$ *for some* $\mathbf{\Phi} \in L^2(\mathbf{R}; \mathbf{C}^{N\times N})$*. Assume the real non-negative eigenvalues of* $\mathbf{E}_{\mathbf{\Phi}}(\omega)$ *are* $\lambda_k(\omega)$, $k = 1, 2, \cdots N$*, and let* $\alpha(\omega) = \min_k \lambda_k(\omega)$, $\beta(\omega) = \max_k \lambda_k(\omega)$*. The sequence* $\{\mathbf{\Phi}_k\}$ *is a Riesz basis for* $\mathbf{V}_0$ *if and only if there are positive constants* A *and* B *such that*

$$A \le \alpha(\omega) \le \beta(\omega) \le B, \ \ on\ \ [0, 2\pi].$$

In particular, the sequence $\{\mathbf{\Phi}_k\}$ *is an orthonormal basis of* $\mathbf{V}_0$ *if and only if*

$$\mathbf{E}_{\mathbf{\Phi}}(\omega) = \mathbf{I}, \ \ on\ \ [0, 2\pi]$$

where $\mathbf{I}$ *is an identity matrix.*

Proof. The sequence $\{\mathbf{\Phi}_k\}$ is a Riesz basis for $\mathbf{V}_0$, the $\mathbf{E}_{\mathbf{\Phi}}(\omega)$ is self-adjoint and positive definite matrix on $[0, 2\pi]$, and thus has real eigenvalues $\lambda_k(\omega) > 0$, $k = 1, 2, \cdots N$ on $[0, 2\pi]$. The results are easy to prove. □

The following theorem introduces the dual Riesz basis $\{\mathbf{\Phi}_k\}$ and $\left\{\tilde{\mathbf{\Phi}}_k\right\}$.

Theorem 3.4. *Let* $\mathbf{V}_0 \equiv \overline{span}\{\mathbf{\Phi}_k(t) : \mathbf{\Phi}_k(t) = \mathbf{\Phi}(t-k), k \in \mathbf{Z}\}$ *be a closed subspace of* $L^2(\mathbf{R};\mathbf{C}^{N\times N})$ *for some* $\mathbf{\Phi} \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$. *Suppose that the sequence* $\{\mathbf{\Phi}_k\}$ *is a Riesz basis for* $\mathbf{V}_0$. *Let* $\tilde{\mathbf{\Phi}} \in L^2(\mathbf{R};\mathbf{C}^{N\times N})$ *be defined by*

$$\tilde{\mathbf{\Phi}}(\omega) \equiv \mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega)\mathbf{\Phi}(\omega). \tag{3.6}$$

for $\omega \in \mathbf{R}$. *Then* $\tilde{\mathbf{\Phi}} \in \mathbf{V}_0$, *and*

$$\sum_k \hat{\mathbf{\Phi}}(\omega + 2\pi k)\,\hat{\tilde{\mathbf{\Phi}}}^*(\omega + 2\pi k) = \mathbf{I}. \tag{3.7}$$

Furthermore,

$$\left\langle \mathbf{\Phi}_k, \tilde{\mathbf{\Phi}}_l \right\rangle = \delta(k-l)\,\mathbf{I}_N,$$

for all $k, l \in \mathbf{Z}$.

Proof. Since the sequence $\{\mathbf{\Phi}_k\}$ is a Riesz basis for $\mathbf{V}_0$, Theorem 3.3 shows that the eigenvalues $\lambda(\omega)$ of $\mathbf{E}_{\mathbf{\Phi}}(\omega)$ satisfy

$$A \le \lambda(\omega) \le B.$$

Therefore, the eigenvalues of $\mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega)$ are bounded by $1/B$ and $1/A$. It follows from the self-adjointness of $\mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega)$ that its coefficients are bounded and in $L^2(\mathbf{R};\mathbf{C}^{N\times N})$. Thus, $\mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega)$ has a matrix Fourier series expansion of the form

$$\mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega) = \sum_k C_k e^{-i\omega k}, \tag{3.8}$$

where $\{C_k\}_{k\in\mathbf{Z}} \in l^2(\mathbf{Z})$. From (3.6) and (3.8) we get

$$\tilde{\mathbf{\Phi}}(t) = \sum_k C_k \mathbf{\Phi}(t-k),$$

which shows that $\tilde{\mathbf{\Phi}} \in \mathbf{V}_0$. The identity (3.7) follows directly from (3.6) and the fact that $\mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega)$ is 2π-periodic. By Parseval's Identity and (3.7) we have that

$$\left\langle \mathbf{\Phi}_k, \tilde{\mathbf{\Phi}}_l \right\rangle = \frac{1}{2\pi}\int_{\mathbf{R}} e^{i\omega(l-k)}\hat{\mathbf{\Phi}}(\omega)\,\hat{\tilde{\mathbf{\Phi}}}^*(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\sum_{j\in\mathbf{Z}}\int_0^{2\pi} e^{i\omega(l-k)}\hat{\mathbf{\Phi}}(\omega + 2\pi j)\,\hat{\tilde{\mathbf{\Phi}}}^*(\omega + 2\pi j)\,d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega(l-k)} \sum_{j\in\mathbf{Z}} \hat{\mathbf{\Phi}}(\omega+2\pi j)\,\hat{\tilde{\mathbf{\Phi}}}^*(\omega+2\pi j)d\omega$$

$$= \delta(k-l)\,\mathbf{I}_N,$$

for all $k, l \in \mathbf{Z}$. □

We give an example that satisfies the conditions of matrix-valued frame and calculate the dual Riesz basis .

Example 3.5. Let $\hat{\mathbf{\Phi}}(\omega) = \xi(\omega)\chi_{[-2\pi a, 2\pi a)}(\omega)$, $0 < a < \frac{1}{2}$, where $\xi(\omega) \in L^2(\mathbf{R})$ is a non-vanishing function on $[-2\pi a, 2\pi a)$. Then

$$\mathbf{E}_{\mathbf{\Phi}}(\omega) = |\xi(\omega)|^2 \chi_{[-2\pi a, 2\pi a)}(\omega) \text{ on } [-\pi, \pi).$$

It is easy to verify that $\{\mathbf{\Phi}_k\}$ is a frame for $\mathbf{V}_0$. By Theorem 3.3, $\{\mathbf{\Phi}_k\}$ is not a Riesz basis for a subspace $\mathbf{V}_0$.

Note that if $a \geq \frac{1}{2}$, then $\{\mathbf{\Phi}_k\}$ is a Riesz basis for $\mathbf{V}_0$. The matrix-valued dual Riesz basis $\tilde{\mathbf{\Phi}}(\omega)$ is given by

$$\tilde{\mathbf{\Phi}}(\omega) \equiv \mathbf{E}_{\mathbf{\Phi}}^{-1}(\omega)\mathbf{\Phi}(\omega) = \frac{\xi(\omega)}{|\xi(\omega)|^2}\chi_{[-2\pi a, 2\pi a)}(\omega).$$

4. Summary

In this paper, matrix-valued wavelet frames are introduced. The necessary and sufficient condition for matrix-valued functions to form a frame is studied. Some relationships and properties are derived about the relevant matrix-valued functions and the calculation method of matrix-valued dual Riesz basis is given. These conclusions are useful in providing theoretical basis for constructing matrix-valued frames.

Acknowledgments

The authors would like to thank the reviewers for their very helpful comments, which greatly improved the quality of this paper. This work was supported in by the China National Natural Science Foundation under Contract 60972089, and by state scholarship fund, the China Scholarship Council (CSC).

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