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# AUTOMATIC CONTINUITY OF HIGHER DERIVATIONS ON JB\*-ALGEBRAS

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ABSTRACT. In this paper we study higher derivations from  $JB^*$ -algebras into Banach Jordan algebras. We show that every higher derivation  $\{d_m\}$  from a  $JB^*$ -algebra  $\mathcal{A}$  into a  $JB^*$ -algebra  $\mathcal{B}$  is continuous provided that  $d_0$  is a \*-homomorphism. Also it is proved that every Jordan higher derivation from a commutative  $C^*$ -algebra or from a  $C^*$ -algebra which has minimal idempotents and is the closure of its socle is continuous.

## 1. Introduction

Let  $\mathcal{A}$  and  $\mathcal{B}$  be algebras (associative or non-associative). By a higher derivation of rank k (k might be  $\infty$ ) we mean a family of linear mappings  $\{d_m\}_{m=0}^k$  from  $\mathcal{A}$  into  $\mathcal{B}$  such that

$$d_m(ab) = \sum_{j=0}^m d_j(a) d_{m-j}(b), \quad (a, b \in \mathcal{A}, \quad m = 0, 1, 2, \dots, k).$$

It is clear that  $d_0$  is a homomorphism. Higher derivations were introduced by Hasse and Schmidt [8], and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find some of algebraic results concerning these mappings in [1, 4, 6, 14, 16, 17]. They are also studied in other contexts. In [19] higher derivations are applied to study

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generic solving of higher differential equations.

A standard example of a higher derivation of rank k is the family  $\{\frac{D^m}{m!}\}_{m=0}^k$ , where D is an ordinary derivation of an algebra  $\mathcal{A}$ .

If  $\mathcal{A}$  and  $\mathcal{B}$  are normed algebras then a higher derivation  $\{d_m\}$  is said to be continuous, whenever every  $d_m$  is continuous. It is known that every derivation on a semisimple Banach algebra is continuous [13]. Ringrose [15] proved that every derivation from a  $C^*$ -algebra  $\mathcal{A}$  into a Banach  $\mathcal{A}$ -module is continuous. In [9] derivations from  $JB^*$ -algebras into Banach Jordan modules were studied and continuity of these mappings were proved in certain cases. Loy in [12] proved that if  $\mathcal{A}$  is an (F)-algebra which is a subalgebra of a Banach algebra  $\mathcal{B}$  of power series, then every higher derivation  $\{d_m\} : \mathcal{A} \to \mathcal{B}$  is automatically continuous. Jewell [11], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that  $\ker(d_0) \subseteq \ker(d_m)$ , for all  $m \geq 1$ . Villena [20] proved that every higher derivation from a unital Banach algebra  $\mathcal{A}$  into  $\mathcal{A}/\mathcal{P}$ , where  $\mathcal{P}$  is a primitive ideal of  $\mathcal{A}$  with infinite codimension, is continuous. Also the range problem of continuous higher derivations was studied in [14].

In this paper we study automatic continuity of higher derivations from  $JB^*$ -algebras. Section 2 is devoted to some concepts which are needed in the sequel. In Section 3 we prove that a higher derivation from a  $JB^*$ -algebra into another  $JB^*$ -algebra is continuous provided that  $d_0$ is a \*-homomorphism. Also we will show that every (Jordan) higher derivation from a commutative  $C^*$ -algebra or from a  $C^*$ -algebra which has minimal idempotents and is the closure of its socle (e. g.  $\mathcal{K}(\mathcal{H})$ ) into a Banach Jordan algebra is continuous. These are in fact generalizations of some results in [9].

### 2. Preliminaries

Let  $\mathcal{A}$  be a Jordan algebra and let  $\mathcal{X}$  be a vector space over the same field as  $\mathcal{A}$ . Then  $\mathcal{X}$  is said to be a Jordan  $\mathcal{A}$ -module if there is a pair of bilinear mappings (called module operations),  $(a, x) \mapsto a.x, (a, x) \mapsto x.a$ , from  $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$  such that for all  $a, b \in \mathcal{A}$  and all  $x \in \mathcal{X}$  the following conditions hold:

(i) 
$$a.x - x.a,$$
  
(ii)  $a.(a^2.x) = a^2.(a.x);$   
(iii)  $2((x.a).b).a + x.(a^2.b) = 2(x.a)(a.b) + (x.b).a^2.$ 

A linear subspace  $\mathcal{S}$  of  $\mathcal{X}$  is called a submodule if

$$\mathcal{AS} := \{a.x: a \in \mathcal{A}, x \in \mathcal{S}\} \subseteq \mathcal{S}.$$

If  $\mathcal{A}$  is a Banach Jordan algebra and  $\mathcal{X}$  is a Banach space which is a Jordan  $\mathcal{A}$ -module then  $\mathcal{X}$  is said to be a weak Jordan  $\mathcal{A}$ -module whenever the mapping  $x \mapsto a.x$ , from  $\mathcal{X} \longrightarrow \mathcal{X}$  is continuous, for all  $a \in \mathcal{A}$ ; and  $\mathcal{X}$  is called a Banach Jordan  $\mathcal{A}$ -module if the mapping  $(a, x) \mapsto a.x$ , from  $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$  is continuous, or equivalently, if there exists M > 0 such that  $|| a.x || \leq M || a || || x || (a \in \mathcal{A}, x \in \mathcal{X})$ .

**Example 2.1.** (i) Every Banach Jordan algebra  $\mathcal{A}$  is a Banach Jordan  $\mathcal{A}$ -module whenever we consider its own product as the module operation.

(ii) If  $\mathcal{A}$  and  $\mathcal{B}$  are Jordan algebras and  $\theta : \mathcal{A} \longrightarrow \mathcal{B}$  is a homomorphism, then  $\mathcal{B}$  can be considered as a Jordan  $\mathcal{A}$ -module with module operation  $a.b = \theta(a)b$   $(a \in \mathcal{A}, b \in \mathcal{B}).$ 

In this case we will say that  $\mathcal{B}$  is an  $\mathcal{A}$ -module via the homomorphism  $\theta$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are Banach Jordan algebras then it is easy to see that  $\mathcal{B}$  is a weak Jordan  $\mathcal{A}$ -module.

(iii) The topological dual  $\mathcal{A}^*$  of  $\mathcal{A}$ , with module operation  $(a, f) \mapsto a.f$  defined by

$$(a.f)(b) = f(ab) \qquad (a, b \in \mathcal{A}, f \in \mathcal{A}^*),$$

is a Banach Jordan  $\mathcal{A}$ -module.

(iv) If  $\mathcal{A}$  is a Banach algebra and  $\mathcal{X}$  is a Banach (respectively weak)  $\mathcal{A}$ -module, then we may consider  $\mathcal{A}$  as a Jordan algebra with Jordan product  $(a, b) \mapsto \frac{ab+ba}{2}, \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$ . Then  $\mathcal{X}$  with the module operation  $a.x = \frac{ax+xa}{2}$  is a Banach (respectively weak) Jordan  $\mathcal{A}$ -module. Here the mappings  $(a, x) \mapsto ax$  and  $(a, x) \mapsto xa, \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ , denote the associative module operations of  $\mathcal{A}$  on  $\mathcal{X}$ .

Let  $\mathcal{X}$  and  $\mathcal{Y}$  be Jordan  $\mathcal{A}$ -modules. Then a linear mapping  $T : \mathcal{X} \longrightarrow \mathcal{Y}$  is said to be a module homomorphism if T(a.x) = a.T(x) ( $a \in \mathcal{A}, x \in \mathcal{X}$ ). In Example 2.1 (ii),  $\theta$  is a module homomorphism.

Let  $\mathcal{A}$  be a Jordan algebra and let  $\mathcal{X}$  be a Jordan  $\mathcal{A}$ -module. Then  $\mathcal{A} \oplus \mathcal{X}$  with product  $(a_1 + x_1)(a_2 + x_2) = a_1a_2 + a_1.x_2 + a_2.x_1$ , is a Jordan algebra which is called the split null extension of  $\mathcal{A}$  and  $\mathcal{X}$ . In fact a linear space  $\mathcal{X}$  is a Jordan  $\mathcal{A}$ -module if and only if this split null extension is a Jordan algebra [10].

Corresponding to  $(a, 0) \in \mathcal{A} \oplus \mathcal{X}$  with  $a \in \mathcal{A}$ , as in any Jordan algebra, we define the linear operators  $R_a$  and  $U_a$  on  $\mathcal{A} \oplus \mathcal{X}$  as follows

 $R_a(u) = au, \quad U_a(u) = 2a(au) - a^2u \quad (u \in \mathcal{A} \oplus \mathcal{X}).$ 

We feel free to use the notation  $R_a$  and  $U_a$  for the same operators on  $\mathcal{A}$ . For every x, y in a Jordan algebra, set  $[R_x, R_y] := R_x R_y - R_y R_x$ . We recall that each x, y, z in a Jordan algebra satisfy

$$[R_{xy}, R_z] + [R_{xz}, R_y] + [R_{yz}, R_x] = 0, (2.1)$$

which is the identity (O1) in Section 1.7 of [10]. For a submodules of  $\mathcal{X}$ , set

$$\mathcal{R}(\mathcal{S}) := \{ a \in \mathcal{A} : R_a(x) = 0 \quad \text{for all} \quad x \in \mathcal{S} \}, \\ \mathcal{Q}(\mathcal{S}) := \{ a \in \mathcal{A} : U_a(x) = 0 \quad \text{for all} \quad x \in \mathcal{S} \}, \\ \mathcal{I}(\mathcal{S}) := \{ a \in \mathcal{R}(\mathcal{S}) : ab \in \mathcal{R}(\mathcal{S}) \quad \text{for all} \quad b \in \mathcal{A} \}.$$

Note that if S is a submodule, then it is an ideal of  $\mathcal{A} \oplus \mathcal{X}$ , and  $\mathcal{I}(S)$  is actually ann(S) in view of Zelmanov, which is an ideal by Lemma 3(b) of [21]. Here we give the proof for the sake of convenience.

**Lemma 2.2.** Let  $\mathcal{A}$  be a Jordan algebra and let  $\mathcal{X}$  be a Jordan  $\mathcal{A}$ -module. If  $\mathcal{S}$  is a submodule then (i)  $\mathcal{I}(\mathcal{S})$  is the largest ideal of  $\mathcal{A}$  contained in  $\mathcal{R}(\mathcal{S})$ ; (ii)  $\mathcal{R}(\mathcal{S}) \cap \mathcal{Q}(\mathcal{S}) = \{a \in \mathcal{A}: a^2 \in \mathcal{R}(\mathcal{S})\};$ (iii)  $\mathcal{I}(\mathcal{S}) \subseteq \mathcal{R}(\mathcal{S}) \cap \mathcal{Q}(\mathcal{S}).$ 

**Proof.** (i) It is easy to see that each ideal of  $\mathcal{A}$  contained in  $\mathcal{R}(\mathcal{S})$  is a subset of  $\mathcal{I}(\mathcal{S})$ . We show that  $\mathcal{I}(\mathcal{S})$  is an ideal. Suppose that  $a \in \mathcal{I}(\mathcal{S})$  and  $b \in \mathcal{A}$ . Then by definition of  $\mathcal{I}(\mathcal{S})$ ,  $ab \in \mathcal{R}(\mathcal{S})$ . Now to see that  $ab \in \mathcal{I}(\mathcal{S})$  it is enough to show that  $(ab)c \in \mathcal{R}(\mathcal{S})$ , for each  $c \in \mathcal{A}$ . We consider (2.1) in the Jordan algebra  $\mathcal{A} \oplus \mathcal{X}$ , for  $x = a, y = b, z \in \mathcal{S}$  and take  $c \in \mathcal{A}$ . Since  $\mathcal{S}$  is a submodule and  $a \in \mathcal{I}(\mathcal{S})$ , it follows that  $R_z R_{ab}(c) = 0$ , or equivalently,  $(ab)c \in \mathcal{R}(\mathcal{S})$ . Parts (ii) and (iii) are easily verified.

# 3. Automatic continuity of Higher derivations from $JB^*$ -algebras

First of all we recall that a real Banach Jordan algebra  $\mathcal{A}$  is a JB-algebra whenever  $||a^2|| = ||a||^2$  and  $||a^2|| \leq ||a^2 + b^2||$ , for all  $a, b \in$ 

 $\mathcal{A}$ . A complex Banach Jordan algebra  $\mathcal{A}$  is said to be a  $JB^*$ -algebra whenever there is an algebra involution \* on  $\mathcal{A}$  such that  $||a^*|| = ||a||$ and  $||U_a(a^*)|| = ||a||^3$ , for all  $a \in \mathcal{A}$ . For a subset  $\mathcal{C}$  of a  $JB^*$ -algebra  $\mathcal{A}$ , set  $\mathcal{C}_h := \{a \in \mathcal{C}: a = a^*\}$ . Then  $\mathcal{A}_h$  is a JB-algebra and  $\mathcal{A} = \mathcal{A}_h + i\mathcal{A}_h$ . If  $a \in \mathcal{A}_h$  then  $C^*(a)$ , the  $JB^*$ -subalgebra of  $\mathcal{A}$  generated by a (or by a, 1 if  $\mathcal{A}$  is unital), is a  $C^*$ -algebra. Clearly each  $C^*$ -algebra with respect to its Jordan product is a  $JB^*$ -algebra. The reader is referred to [7] for more details on JB-algebras and  $JB^*$ -algebras. From now on throughout this section we assume that  $\mathcal{A}$  is a unital  $JB^*$ -algebra,  $\mathcal{B}$  is a Banach Jordan algebra and  $\{d_m\}$  is a higher derivation of infinite rank from  $\mathcal{A}$ into  $\mathcal{B}$  with continuous  $d_0$ . For each  $m = 0, 1, 2, \ldots$ , set

$$\mathcal{S}_m := \{ b \in \mathcal{B} : \exists \{a_n\} \subseteq \mathcal{A} \text{ s.t. } a_n \to 0 \text{ and } d_m(a_n) \to b \},\$$

which is called the separating space of  $d_m$ . This is a closed linear subspace of  $\mathcal{B}$  ([5], Theorem 5.1.2) and by the closed graph theorem  $d_m$  is continuous if and only if  $\mathcal{S}_m = \{0\}$ . Therefore  $\{d_m\}$  is continuous if and only if  $\mathcal{S}_m = \{0\}$ , for all  $m \geq 0$ . If we consider  $\mathcal{B}$  as a Jordan  $\mathcal{A}$ -module via the homomorphism  $d_0$  as in Example 2.1 (ii), then  $d_1$  would be a derivation from  $\mathcal{A}$  into  $\mathcal{B}$ . With the assumption on  $d_0$  we have  $\mathcal{S}_0 = \{0\}$ and it is easy to see that  $\mathcal{S}_1$  is a submodule of  $\mathcal{B}$ . In general  $\mathcal{S}_m$  is not a submodule for  $m \geq 2$ , but if  $d_o, d_1, \ldots, d_{m-1}$  are assumed to be continuous, then  $d_m$  would be an intertwining map and hence  $\mathcal{S}_m$  is a submodule. Using the same notations as in Section 2, set  $\mathcal{R}_m := \mathcal{R}(\mathcal{S}_m)$ ,  $\mathcal{Q}_m := \mathcal{Q}(\mathcal{S}_m)$  and  $\mathcal{I}_m := \mathcal{I}(\mathcal{S}_m)$ . If  $d_0, \ldots, d_{m-1}$  are continuous then we have

$$\mathcal{R}_m = \{ a \in \mathcal{A}: R_a d_m \quad \text{is continuous} \}$$
  
=  $\{ a \in \mathcal{A}: d_m R_a \quad \text{is continuous} \},$ 

and

$$\mathcal{Q}_m = \{ a \in \mathcal{A} : U_a d_m \quad \text{is continuous} \} \\= \{ a \in \mathcal{A} : d_m U_a \quad \text{is continuous} \}.$$

Before we prove the next lemma, we recall that a subalgebra  $\mathcal{C}$  of a Jordan algebra  $\mathcal{A}$  is said to be strongly associative if  $[R_a, R_b] = 0$ , for all  $a, b \in \mathcal{C}$ . By Example 1.8.1 of [10], for each  $a \in \mathcal{A}$ , the subalgebra of  $\mathcal{A}$  generated by a, (or by a, 1 if  $\mathcal{A}$  is unital) is strongly associative and by ([10] Lemma 1.8.8), if a, b lie in a strongly associative subalgebra, then  $U_{ab} = U_a U_b$ .

**Lemma 3.1.** Let  $\mathcal{A}$  be a  $JB^*$ -algebra. Suppose that  $\mathcal{X}$  is a Banach Jordan  $\mathcal{A}$ -module,  $\mathcal{Y}$  is a weak Jordan  $\mathcal{A}$ -module and  $T : \mathcal{X} \longrightarrow \mathcal{Y}$  is

a module homomorphism. If  $a \in \mathcal{A}_h$ , and  $\{f_n\} \subseteq C^*(a)$  is such that  $f_i f_j = 0 \ (i \neq j)$ , then  $U_{f_n^2}T$  is continuous for all but a finite number of n's.

**Proof.** Suppose that  $U_{f_n}{}^2T$  is discontinuous for infinitely many *n*'s. By considering a subsequence we may assume that  $U_{f_n}{}^2T$  is discontinuous for each *n*. Let  $M_n$  and  $K_n$  be the norms of the bounded linear operators  $x \mapsto U_{f_n}(x), \mathcal{X} \longrightarrow \mathcal{X}$ , and  $y \mapsto U_{f_n}(y), \mathcal{Y} \longrightarrow \mathcal{Y}$ , respectively. Note that  $M_n, K_n > 0$  for each n; otherwise  $U_{f_n}{}^2T = TU_{f_n}{}^2 = 0$  which is continuous. Choose a sequence  $\{x_n\}$  in  $\mathcal{X}$  such that

$$||x_n|| \le 2^{-n}/M_n,$$
  
 $||U_{f_n^2}T(x_n)|| \ge nK_n.$ 

Take  $z = \sum_{n=1}^{\infty} U_{f_n}(x_n)$ . By strong associativity of  $C^*(a)$  as a subalgebra of  $\mathcal{A} \oplus \mathcal{X}$  and  $\mathcal{A} \oplus \mathcal{Y}$ , we have  $U_{f_i}U_{f_j} = U_{f_if_j} = 0$   $(i \neq j)$ , on  $\mathcal{A} \oplus \mathcal{X}$  and  $\mathcal{A} \oplus \mathcal{Y}$ . Since T is a module homomorphism,  $K_n ||T(z)|| \geq ||U_{f_n}(T(z))|| = ||TU_{f_n}(z)|| = ||T(U_{f_n^2}(x_n))|| = ||U_{f_n^2}(Tx_n)|| \geq nK_n$ . Therefore  $||T(z)|| \geq n$  for each n, which is impossible. So the result holds.

**Remark 3.2.** Suppose that  $\mathcal{B}$  is a Jordan algebra. Then  $\mathcal{B}_m := \underbrace{\mathcal{B} \oplus \mathcal{B} \oplus \ldots \oplus \mathcal{B}}_{m+1}$  is a Jordan algebra with the product defined by

$$(x_0, x_1, \dots, x_m)(y_0, y_1, \dots, y_m) = (x_0 y_0, x_0 y_1 + x_1 y_0, \dots, \sum_{i=0}^m x_i y_{m-i}),$$

for all  $(x_0, x_1, \ldots, x_m), (y_0, y_1, \ldots, y_m) \in \mathcal{B}_m$ . Clearly, this product is commutative. Suppose that  $\bar{x} = (x_0, x_1, \ldots, x_m), \bar{y} = (y_0, y_1, \ldots, y_m) \in \mathcal{B}_m$ . Then the  $k^{th}$  entries of  $\bar{x}(\bar{x}^2\bar{y})$  and  $\bar{x}^2(\bar{x}\bar{y})$  are

$$\sum_{l=0}^{k} x_l \Big( \sum_{j=0}^{k-l} \sum_{i=0}^{j} (x_j x_{j-i}) y_{k-j-l} \Big),$$
(3.1)

and

$$\sum_{l=0}^{k} \sum_{i=0}^{l} \sum_{j=0}^{k-l} (x_l x_{l-i}) (x_j y_{k-j-l}), \qquad (3.2)$$

respectively. By identities (O2) and (O3) in Section 1.7 of [10], (3.1) and (3.2) are equal, and hence  $\mathcal{B}_m$  is a Jordan algebra. Furthermore, let  $\mathcal{B}$  be

a Banach Jordan algebra. Define a norm on  $\mathcal{B}_m$  by  $\|(x_0, x_1, \ldots, x_m)\|_0 = \sum_{i=0}^m \|x_i\|$ . Then  $\|.\|_0$  is a complete norm on  $\mathcal{B}_m$  and it is easy to see that

$$\|(x_0, x_1, \dots, x_m)(y_0, y_1, \dots, y_m)\|_0$$
  
$$\leq \|(x_0, x_1, \dots, x_m)\|_0 \|(y_0, y_1, \dots, y_m)\|_0,$$

for all  $(x_0, x_1, \ldots, x_m), (y_0, y_1, \ldots, y_m) \in \mathcal{B}_m$ . Therefore  $\mathcal{B}_m$  is a Banach Jordan algebra.

**Lemma 3.3.** Suppose that  $\mathcal{A}$  is a  $JB^*$ -algebra and  $\mathcal{B}$  is a Banach Jordan algebra. Let  $\{d_m\} : \mathcal{A} \longrightarrow \mathcal{B}$  be a higher derivation with continuous  $d_0$ . Let  $a \in \mathcal{A}_h$  and let  $\{f_n\} \subseteq C^*(a)$  be such that  $f_i f_j = 0 \ (i \neq j)$ . Then for each  $m = 0, 1, 2, \ldots$ , we have  $f_n^2 \in \mathcal{Q}_m$ , for all but a finite number of n's.

**Proof.** Consider a fixed m, and let  $\mathcal{B}_m$  be as in Remark 3.2. We define  $\theta_m \colon \mathcal{A} \longrightarrow \mathcal{B}_m$ ,

$$a \mapsto (d_0(a), d_1(a), \ldots, d_m(a)).$$

Then  $\theta_m$  is a homomorphism and  $\mathcal{B}_m$  is a weak Jordan  $\mathcal{A}$ -module via the homomorphism  $\theta_m$ . Also as in Example 2.1 (ii),  $\theta_m$  is a module homomorphism. We have  $U_{f_i}U_{f_j} = U_{f_if_j} = 0$   $(i \neq j)$ , on the split null extension of  $\mathcal{A}$  and  $\mathcal{B}_m$ . Hence by Lemma 3.1,  $U_{f_n^2}\theta_m$  is continuous for all but a finite number of *n*'s. Thus for such *n*'s,  $U_{f_n^2}d_1, \ldots, U_{f_n^2}d_m$  are continuous and it follows that  $f_n^2 \in \mathcal{Q}_m$ , for all but a finite number of *n*'s.

**Theorem 3.4.** Let  $\mathcal{A}$  be a  $JB^*$ -algebra and let  $\mathcal{B}$  be a Jordan Banach algebra. Suppose that  $\{d_m\}$  is a higher derivation from  $\mathcal{A}$  into  $\mathcal{B}$  with continuous  $d_0$ . Then the following assertions hold.

(i) If  $a \in \mathcal{A}_h$  and  $\Delta$  is the maximal ideal space of  $C^*(a)$ , then for every  $m = 1, 2, \ldots$ , the set  $F_m = \{\lambda \in \Delta : \lambda(\mathcal{Q}_m \cap C^*(a)) = \{0\}\}$  is finite.

(ii) If  $\mathcal{I}$  is a closed ideal of  $\mathcal{A}$  containing  $\mathcal{Q}_m$ , then every element in the JB-algebra  $\left(\frac{\mathcal{A}}{\mathcal{I}}\right)_h$  has finite spectrum.

(iii) If  $d_1, \ldots, d_{m-1}$  are continuous and  $\mathcal{K}$  is a closed ideal of  $\mathcal{A}$  contained in  $\mathcal{Q}_m$ , then  $d_m \mid_{\mathcal{K}}$  is continuous.

(iv) If  $d_1, \ldots, d_{m-1}$  are continuous and  $\mathcal{L}$  is an ideal of  $\mathcal{A}$  such that  $d_m \mid_{\mathcal{L}}$  is continuous, then  $\mathcal{L} \subseteq \mathcal{I}_m \subseteq \mathcal{Q}_m$ .

**Proof.** (i) If  $F_m$  is infinite, then we may find an infinite sequence  $\{\lambda_k\} \subseteq \Delta$  and a sequence  $\{V_k\}$  of open subsets of  $\Delta$  such that  $V_j \cap V_k = \emptyset$   $(j \neq k)$ , and  $\lambda_k \in V_k$ , for each k. For every  $k \in \mathbb{N}$ , choose  $f_k \in C^*(a)$  such that  $f_k(\lambda_k) \neq 0$  and  $f_k(\Delta \setminus V_k) = \{0\}$ . Then  $f_k f_j = 0$   $(k \neq j)$ , and  $f_k^2 \notin \mathcal{Q}_m$  which contradicts Lemma 3.3.

(ii) Let  $\mathcal{I}$  be a closed ideal in  $\mathcal{A}$  such that  $\mathcal{Q}_m \subseteq \mathcal{I}$ , for all  $m = 0, 1, 2, \ldots$ . For each  $a \in \mathcal{A}_h$ , we have

 $\{\lambda \in \Delta : \lambda (I \cap C^*(a)) = \{0\}\} \subseteq \{\lambda \in \Delta : \lambda (\mathcal{Q}_m \cap C^*(a)) = \{0\}\}.$ Hence by (i) the left hand side is a finite set and as in Theorem 12.2 of [18],  $\frac{C^*(a)}{C^*(a) \cap \mathcal{I}}$  is finite dimensional, and since the closed \*-subalgebra of  $\frac{\mathcal{A}}{\mathcal{I}}$  generated by a and 1 is isomorphic to  $\frac{C^*(a)}{C^*(a) \cap \mathcal{I}}$ , the result holds.

(iii) We show that  $d_m$  is bounded on bounded subsets of  $\mathcal{K}_h$ . On the contrary suppose that there is a sequence  $\{a_n\} \subseteq \mathcal{K}_h$  such that  $a_n \to 0$  and  $\parallel d_m(a_n) \parallel \to \infty$ . We may assume that  $\sum_{n=1}^{\infty} \parallel a_n \parallel^2 \leq 1$ . Let  $b = (\sum_{n=1}^{\infty} a_n^2)^{1/8}$ . Then  $b \geq 0$ ,  $\parallel b \parallel \leq 1$  and  $a_n^2 \leq b^8$   $(n \in \mathbb{N})$ . By [9] Lemma 1.7, for each  $n \in \mathbb{N}$  there exists  $u_n \in \mathcal{K}_h$  such that  $\parallel u_n \parallel \leq 2 \parallel b^{1/4} \parallel \leq 2$  and  $a_n = U_b(u_n)$ . Hence  $d_m(a_n) = d_m U_b(u_n)$ . Since  $\mathcal{K} \subseteq \mathcal{Q}_m$ , we have  $b \in \mathcal{Q}_m$  and so  $d_m U_b$  is continuous. Now it follows that  $\parallel d_m(a_n) \parallel \leq \parallel d_m U_b \parallel \parallel u_n \parallel \leq 2 \parallel d_m U_b \parallel$ , which is a contradiction.

(iv) Suppose that  $d_m \mid_{\mathcal{L}}$  is continuous. Take  $a \in \mathcal{S}_m$ . Then there is a sequence  $\{a_n\} \subseteq \mathcal{A}$  such that  $a_n \to 0$  and  $d_m(a_n) \to a$ . Let  $b \in \mathcal{L}$ . Since  $d_1, \ldots, d_{m-1}$  are continuous it follows that

$$d_m(ba_n) = d_0(b)d_m(a_n) + d_1(b)d_{m-1}(a_n) + \ldots + d_m(b)d_0(a_n) \to ba.$$

Since  $ba_n \in \mathcal{L}$  and  $d_m \mid_{\mathcal{L}}$  is continuous, ba = 0. This means that  $b \in \mathcal{R}_m$ and hence  $\mathcal{L} \subseteq \mathcal{R}_m$ . But  $\mathcal{I}_m$  is the largest ideal of  $\mathcal{A}$  contained in  $\mathcal{R}_m$ , so we have  $\mathcal{L} \subseteq \mathcal{I}_m \subseteq \mathcal{Q}_m$ .

**Corollary 3.5.** Let  $\mathcal{A}$  be a  $JB^*$ -algebra and let  $\mathcal{B}$  be a Banach Jordan algebra. Suppose that  $\{d_m\}$  is a higher derivation from  $\mathcal{A}$  into  $\mathcal{B}$  with continuous  $d_0$ . If  $\mathcal{K}$  is a closed ideal of  $\mathcal{A}$  contained in  $\bigcap \mathcal{Q}_m$ , then  $d_m \mid_{\mathcal{K}}$  is continuous for all m.

**Proof**. Similar to the proof of Theorem 3.4 (iii).

**Theorem 3.6.** Let  $\{d_m\}$  be a higher derivation of a  $JB^*$ -algebra  $\mathcal{A}$  into a Banach Jordan algebra  $\mathcal{B}$  such that  $d_0$  is continuous. Then  $\{d_m\}$  is

continuous if and only if  $(\mathcal{Q}_m)_h := \{a \in \mathcal{Q}_m : a = a^*\}$  is a real linear subspace of  $\mathcal{A}_h$ , for all  $m \in \mathbb{N}$ .

**Proof.** If  $\{d_m\}$  is continuous then  $\mathcal{Q}_m = \mathcal{A}$ , and so  $(\mathcal{Q}_m)_h$  is real linear. Conversely let  $(\mathcal{Q}_m)_h$  be real linear. Since  $d_1$  is a derivation, by ([9] Theorem 2.2),  $d_1$  is continuous. Suppose by induction that each  $d_i (i < m)$  is continuous. Then  $\mathcal{S}_m$  is a submodule of  $\mathcal{B}$ , and  $U_{\mathcal{A}_h} (\mathcal{Q}_m)_h$  $\subseteq (\mathcal{Q}_m)_h$ , hence  $(\mathcal{Q}_m)_h$  is an ideal of  $\mathcal{A}_h$ . By Theorem 3.4 (iii),  $d_m$  is continuous on  $(\mathcal{Q}_m)_h \oplus i(\mathcal{Q}_m)_h$ . Hence  $(\mathcal{Q}_m)_h \oplus i(\mathcal{Q}_m)_h \subseteq \mathcal{I}_m \subseteq \mathcal{Q}_m$  and so  $\mathcal{I}_m = (\mathcal{Q}_m)_h \oplus i(\mathcal{Q}_m)_h$ . Let  $\pi : \mathcal{A} \to \frac{\mathcal{A}}{\mathcal{I}_m}$  be the canonical quotient map. By Theorem 3.4 (ii) every element in  $\left(\frac{A}{I_m}\right)_h$  has finite spectrum. But  $\left(\frac{A}{I_{\rm m}}\right)_h = \frac{A_{\rm h}}{(I_{\rm m})_{\rm h}}$  is a semisimple real Banach Jordan algebra in which every element has non-empty finite spectrum and by [2] it is reduced, that is, there exist idempotents  $\pi(e_1), \ldots, \pi(e_n) \in \left(\frac{\mathcal{A}}{\mathcal{I}_m}\right)_h$  such that  $\pi(e_i)\pi(e_j) =$  $0, (i \neq j), \sum_{i=1}^{n} \pi(e_i) = 1$ , and  $U_{\pi(e_i)} \left(\frac{\mathcal{A}}{\mathcal{I}_m}\right)_h^m = \mathbb{R}\pi(e_i), (i = 1, \dots, n)$ . Since each  $\pi(e_i)$  is self-adjoint,  $\pi(e_i^*e_i) = \pi(e_i), (i = 1, \dots, n)$ , and so  $\pi(e_1^*e_1), \ldots, \pi(e_n^*e_n)$  are idempotents in  $\left(\frac{\mathcal{A}}{\mathcal{I}_m}\right)_h$  with sum 1 such that  $\pi(e_i^*e_i)\pi(e_j^*e_j) = 0$ ,  $(i \neq j)$ . Hence by replacing  $e_i$  with  $e_i^*e_i$ , if necessary, we may assume that each  $e_i$  is self-adjoint. Suppose that  $\{a_k\} \subseteq \mathcal{A}_h \text{ and } a_k \to 0.$  Then  $\pi(a_k) \to 0$ , and for each  $i = 1, \ldots, n$ , and each  $k \in \mathbb{N}$ , there exists  $\lambda_{ik} \in \mathbb{R}$  such that

$$U_{\pi(e_i)}(\pi(a_k)) = \lambda_{ik}\pi(e_i). \tag{3.3}$$

Hence  $\lambda_{ik}\pi(e_i) \to 0$  as  $k \to \infty$ , and so  $\lambda_{ik} \to 0$  as  $k \to \infty$ . By (3.3) we have

$$U_{e_i}(a_k) - \lambda_{ik} e_i \in \mathcal{I}_m, \qquad (i = 1, \dots, n, \ k \in \mathbb{N}),$$

and by continuity of  $d_m \mid_{\mathcal{I}_m}$ ,  $\lim_{k\to\infty} d_m (U_{e_i}(a_k) - \lambda_{ik}e_i) = 0$ . Since  $\lim_{k\to\infty} \lambda_{ik} = 0$ , we have  $\lim_{k\to\infty} d_m U_{e_i}(a_k) = 0$ . Therefore  $d_m U_{e_i}$  is continuous for  $i = 1, \ldots, n$ , and  $e_1, \ldots, e_n \in (\mathcal{Q}_m)_h$ . So  $e_1 + \ldots + e_n \in \mathcal{I}_m = (\mathcal{Q}_m)_h \oplus i(\mathcal{Q}_m)_h$ . Since  $\pi(e_1 + \ldots + e_n)$  is the identity of  $\frac{\mathcal{A}}{\mathcal{I}_m}$ ,  $\mathcal{A} = \mathcal{I}_m$  and  $d_m$  is continuous on  $\mathcal{A}$ .

**Lemma 3.7.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $JB^*$ -algebras and let  $\phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a \*-homomorphism, that is  $\phi(a^*) = (\phi(a))^*$   $(a \in \mathcal{A})$ . Consider  $\mathcal{B}$  as a Banach Jordan  $\mathcal{A}$ -module via the homomorphism  $\phi$ . If  $\mathcal{S}$  is a submodule of  $\mathcal{B}$ , then  $\mathcal{Q}(\mathcal{S}) = \mathcal{I}(\mathcal{S})$ .

**Proof.** We show that  $(\mathcal{Q}(\mathcal{S}))_h = (\mathcal{I}(\mathcal{S}))_h$ . Consider the identities

$$(U_x(y^2))^2 = U_x U_y U_y(x^2), (3.4)$$

$$(xy)^{2} = \frac{1}{2}yU_{x}(y) + \frac{1}{4}U_{x}(y^{2}) + \frac{1}{4}U_{y}(x^{2}), \qquad (3.5)$$

which are valid in any Jordan algebra, see [10], p. 37 for the first one. The second holds by the fact that any Jordan algebra generated by two elements is special, see Shirsov-Cohen's theorem, [7] Theorem 2.4.14. Now, if  $a \in \mathcal{Q}(S)_h$  then by setting  $x = \phi(a) \in \mathcal{B}_h$  in (3.4) and (3.5), we have  $(\phi(a)b)^2 = 0$  ( $a \in \mathcal{Q}(S)$ ,  $b \in S$ ). Thus  $\phi(a)b = 0$  ( $a \in \mathcal{Q}(S)$ ,  $b \in S$ ). Therefore  $a \in \mathcal{R}(S)$  and it follows that  $(\mathcal{Q}(S))_h \subseteq (\mathcal{R}(S))_h$ . So  $(\mathcal{Q}(S))_h = (\mathcal{Q}(S))_h \cap (\mathcal{R}(S))_h = (\mathcal{I}(S))_h$ , by ([9], Theorem 1.4).

**Corollary 3.8.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $JB^*$ -algebras, and let  $\{d_m\} : \mathcal{A} \longrightarrow \mathcal{B}$ be a higher derivation for which  $d_0$  is a \*-homomorphism. Then  $\{d_m\}$ is continuous.

**Proof.** Since  $d_0$  is a \*-homomorphism, it is automatically continuous. Note that  $S_1$  is a submodule of  $\mathcal{B}$ , thus by Lemma 3.7,  $\mathcal{Q}_1$  is a linear subspace of  $\mathcal{A}$  and hence by ([9] Theorem 2.2),  $d_1$  is continuous. Fix m, suppose that each  $d_i(i < m)$  is continuous. Therefore  $S_m$  is a submodule and again by Lemma 3.7,  $\mathcal{Q}_m$  is a linear subspace of  $\mathcal{A}$ , and hence by Theorem 3.6,  $d_m$  is continuous.

In the next few results, by a Jordan higher derivation from a  $C^*$ -algebra  $\mathcal{A}$  we mean a higher derivation from  $\mathcal{A}$ , with its Jordan product, into a Banach Jordan algebra. Obviously each higher derivation (with respect to the associative product) is also a Jordan derivation.

As a consequence of Corollary 3.8 each higher derivation, or each Jordan higher derivation between  $C^*$ -algebras, is continuous provided that  $d_0$  is a \*-homomorphism. In the next results  $d_0$  is not assumed to be a \*-homomorphism.

**Theorem 3.9.** Let  $\mathcal{A}$  be a commutative  $C^*$ -algebra, and let  $\mathcal{B}$  be a Banach Jordan algebra. If  $\{d_m\} : \mathcal{A} \to \mathcal{B}$  is a Jordan higher derivation such that  $d_0$  is continuous, then  $\{d_m\}$  is continuous.

**Proof.** By ([9], Theorem 2.4) of ,  $d_1$  is continuous. Suppose that  $d_1, \ldots, d_{m-1}$  are continuous. Then  $S_m$  is a submodule. We show that  $(Q_m)_h =$ 

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 $(\mathcal{I}_m)_h$ . Let  $a \in (\mathcal{Q}_m)_h$ . We have  $a^2\mathcal{A} = a\mathcal{A}a = U_a\mathcal{A} \subseteq \mathcal{Q}_m$ , and hence  $a^2\mathcal{A} \subseteq \mathcal{Q}_m$ . Since  $\mathcal{I}_m$  is the largest ideal of  $\mathcal{A}$  contained in  $\mathcal{Q}_m$ ,  $a^2\mathcal{A} \subseteq \mathcal{I}_m$ . Therefore  $a^4 \in \mathcal{I}_m$  and since  $a = a^*$ , we have  $a \in \mathcal{I}_m$ .  $\Box$ Before proving the next result, we recall that if  $\mathcal{A}$  is an associative algebra with associative product  $(a, b) \mapsto ab$ , and the Jordan product  $(a, b) \mapsto \frac{ab+ba}{2}$ , then  $U_a(b) = aba \quad (a, b \in \mathcal{A})$ .

**Theorem 3.10.** Let  $\mathcal{A}$  be a  $C^*$ -algebra with minimal idempotents, and let  $\{d_m\}$  be a Jordan higher derivation from  $\mathcal{A}$  to a Banach Jordan algebra  $\mathcal{B}$ . If  $d_0, \ldots, d_{m-1}$  are continuous on  $\mathcal{A}$ , then  $\{d_m\}$  is continuous on  $\overline{\operatorname{soc}(\mathcal{A})}$ .

**Proof.** By ([3], Theorem 30.10),  $\operatorname{soc}(\mathcal{A})$  exists. Let M denote the set of all minimal idempotents of  $\mathcal{A}$ . Then

$$\operatorname{soc}(\mathcal{A}) = \sum_{e \in M} e\mathcal{A} = \sum_{e \in M} \mathcal{A}e,$$
(3.6)

where by  $\sum$  we mean the algebraic sum. Since  $d_0, \ldots, d_{m-1}$  are continuous, we have

 $\mathcal{Q}_m = \{a \in \mathcal{A} : U_a d_m \text{ is continuous}\} = \{a \in \mathcal{A} : d_m U_a \text{ is continuous}\}(3.7)$ Suppose that  $a \in \text{soc}(\mathcal{A})_h$ , then there exist  $b_1, \ldots, b_n \in \mathcal{A}$ , and  $e_1, \ldots, e_n \in \mathcal{M}$  such that  $a = e_1 b_1 + \ldots + e_n b_n$ , and hence  $a^* = b_1^* e_1^* + \ldots + b_n^* e_n^* = a$ . So

$$U_a(b) = aba = \sum_{i=1}^n \sum_{j=1}^n e_i b_i b b_j^* e_j^* \qquad (b \in \mathcal{A}).$$
(3.8)

We know that the adjoint of a minimal idempotent is also a minimal idempotent, hence by ([3], Theorem 31.6),  $\dim(e_i\mathcal{A}e_j^*) \leq 1$ , for  $i, j = 1, \ldots, n$ . By (3.8) we have,  $U_a(\mathcal{A}) \subseteq \sum_{i=1}^n \sum_{j=1}^n e_i\mathcal{A}e_j^*$ , thus  $\dim(U_a(\mathcal{A})) < \infty$  and  $d_m$  is continuous on  $U_a(\mathcal{A})$ . This shows that  $d_mU_a$  is continuous on  $\mathcal{A}$ , and hence by (3.7),  $a \in \mathcal{Q}_m$ . It follows that  $\operatorname{soc}(\mathcal{A})_h \subseteq \mathcal{Q}_m$ , and since  $\mathcal{Q}_m$  is closed,  $(\operatorname{soc}(\mathcal{A})_h) \subseteq \mathcal{Q}_m$ . By (3.6)  $\operatorname{soc}(\mathcal{A})$  is an \*-ideal, hence  $(\operatorname{soc}(\mathcal{A}))_h = (\operatorname{soc}(\mathcal{A})_h)$ . Now the same argument as in Theorem 3.4 (iii) implies that  $d_m$  is continuous on  $\operatorname{soc}(\mathcal{A})$ .  $\Box$ 

**Corollary 3.11.** If  $\mathcal{A}$  is a  $C^*$ -algebra with minimal idempotents such that  $\overline{\operatorname{soc}(\mathcal{A})} = \mathcal{A}$ , then each Jordan higher derivation from  $\mathcal{A}$  into a Banach Jordan algebra  $\mathcal{B}$  with continuous  $d_0$  is continuous. In particular,

if  $\mathcal{A} = \mathcal{K}(\mathcal{H})$ ), the C<sup>\*</sup>-algebra of all compact operators on a Hilbert space  $\mathcal{H}$ , then every Jordan higher derivation from  $\mathcal{A}$  into a Banach Jordan algebra  $\mathcal{B}$  with continuous  $d_0$ , is continuous.

**Proof.** By the hypothesis,  $d_0$  is continuous on  $\mathcal{A}$ . Suppose by induction that  $d_0, \ldots, d_{m-1}$  are continuous on  $\mathcal{A}$ . Then by Theorem 3.10,  $d_m$  is continuous on  $\operatorname{soc}(\mathcal{A}) = \mathcal{A}$ . The last assertion follows by the fact that  $\operatorname{soc}(\mathcal{K}(\mathcal{H}))$  is  $\mathcal{F}(\mathcal{H})$ , the ideal of finite rank bounded operators on  $\mathcal{H}$ , which is dense in  $\mathcal{K}(\mathcal{H})$ .

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