# AUTOMATIC CONTINUITY OF HIGHER DERIVATIONS ON $J B^{*}$-ALGEBRAS 

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#### Abstract

In this paper we study higher derivations from $J B^{*}$-algebras into Banach Jordan algebras. We show that every higher derivation $\left\{d_{m}\right\}$ from a $J B^{*}$-algebra $\mathcal{A}$ into a $J B^{*}$-algebra $\mathcal{B}$ is continuous provided that $d_{0}$ is a $*$-homomorphism. Also it is proved that every Jordan higher derivation from a commutative $C^{*}$-algebra or from a $C^{*}$-algebra which has minimal idempotents and is the closure of its socle is continuous.


## 1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras (associative or non-associative). By a higher derivation of rank $k$ ( $k$ might be $\infty$ ) we mean a family of linear mappings $\left\{d_{m}\right\}_{m=0}^{k}$ from $\mathcal{A}$ into $\mathcal{B}$ such that

$$
d_{m}(a b)=\sum_{j=0}^{m} d_{j}(a) d_{m-j}(b), \quad(a, b \in \mathcal{A}, \quad m=0,1,2, \ldots, k) .
$$

It is clear that $d_{0}$ is a homomorphism. Higher derivations were introduced by Hasse and Schmidt [8], and algebraists sometimes call them Hasse-Schmidt derivations. The reader may find some of algebraic results concerning these mappings in $[1,4,6,14,16,17]$. They are also studied in other contexts. In [19] higher derivations are applied to study

[^0]generic solving of higher differential equations.
A standard example of a higher derivation of rank $k$ is the family $\left\{\frac{\mathrm{D}^{\mathrm{m}}}{\mathrm{m}!}\right\}_{m=0}^{k}$, where $D$ is an ordinary derivation of an algebra $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{B}$ are normed algebras then a higher derivation $\left\{d_{m}\right\}$ is said to be continuous, whenever every $d_{m}$ is continuous. It is known that every derivation on a semisimple Banach algebra is continuous [13]. Ringrose [15] proved that every derivation from a $C^{*}$-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module is continuous. In [9] derivations from $J B^{*}$-algebras into Banach Jordan modules were studied and continuity of these mappings were proved in certain cases. Loy in [12] proved that if $\mathcal{A}$ is an $(F)$-algebra which is a subalgebra of a Banach algebra $\mathcal{B}$ of power series, then every higher derivation $\left\{d_{m}\right\}: \mathcal{A} \rightarrow \mathcal{B}$ is automatically continuous. Jewell [11], showed that a higher derivation from a Banach algebra onto a semisimple Banach algebra is continuous provided that $\operatorname{ker}\left(d_{0}\right) \subseteq \operatorname{ker}\left(d_{m}\right)$, for all $m \geq 1$. Villena [20] proved that every higher derivation from a unital Banach algebra $\mathcal{A}$ into $\mathcal{A} / \mathcal{P}$, where $\mathcal{P}$ is a primitive ideal of $\mathcal{A}$ with infinite codimension, is continuous. Also the range problem of continuous higher derivations was studied in [14].

In this paper we study automatic continuity of higher derivations from $J B^{*}$-algebras. Section 2 is devoted to some concepts which are needed in the sequel. In Section 3 we prove that a higher derivation from a $J B^{*}$-algebra into another $J B^{*}$-algebra is continuous provided that $d_{0}$ is a $*$-homomorphism. Also we will show that every (Jordan) higher derivation from a commutative $C^{*}$-algebra or from a $C^{*}$-algebra which has minimal idempotents and is the closure of its socle (e. g. $\mathcal{K}(\mathcal{H})$ ) into a Banach Jordan algebra is continuous. These are in fact generalizations of some results in [9].

## 2. Preliminaries

Let $\mathcal{A}$ be a Jordan algebra and let $\mathcal{X}$ be a vector space over the same field as $\mathcal{A}$. Then $\mathcal{X}$ is said to be a Jordan $\mathcal{A}$-module if there is a pair of bilinear mappings (called module operations), $(a, x) \mapsto a . x,(a, x) \mapsto x . a$, from $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ such that for all $a, b \in \mathcal{A}$ and all $x \in \mathcal{X}$ the following conditions hold:
(i) $a \cdot x=x \cdot a$;
(ii) $\quad a \cdot\left(a^{2} \cdot x\right)=a^{2} \cdot(a \cdot x)$;
(iii) $2((x \cdot a) \cdot b) \cdot a+x \cdot\left(a^{2} \cdot b\right)=2(x \cdot a)(a \cdot b)+(x \cdot b) \cdot a^{2}$.

A linear subspace $\mathcal{S}$ of $\mathcal{X}$ is called a submodule if

$$
\mathcal{A S}:=\{a . x: a \in \mathcal{A}, x \in \mathcal{S}\} \subseteq \mathcal{S} .
$$

If $\mathcal{A}$ is a Banach Jordan algebra and $\mathcal{X}$ is a Banach space which is a Jordan $\mathcal{A}$-module then $\mathcal{X}$ is said to be a weak Jordan $\mathcal{A}$-module whenever the mapping $x \mapsto a . x$, from $\mathcal{X} \longrightarrow \mathcal{X}$ is continuous, for all $a \in \mathcal{A}$; and $\mathcal{X}$ is called a Banach Jordan $\mathcal{A}$-module if the mapping $(a, x) \mapsto a . x$, from $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ is continuous, or equivalently, if there exists $M>0$ such that $\|a \cdot x\| \leq M\|a\|\|x\| \quad(a \in \mathcal{A}, x \in \mathcal{X})$.

Example 2.1. (i) Every Banach Jordan algebra $\mathcal{A}$ is a Banach Jordan $\mathcal{A}$-module whenever we consider its own product as the module operation.
(ii) If $\mathcal{A}$ and $\mathcal{B}$ are Jordan algebras and $\theta: \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism, then $\mathcal{B}$ can be considered as a Jordan $\mathcal{A}$-module with module operation

$$
a . b=\theta(a) b \quad(a \in \mathcal{A}, b \in \mathcal{B})
$$

In this case we will say that $\mathcal{B}$ is an $\mathcal{A}$-module via the homomorphism $\theta$. If $\mathcal{A}$ and $\mathcal{B}$ are Banach Jordan algebras then it is easy to see that $\mathcal{B}$ is a weak Jordan $\mathcal{A}$-module.
(iii) The topological dual $\mathcal{A}^{*}$ of $\mathcal{A}$, with module operation $(a, f) \mapsto a . f$ defined by

$$
(a . f)(b)=f(a b) \quad\left(a, b \in \mathcal{A}, f \in \mathcal{A}^{*}\right)
$$

is a Banach Jordan $\mathcal{A}$-module.
(iv) If $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ is a Banach (respectively weak) $\mathcal{A}$-module, then we may consider $\mathcal{A}$ as a Jordan algebra with Jordan product $(a, b) \mapsto \frac{a b+b a}{2}, \mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$. Then $\mathcal{X}$ with the module operation $a . x=\frac{a x+x a}{2}$ is a Banach (respectively weak) Jordan $\mathcal{A}$-module. Here the mappings $(a, x) \mapsto a x$ and $(a, x) \mapsto x a, \mathcal{A} \times \mathcal{X} \rightarrow \mathcal{X}$, denote the associative module operations of $\mathcal{A}$ on $\mathcal{X}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Jordan $\mathcal{A}$-modules. Then a linear mapping $T: \mathcal{X} \longrightarrow$ $\mathcal{Y}$ is said to be a module homomorphism if $T(a . x)=a . T(x)(a \in \mathcal{A}, x \in$ $\mathcal{X})$. In Example 2.1 (ii), $\theta$ is a module homomorphism.

Let $\mathcal{A}$ be a Jordan algebra and let $\mathcal{X}$ be a Jordan $\mathcal{A}$-module. Then $\mathcal{A} \oplus \mathcal{X}$ with product $\left(a_{1}+x_{1}\right)\left(a_{2}+x_{2}\right)=a_{1} a_{2}+a_{1} \cdot x_{2}+a_{2} \cdot x_{1}$, is a Jordan algebra which is called the split null extension of $\mathcal{A}$ and $\mathcal{X}$. In fact a linear space $\mathcal{X}$ is a Jordan $\mathcal{A}$-module if and only if this split null extension is a Jordan algebra [10].

Corresponding to $(a, 0) \in \mathcal{A} \oplus \mathcal{X}$ with $a \in \mathcal{A}$, as in any Jordan algebra, we define the linear operators $R_{a}$ and $U_{a}$ on $\mathcal{A} \oplus \mathcal{X}$ as follows

$$
R_{a}(u)=a u, \quad U_{a}(u)=2 a(a u)-a^{2} u \quad(u \in \mathcal{A} \oplus \mathcal{X}) .
$$

We feel free to use the notation $R_{a}$ and $U_{a}$ for the same operators on $\mathcal{A}$. For every $x, y$ in a Jordan algebra, set $\left[R_{x}, R_{y}\right]:=R_{x} R_{y}-R_{y} R_{x}$. We recall that each $x, y, z$ in a Jordan algebra satisfy

$$
\begin{equation*}
\left[R_{x y}, R_{z}\right]+\left[R_{x z}, R_{y}\right]+\left[R_{y z}, R_{x}\right]=0 \tag{2.1}
\end{equation*}
$$

which is the identity (O1) in Section 1.7 of [10]. For a submodules of $\mathcal{X}$, set

$$
\begin{aligned}
\mathcal{R}(\mathcal{S}) & :=\left\{a \in \mathcal{A}: R_{a}(x)=0\right. \\
\mathcal{L}(\mathcal{S}): & \text { for all } \\
=\left\{a \in \mathcal{A}: U_{a}(x)=0\right. & \text { for all } \\
\mathcal{I}(\mathcal{S}) & :=\{a \in \mathcal{R}(\mathcal{S}): a b \in \mathcal{R}(\mathcal{S}) \\
\text { for all } & b \in \mathcal{A}\} .
\end{aligned}
$$

Note that if $\mathcal{S}$ is a submodule, then it is an ideal of $\mathcal{A} \oplus \mathcal{X}$, and $\mathcal{I}(\mathcal{S})$ is actually $\operatorname{ann}(\mathcal{S})$ in view of Zelmanov, which is an ideal by Lemma 3(b) of [21]. Here we give the proof for the sake of convenience.

Lemma 2.2. Let $\mathcal{A}$ be a Jordan algebra and let $\mathcal{X}$ be a Jordan $\mathcal{A}$-module. If $\mathcal{S}$ is a submodule then
(i) $\mathcal{I}(\mathcal{S})$ is the largest ideal of $\mathcal{A}$ contained in $\mathcal{R}(\mathcal{S})$;
(ii) $\mathcal{R}(\mathcal{S}) \cap \mathcal{Q}(\mathcal{S})=\left\{a \in \mathcal{A}: a^{2} \in \mathcal{R}(\mathcal{S})\right\}$;
(iii) $\mathcal{I}(\mathcal{S}) \subseteq \mathcal{R}(\mathcal{S}) \cap \mathcal{Q}(\mathcal{S})$.

Proof. (i) It is easy to see that each ideal of $\mathcal{A}$ contained in $\mathcal{R}(\mathcal{S})$ is a subset of $\mathcal{I}(\mathcal{S})$. We show that $\mathcal{I}(\mathcal{S})$ is an ideal. Suppose that $a \in \mathcal{I}(\mathcal{S})$ and $b \in \mathcal{A}$. Then by definition of $\mathcal{I}(\mathcal{S}), a b \in \mathcal{R}(\mathcal{S})$. Now to see that $a b \in \mathcal{I}(\mathcal{S})$ it is enough to show that $(a b) c \in \mathcal{R}(\mathcal{S})$, for each $c \in \mathcal{A}$. We consider (2.1) in the Jordan algebra $\mathcal{A} \oplus \mathcal{X}$, for $x=a, y=b, z \in \mathcal{S}$ and take $c \in \mathcal{A}$. Since $\mathcal{S}$ is a submodule and $a \in \mathcal{I}(\mathcal{S})$, it follows that $R_{z} R_{a b}(c)=0$, or equivalently, $(a b) c \in \mathcal{R}(\mathcal{S})$. Parts (ii) and (iii) are easily verified.

## 3. Automatic continuity of Higher derivations from $J B^{*}$-algebras

First of all we recall that a real Banach Jordan algebra $\mathcal{A}$ is a $J B$-algebra whenever $\left\|a^{2}\right\|=\|a\|^{2}$ and $\left\|a^{2}\right\| \leq\left\|a^{2}+b^{2}\right\|$, for all $a, b \in$
$\mathcal{A}$. A complex Banach Jordan algebra $\mathcal{A}$ is said to be a $J B^{*}$-algebra whenever there is an algebra involution $*$ on $\mathcal{A}$ such that $\left\|a^{*}\right\|=\|a\|$ and $\left\|U_{a}\left(a^{*}\right)\right\|=\|a\|^{3}$, for all $a \in \mathcal{A}$. For a subset $\mathcal{C}$ of a $J B^{*}$-algebra $\mathcal{A}$, set $\mathcal{C}_{h}:=\left\{a \in \mathcal{C}: a=a^{*}\right\}$. Then $\mathcal{A}_{h}$ is a $J B$-algebra and $\mathcal{A}=\mathcal{A}_{h}+i \mathcal{A}_{h}$. If $a \in \mathcal{A}_{h}$ then $C^{*}(a)$, the $J B^{*}$-subalgebra of $\mathcal{A}$ generated by $a$ (or by $a, 1$ if $\mathcal{A}$ is unital), is a $C^{*}$-algebra. Clearly each $C^{*}$-algebra with respect to its Jordan product is a $J B^{*}$-algebra. The reader is referred to [7] for more details on $J B$-algebras and $J B^{*}$-algebras. From now on throughout this section we assume that $\mathcal{A}$ is a unital $J B^{*}$-algebra, $\mathcal{B}$ is a Banach Jordan algebra and $\left\{d_{m}\right\}$ is a higher derivation of infinite rank from $\mathcal{A}$ into $\mathcal{B}$ with continuous $d_{0}$. For each $m=0,1,2, \ldots$, set

$$
\mathcal{S}_{m}:=\left\{b \in \mathcal{B}: \exists\left\{a_{n}\right\} \subseteq \mathcal{A} \text { s.t. } a_{n} \rightarrow 0 \text { and } d_{m}\left(a_{n}\right) \rightarrow b\right\}
$$

which is called the separating space of $d_{m}$. This is a closed linear subspace of $\mathcal{B}$ ([5], Theorem 5.1.2) and by the closed graph theorem $d_{m}$ is continuous if and only if $\mathcal{S}_{m}=\{0\}$. Therefore $\left\{d_{m}\right\}$ is continuous if and only if $\mathcal{S}_{m}=\{0\}$, for all $m \geq 0$. If we consider $\mathcal{B}$ as a Jordan $\mathcal{A}$-module via the homomorphism $d_{0}$ as in Example 2.1 (ii), then $d_{1}$ would be a derivation from $\mathcal{A}$ into $\mathcal{B}$. With the assumption on $d_{0}$ we have $\mathcal{S}_{0}=\{0\}$ and it is easy to see that $\mathcal{S}_{1}$ is a submodule of $\mathcal{B}$. In general $\mathcal{S}_{m}$ is not a submodule for $m \geq 2$, but if $d_{o}, d_{1}, \ldots d_{m-1}$ are assumed to be continuous, then $d_{m}$ would be an intertwining map and hence $\mathcal{S}_{m}$ is a submodule. Using the same notations as in Section 2, set $\mathcal{R}_{m}:=\mathcal{R}\left(\mathcal{S}_{m}\right)$, $\mathcal{Q}_{m}:=\mathcal{Q}\left(\mathcal{S}_{m}\right)$ and $\mathcal{I}_{m}:=\mathcal{I}\left(\mathcal{S}_{m}\right)$. If $d_{0}, \ldots, d_{m-1}$ are continuous then we have

$$
\begin{gathered}
\mathcal{R}_{m}=\left\{a \in \mathcal{A}: R_{a} d_{m} \quad \text { is continuous }\right\} \\
=\left\{a \in \mathcal{A}: d_{m} R_{a} \quad \text { is continuous }\right\}
\end{gathered}
$$

and

$$
\begin{aligned}
& \mathcal{Q}_{m}=\left\{a \in \mathcal{A}: U_{a} d_{m} \quad \text { is continuous }\right\} \\
& =\left\{a \in \mathcal{A}: d_{m} U_{a} \quad \text { is continuous }\right\} .
\end{aligned}
$$

Before we prove the next lemma, we recall that a subalgebra $\mathcal{C}$ of a Jordan algebra $\mathcal{A}$ is said to be strongly associative if $\left[R_{a}, R_{b}\right]=0$, for all $a, b \in \mathcal{C}$. By Example 1.8 .1 of [10], for each $a \in \mathcal{A}$, the subalgebra of $\mathcal{A}$ generated by $a$, (or by $a, 1$ if $\mathcal{A}$ is unital) is strongly associative and by ([10] Lemma 1.8.8), if $a, b$ lie in a strongly associative subalgebra, then $U_{a b}=U_{a} U_{b}$.

Lemma 3.1. Let $\mathcal{A}$ be a JB*-algebra. Suppose that $\mathcal{X}$ is a Banach Jordan $\mathcal{A}$-module, $\mathcal{Y}$ is a weak Jordan $\mathcal{A}$-module and $T: \mathcal{X} \longrightarrow \mathcal{Y}$ is
a module homomorphism. If $a \in \mathcal{A}_{h}$, and $\left\{f_{n}\right\} \subseteq C^{*}(a)$ is such that $f_{i} f_{j}=0(i \neq j)$, then $U_{f_{n}} 2 T$ is continuous for all but a finite number of $n$ 's.

Proof. Suppose that $U_{f_{n}} 2 T$ is discontinuous for infinitely many $n$ 's. By considering a subsequence we may assume that $U_{f_{n}}{ }^{2} T$ is discontinuous for each $n$. Let $M_{n}$ and $K_{n}$ be the norms of the bounded linear operators $x \mapsto U_{f_{n}}(x), \mathcal{X} \longrightarrow \mathcal{X}$, and $y \mapsto U_{f_{n}}(y), \mathcal{Y} \longrightarrow \mathcal{Y}$, respectively. Note that $M_{n}, K_{n}>0$ for each n ; otherwise $U_{f_{n}}{ }^{2} T=T U_{f_{n}}{ }^{2}=0$ which is continuous. Choose a sequence $\left\{x_{n}\right\}$ in $\mathcal{X}$ such that

$$
\begin{array}{r}
\left\|x_{n}\right\| \leq 2^{-n} / M_{n} \\
\left\|U_{f_{n}} 2 T\left(x_{n}\right)\right\| \geq n K_{n}
\end{array}
$$

Take $z=\sum_{n=1}^{\infty} U_{f_{n}}\left(x_{n}\right)$. By strong associativity of $C^{*}(a)$ as a subalgebra of $\mathcal{A} \oplus \mathcal{X}$ and $\mathcal{A} \oplus \mathcal{Y}$, we have $U_{f_{i}} U_{f_{j}}=U_{f_{i} f_{j}}=0(i \neq j)$, on $\mathcal{A} \oplus \mathcal{X}$ and $\mathcal{A} \oplus \mathcal{Y}$. Since $T$ is a module homomorphism, $K_{n}\|T(z)\| \geq$ $\left\|U_{f_{n}}(T(z))\right\|=\left\|T U_{f_{n}}(z)\right\|=\left\|T\left(U_{f_{n}}{ }^{2}\left(x_{n}\right)\right)\right\|=\left\|U_{f_{n}}{ }^{2}\left(T x_{n}\right)\right\| \geq n K_{n}$. Therefore $\|T(z)\| \geq n$ for each $n$, which is impossible. So the result holds.

Remark 3.2. Suppose that $\mathcal{B}$ is a Jordan algebra. Then $\mathcal{B}_{m}:=$ $\underbrace{\mathcal{B} \oplus \mathcal{B} \oplus \ldots \oplus \mathcal{B}}_{m+1}$ is a Jordan algebra with the product defined by

$$
\left(x_{0}, x_{1}, \ldots, x_{m}\right)\left(y_{0}, y_{1}, \ldots, y_{m}\right)=\left(x_{0} y_{0}, x_{0} y_{1}+x_{1} y_{0}, \ldots, \sum_{i=0}^{m} x_{i} y_{m-i}\right)
$$

for all $\left(x_{0}, x_{1}, \ldots, x_{m}\right),\left(y_{0}, y_{1}, \ldots, y_{m}\right) \in \mathcal{B}_{m}$. Clearly, this product is commutative. Suppose that $\bar{x}=\left(x_{0}, x_{1}, \ldots, x_{m}\right), \bar{y}=\left(y_{0}, y_{1}, \ldots, y_{m}\right) \in$ $\mathcal{B}_{m}$. Then the $k^{\text {th }}$ entries of $\bar{x}\left(\bar{x}^{2} \bar{y}\right)$ and $\bar{x}^{2}(\bar{x} \bar{y})$ are

$$
\begin{equation*}
\sum_{l=0}^{k} x_{l}\left(\sum_{j=0}^{k-l} \sum_{i=0}^{j}\left(x_{j} x_{j-i}\right) y_{k-j-l}\right) \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{l=0}^{k} \sum_{i=0}^{l} \sum_{j=0}^{k-l}\left(x_{l} x_{l-i}\right)\left(x_{j} y_{k-j-l}\right) \tag{3.2}
\end{equation*}
$$

respectively. By identities (O2) and (O3) in Section 1.7 of [10], (3.1) and (3.2) are equal, and hence $\mathcal{B}_{m}$ is a Jordan algebra. Furthermore, let $\mathcal{B}$ be
a Banach Jordan algebra. Define a norm on $\mathcal{B}_{m}$ by $\left\|\left(x_{0}, x_{1}, \ldots, x_{m}\right)\right\|_{0}=$ $\sum_{i=0}^{m}\left\|x_{i}\right\|$. Then $\|\cdot\|_{0}$ is a complete norm on $\mathcal{B}_{m}$ and it is easy to see that

$$
\begin{gathered}
\left\|\left(x_{0}, x_{1}, \ldots, x_{m}\right)\left(y_{0}, y_{1}, \ldots, y_{m}\right)\right\|_{0} \\
\leq\left\|\left(x_{0}, x_{1}, \ldots, x_{m}\right)\right\|_{o}\left\|\left(y_{0}, y_{1}, \ldots, y_{m}\right)\right\|_{0},
\end{gathered}
$$

for all $\left(x_{0}, x_{1}, \ldots, x_{m}\right),\left(y_{0}, y_{1}, \ldots, y_{m}\right) \in \mathcal{B}_{m}$. Therefore $\mathcal{B}_{m}$ is a Banach Jordan algebra.

Lemma 3.3. Suppose that $\mathcal{A}$ is a $J B^{*}$-algebra and $\mathcal{B}$ is a Banach Jordan algebra. Let $\left\{d_{m}\right\}: \mathcal{A} \longrightarrow \mathcal{B}$ be a higher derivation with continuous $d_{0}$. Let $a \in \mathcal{A}_{h}$ and let $\left\{f_{n}\right\} \subseteq C^{*}(a)$ be such that $f_{i} f_{j}=0(i \neq j)$. Then for each $m=0,1,2, \ldots$, we have $f_{n}^{2} \in \mathcal{Q}_{m}$, for all but a finite number of $n$ 's.

Proof. Consider a fixed $m$, and let $\mathcal{B}_{m}$ be as in Remark 3.2. We define

$$
\begin{gathered}
\theta_{m}: \mathcal{A} \longrightarrow \mathcal{B}_{m}, \\
a \mapsto\left(d_{0}(a), d_{1}(a), \ldots, d_{m}(a)\right) .
\end{gathered}
$$

Then $\theta_{m}$ is a homomorphism and $\mathcal{B}_{m}$ is a weak Jordan $\mathcal{A}$-module via the homomorphism $\theta_{m}$. Also as in Example 2.1 (ii), $\theta_{m}$ is a module homomorphism. We have $U_{f_{i}} U_{f_{j}}=U_{f_{i} f_{j}}=0 \quad(i \neq j)$, on the split null extension of $\mathcal{A}$ and $\mathcal{B}_{m}$. Hence by Lemma 3.1, $U_{f_{n}{ }^{2}} \theta_{m}$ is continuous for all but a finite number of $n$ 's. Thus for such $n$ 's, $U_{f_{n}^{2}} d_{1}, \ldots, U_{f_{n}^{2}} d_{m}$ are continuous and it follows that $f_{n}^{2} \in \mathcal{Q}_{m}$, for all but a finite number of $n$ 's.

Theorem 3.4. Let $\mathcal{A}$ be a $J B^{*}$-algebra and let $\mathcal{B}$ be a Jordan Banach algebra. Suppose that $\left\{d_{m}\right\}$ is a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ with continuous $d_{0}$. Then the following assertions hold.
(i) If $a \in \mathcal{A}_{h}$ and $\Delta$ is the maximal ideal space of $C^{*}(a)$, then for every $m=1,2, \ldots$, the set $F_{m}=\left\{\lambda \in \Delta: \lambda\left(\mathcal{Q}_{m} \cap C^{*}(a)\right)=\{0\}\right\}$ is finite.
(ii) If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ containing $\mathcal{Q}_{m}$, then every element in the JB-algebra $\left(\frac{\mathcal{A}}{\mathcal{I}}\right)_{h}$ has finite spectrum.
(iii) If $d_{1}, \ldots, d_{m-1}$ are continuous and $\mathcal{K}$ is a closed ideal of $\mathcal{A}$ contained in $\mathcal{Q}_{m}$, then $\left.d_{m}\right|_{\mathcal{K}}$ is continuous.
(iv) If $d_{1}, \ldots, d_{m-1}$ are continuous and $\mathcal{L}$ is an ideal of $\mathcal{A}$ such that $\left.d_{m}\right|_{\mathcal{L}}$ is continuous, then $\mathcal{L} \subseteq \mathcal{I}_{m} \subseteq \mathcal{Q}_{m}$.

Proof. (i) If $F_{m}$ is infinite, then we may find an infinite sequence $\left\{\lambda_{k}\right\} \subseteq \Delta$ and a sequence $\left\{V_{k}\right\}$ of open subsets of $\Delta$ such that $V_{j} \cap V_{k}=$ $\emptyset(j \neq k)$, and $\lambda_{k} \in V_{k}$, for each $k$. For every $k \in \mathbb{N}$, choose $f_{k} \in C^{*}(a)$ such that $f_{k}\left(\lambda_{k}\right) \neq 0$ and $f_{k}\left(\Delta \backslash V_{k}\right)=\{0\}$. Then $f_{k} f_{j}=0(k \neq j)$, and $f_{k}^{2} \notin \mathcal{Q}_{m}$ which contradicts Lemma 3.3.
(ii) Let $\mathcal{I}$ be a closed ideal in $\mathcal{A}$ such that $\mathcal{Q}_{m} \subseteq \mathcal{I}$, for all $m=0,1,2, \ldots$ For each $a \in \mathcal{A}_{h}$, we have
$\left\{\lambda \in \Delta: \lambda\left(\mathrm{I} \cap C^{*}(a)\right)=\{0\}\right\} \subseteq\left\{\lambda \in \Delta: \lambda\left(\mathcal{Q}_{m} \cap C^{*}(a)\right)=\{0\}\right\}$.
Hence by (i) the left hand side is a finite set and as in Theorem 12.2 of [18], $\frac{\mathrm{C}^{*}(\mathrm{a})}{\mathrm{C}^{*}(\mathrm{a}) \cap \mathcal{I}}$ is finite dimensional, and since the closed $*$-subalgebra of $\frac{\mathcal{A}}{\mathcal{I}}$ generated by $a$ and 1 is isomorphic to $\frac{\mathrm{C}^{*}(\mathrm{a})}{\mathrm{C}^{*}(\mathrm{a}) \cap \mathcal{I}}$, the result holds.
(iii) We show that $d_{m}$ is bounded on bounded subsets of $\mathcal{K}_{h}$. On the contrary suppose that there is a sequence $\left\{a_{n}\right\} \subseteq \mathcal{K}_{h}$ such that $a_{n} \rightarrow 0$ and $\left\|d_{m}\left(a_{n}\right)\right\| \rightarrow \infty$. We may assume that $\sum_{n=1}^{\infty}\left\|a_{n}\right\|^{2} \leq 1$. Let $b=\left(\sum_{n=1}^{\infty} a_{n}^{2}\right)^{1 / 8}$. Then $b \geq 0,\|b\| \leq 1$ and $a_{n}^{2} \leq b^{8}(n \in \mathbb{N})$. By [9] Lemma 1.7, for each $n \in \mathbb{N}$ there exists $u_{n} \in \mathcal{K}_{h}$ such that $\left\|u_{n}\right\| \leq 2\left\|b^{1 / 4}\right\| \leq 2$ and $a_{n}=U_{b}\left(u_{n}\right)$. Hence $d_{m}\left(a_{n}\right)=d_{m} U_{b}\left(u_{n}\right)$. Since $\mathcal{K} \subseteq \mathcal{Q}_{m}$, we have $b \in \mathcal{Q}_{m}$ and so $d_{m} U_{b}$ is continuous. Now it follows that $\left\|d_{m}\left(a_{n}\right)\right\| \leq\left\|d_{m} U_{b}\right\|\left\|u_{n}\right\| \leq 2\left\|d_{m} U_{b}\right\|$, which is a contradiction.
(iv) Suppose that $\left.d_{m}\right|_{\mathcal{L}}$ is continuous. Take $a \in \mathcal{S}_{m}$. Then there is a sequence $\left\{a_{n}\right\} \subseteq \mathcal{A}$ such that $a_{n} \rightarrow 0$ and $d_{m}\left(a_{n}\right) \rightarrow a$. Let $b \in \mathcal{L}$. Since $d_{1}, \ldots, d_{m-1}$ are continuous it follows that

$$
d_{m}\left(b a_{n}\right)=d_{0}(b) d_{m}\left(a_{n}\right)+d_{1}(b) d_{m-1}\left(a_{n}\right)+\ldots+d_{m}(b) d_{0}\left(a_{n}\right) \rightarrow b a
$$

Since $b a_{n} \in \mathcal{L}$ and $\left.d_{m}\right|_{\mathcal{L}}$ is continuous, $b a=0$. This means that $b \in \mathcal{R}_{m}$ and hence $\mathcal{L} \subseteq \mathcal{R}_{m}$. But $\mathcal{I}_{m}$ is the largest ideal of $\mathcal{A}$ contained in $\mathcal{R}_{m}$, so we have $\mathcal{L} \subseteq \mathcal{I}_{m} \subseteq \mathcal{Q}_{m}$.

Corollary 3.5. Let $\mathcal{A}$ be a $J B^{*}$-algebra and let $\mathcal{B}$ be a Banach Jordan algebra. Suppose that $\left\{d_{m}\right\}$ is a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ with continuous $d_{0}$. If $\mathcal{K}$ is a closed ideal of $\mathcal{A}$ contained in $\bigcap \mathcal{Q}_{m}$, then $\left.d_{m}\right|_{\mathcal{K}}$ is continuous for all $m$.

Proof. Similar to the proof of Theorem 3.4 (iii).
Theorem 3.6. Let $\left\{d_{m}\right\}$ be a higher derivation of a $J B^{*}$-algebra $\mathcal{A}$ into a Banach Jordan algebra $\mathcal{B}$ such that $d_{0}$ is continuous. Then $\left\{d_{m}\right\}$ is
continuous if and only if $\left(\mathcal{Q}_{m}\right)_{h}:=\left\{a \in \mathcal{Q}_{m}: \quad a=a^{*}\right\}$ is a real linear subspace of $\mathcal{A}_{h}$, for all $m \in \mathbb{N}$.

Proof. If $\left\{d_{m}\right\}$ is continuous then $\mathcal{Q}_{m}=\mathcal{A}$, and so $\left(\mathcal{Q}_{m}\right)_{h}$ is real linear. Conversely let $\left(\mathcal{Q}_{m}\right)_{h}$ be real linear. Since $d_{1}$ is a derivation, by ([9] Theorem 2.2), $d_{1}$ is continuous. Suppose by induction that each $d_{i}(i<m)$ is continuous. Then $\mathcal{S}_{m}$ is a submodule of $\mathcal{B}$, and $U_{\mathcal{A}_{h}}\left(\mathcal{Q}_{m}\right)_{h}$ $\subseteq\left(\mathcal{Q}_{m}\right)_{h}$, hence $\left(\mathcal{Q}_{m}\right)_{h}$ is an ideal of $\mathcal{A}_{h}$. By Theorem 3.4 (iii), $d_{m}$ is continuous on $\left(\mathcal{Q}_{m}\right)_{h} \oplus i\left(\mathcal{Q}_{m}\right)_{h}$. Hence $\left(\mathcal{Q}_{m}\right)_{h} \oplus i\left(\mathcal{Q}_{m}\right)_{h} \subseteq \mathcal{I}_{m} \subseteq \mathcal{Q}_{m}$ and so $\mathcal{I}_{m}=\left(\mathcal{Q}_{m}\right)_{h} \oplus i\left(\mathcal{Q}_{m}\right)_{h}$. Let $\pi: \mathcal{A} \rightarrow \frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}$ be the canonical quotient map. By Theorem 3.4 (ii) every element in $\left(\frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}\right)_{h}$ has finite spectrum. But $\left(\frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}\right)_{h}=\frac{\mathcal{A}_{\mathrm{h}}}{\left(\mathcal{I}_{\mathrm{m}}\right)_{\mathrm{h}}}$ is a semisimple real Banach Jordan algebra in which every element has non-empty finite spectrum and by [2] it is reduced, that is, there exist idempotents $\pi\left(e_{1}\right), \ldots, \pi\left(e_{n}\right) \in\left(\frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}\right)_{h}$ such that $\pi\left(e_{i}\right) \pi\left(e_{j}\right)=$ $0,(i \neq j), \sum_{i=1}^{n} \pi\left(e_{i}\right)=1$, and $U_{\pi\left(e_{i}\right)}\left(\frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}\right)_{h}=\mathbb{R} \pi\left(e_{i}\right),(i=1, \ldots, n)$. Since each $\pi\left(e_{i}\right)$ is self-adjoint, $\pi\left(e_{i}{ }^{*} e_{i}\right)=\pi\left(e_{i}\right),(i=1, \ldots, n)$, and so $\pi\left(e_{1}{ }^{*} e_{1}\right), \ldots, \pi\left(e_{n}{ }^{*} e_{n}\right)$ are idempotents in $\left(\frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}\right)_{h}$ with sum 1 such that $\pi\left(e_{i}{ }^{*} e_{i}\right) \pi\left(e_{j}{ }^{*} e_{j}\right)=0,(i \neq j)$. Hence by replacing $e_{i}$ with $e_{i}{ }^{*} e_{i}$, if necessary, we may assume that each $e_{i}$ is self-adjoint. Suppose that $\left\{a_{k}\right\} \subseteq \mathcal{A}_{h}$ and $a_{k} \rightarrow 0$. Then $\pi\left(a_{k}\right) \rightarrow 0$, and for each $i=1, \ldots, n$, and each $k \in \mathbb{N}$, there exists $\lambda_{i k} \in \mathbb{R}$ such that

$$
\begin{equation*}
U_{\pi\left(e_{i}\right)}\left(\pi\left(a_{k}\right)\right)=\lambda_{i k} \pi\left(e_{i}\right) . \tag{3.3}
\end{equation*}
$$

Hence $\lambda_{i k} \pi\left(e_{i}\right) \rightarrow 0$ as $k \rightarrow \infty$, and so $\lambda_{i k} \rightarrow 0$ as $k \rightarrow \infty$. By (3.3) we have

$$
U_{e_{i}}\left(a_{k}\right)-\lambda_{i k} e_{i} \in \mathcal{I}_{m}, \quad(i=1, \ldots, n, k \in \mathbb{N}),
$$

and by continuity of $\left.d_{m}\right|_{\mathcal{I}_{m}}, \lim _{k \rightarrow \infty} d_{m}\left(U_{e_{i}}\left(a_{k}\right)-\lambda_{i k} e_{i}\right)=0$. Since $\lim _{k \rightarrow \infty} \lambda_{i k}=0$, we have $\lim _{k \rightarrow \infty} d_{m} U_{e_{i}}\left(a_{k}\right)=0$. Therefore $d_{m} U_{e_{i}}$ is continuous for $i=1, \ldots, n$, and $e_{1}, \ldots, e_{n} \in\left(\mathcal{Q}_{m}\right)_{h}$. So $e_{1}+\ldots+e_{n} \in$ $\mathcal{I}_{m}=\left(\mathcal{Q}_{m}\right)_{h} \oplus i\left(\mathcal{Q}_{m}\right)_{h}$. Since $\pi\left(e_{1}+\ldots+e_{n}\right)$ is the identity of $\frac{\mathcal{A}}{\mathcal{I}_{\mathrm{m}}}$, $\mathcal{A}=\mathcal{I}_{m}$ and $d_{m}$ is continuous on $\mathcal{A}$.

Lemma 3.7. Let $\mathcal{A}$ and $\mathcal{B}$ be $J B^{*}$-algebras and let $\phi: \mathcal{A} \longrightarrow \mathcal{B}$ be a *-homomorphism, that is $\phi\left(a^{*}\right)=(\phi(a))^{*} \quad(a \in \mathcal{A})$. Consider $\mathcal{B}$ as a Banach Jordan $\mathcal{A}$-module via the homomorphism $\phi$. If $\mathcal{S}$ is a submodule of $\mathcal{B}$, then $\mathcal{Q}(\mathcal{S})=\mathcal{I}(\mathcal{S})$.

Proof. We show that $(\mathcal{Q}(\mathcal{S}))_{h}=(\mathcal{I}(\mathcal{S}))_{h}$. Consider the identities

$$
\begin{gather*}
\left(U_{x}\left(y^{2}\right)\right)^{2}=U_{x} U_{y} U_{y}\left(x^{2}\right),  \tag{3.4}\\
(x y)^{2}=\frac{1}{2} y U_{x}(y)+\frac{1}{4} U_{x}\left(y^{2}\right)+\frac{1}{4} U_{y}\left(x^{2}\right), \tag{3.5}
\end{gather*}
$$

which are valid in any Jordan algebra, see [10], p. 37 for the first one. The second holds by the fact that any Jordan algebra generated by two elements is special, see Shirsov-Cohen's theorem, [7] Theorem 2.4.14. Now, if $a \in \mathcal{Q}(\mathcal{S})_{h}$ then by setting $x=\phi(a) \in \mathcal{B}_{h}$ in (3.4) and (3.5), we have $(\phi(a) b)^{2}=0 \quad(a \in \mathcal{Q}(\mathcal{S}), b \in \mathcal{S})$. Thus $\phi(a) b=0 \quad(a \in \mathcal{Q}(\mathcal{S})$, $b \in \mathcal{S})$. Therefore $a \in \mathcal{R}(\mathcal{S})$ and it follows that $(\mathcal{Q}(\mathcal{S}))_{h} \subseteq(\mathcal{R}(\mathcal{S}))_{h}$. So $(\mathcal{Q}(\mathcal{S}))_{h}=(\mathcal{Q}(\mathcal{S}))_{h} \cap(\mathcal{R}(\mathcal{S}))_{h}=(\mathcal{I}(\mathcal{S}))_{h}$, by ([9], Theorem 1.4).

Corollary 3.8. Let $\mathcal{A}$ and $\mathcal{B}$ be $J B^{*}$-algebras, and let $\left\{d_{m}\right\}: \mathcal{A} \longrightarrow \mathcal{B}$ be a higher derivation for which $d_{0}$ is a *-homomorphism. Then $\left\{d_{m}\right\}$ is continuous.

Proof. Since $d_{0}$ is a $*$-homomorphism, it is automatically continuous. Note that $\mathcal{S}_{1}$ is a submodule of $\mathcal{B}$, thus by Lemma 3.7, $\mathcal{Q}_{1}$ is a linear subspace of $\mathcal{A}$ and hence by ([9] Theorem 2.2), $d_{1}$ is continuous. Fix $m$, suppose that each $d_{i}(i<m)$ is continuous. Therefore $\mathcal{S}_{m}$ is a submodule and again by Lemma 3.7, $\mathcal{Q}_{m}$ is a linear subspace of $\mathcal{A}$, and hence by Theorem 3.6, $d_{m}$ is continuous.
In the next few results, by a Jordan higher derivation from a $C^{*}$-algebra $\mathcal{A}$ we mean a higher derivation from $\mathcal{A}$, with its Jordan product, into a Banach Jordan algebra. Obviously each higher derivation (with respect to the associative product) is also a Jordan derivation.
As a consequence of Corollary 3.8 each higher derivation, or each Jordan higher derivation between $C^{*}$-algebras, is continuous provided that $d_{0}$ is a $*$-homomorphism. In the next results $d_{0}$ is not assumed to be a *-homomorphism.

Theorem 3.9. Let $\mathcal{A}$ be a commutative $C^{*}$-algebra, and let $\mathcal{B}$ be a Banach Jordan algebra. If $\left\{d_{m}\right\}: \mathcal{A} \rightarrow \mathcal{B}$ is a Jordan higher derivation such that $d_{0}$ is continuous, then $\left\{d_{m}\right\}$ is continuous.

Proof. By ([9], Theorem 2.4) of , $d_{1}$ is continuous. Suppose that $d_{1}, \ldots$, $d_{m-1}$ are continuous. Then $\mathcal{S}_{m}$ is a submodule. We show that $\left(\mathcal{Q}_{m}\right)_{h}=$
$\left(\mathcal{I}_{m}\right)_{h}$. Let $a \in\left(\mathcal{Q}_{m}\right)_{h}$. We have $a^{2} \mathcal{A}=a \mathcal{A} a=U_{a} \mathcal{A} \subseteq \mathcal{Q}_{m}$, and hence $a^{2} \mathcal{A} \subseteq \mathcal{Q}_{m}$. Since $\mathcal{I}_{m}$ is the largest ideal of $\mathcal{A}$ contained in $\mathcal{Q}_{m}$, $a^{2} \mathcal{A} \subseteq \mathcal{I}_{m}$. Therefore $a^{4} \in \mathcal{I}_{m}$ and since $a=a^{*}$, we have $a \in \mathcal{I}_{m}$.
Before proving the next result, we recall that if $\mathcal{A}$ is an associative algebra with associative product $(a, b) \mapsto a b$, and the Jordan product $(a, b) \mapsto \frac{a b+b a}{2}$, then $U_{a}(b)=a b a \quad(a, b \in \mathcal{A})$.

Theorem 3.10. Let $\mathcal{A}$ be a $C^{*}$-algebra with minimal idempotents, and let $\left\{d_{m}\right\}$ be a Jordan higher derivation from $\mathcal{A}$ to a Banach Jordan algebra $\mathcal{B}$. If $d_{0}, \ldots, d_{m-1}$ are continuous on $\mathcal{A}$, then $\left\{d_{m}\right\}$ is continuous on $\overline{\operatorname{soc}(\mathcal{A})}$.

Proof. By ([3], Theorem 30.10) $\operatorname{soc}(\mathcal{A})$ exists. Let $M$ denote the set of all minimal idempotents of $\mathcal{A}$. Then

$$
\begin{equation*}
\operatorname{soc}(\mathcal{A})=\sum_{e \in M} e \mathcal{A}=\sum_{e \in M} \mathcal{A} e, \tag{3.6}
\end{equation*}
$$

where by $\sum$ we mean the algebraic sum. Since $d_{0}, \ldots, d_{m-1}$ are continuous, we have
$\mathcal{Q}_{m}=\left\{a \in \mathcal{A}: U_{a} d_{m}\right.$ is continuous $\}=\left\{a \in \mathcal{A}: d_{m} U_{a}\right.$ is continuous $\}$ (3.7)
Suppose that $a \in \operatorname{soc}(\mathcal{A})_{h}$, then there exist $b_{1}, \ldots, b_{n} \in \mathcal{A}$, and $e_{1}, \ldots, e_{n} \in$ $M$ such that $a=e_{1} b_{1}+\ldots+e_{n} b_{n}$, and hence $a^{*}=b_{1}{ }^{*} e_{1}^{*}+\ldots+b_{n}{ }^{*} e_{n}^{*}=a$. So

$$
\begin{equation*}
U_{a}(b)=a b a=\sum_{i=1}^{n} \sum_{j=1}^{n} e_{i} b_{i} b b_{j}^{*} e_{j}^{*} \quad(b \in \mathcal{A}) . \tag{3.8}
\end{equation*}
$$

We know that the adjoint of a minimal idempotent is also a minimal idempotent, hence by ([3], Theorem 31.6), $\operatorname{dim}\left(e_{i} \mathcal{A} e_{j}{ }^{*}\right) \leq 1$, for $i, j=1, \ldots, n$. By (3.8) we have, $U_{a}(\mathcal{A}) \subseteq \sum_{i=1}^{n} \sum_{j=1}^{n} e_{i} \mathcal{A e}_{j}{ }^{*}$, thus $\operatorname{dim}\left(U_{a}(\mathcal{A})\right)<\infty$ and $d_{m}$ is continuous on $U_{a}(\mathcal{A})$. This shows that $d_{m} U_{a}$ is continuous on $\mathcal{A}$, and hence by (3.7), $a \in \mathcal{Q}_{m}$. It follows that $\operatorname{soc}(\mathcal{A})_{h} \subseteq \mathcal{Q}_{m}$, and since $\mathcal{Q}_{m}$ is closed, $\overline{\left(\operatorname{soc}(\mathcal{A})_{h}\right)} \subseteq \mathcal{Q}_{m}$. By (3.6) $\operatorname{soc}(\mathcal{A})$ is an $*$-ideal, hence $(\overline{\operatorname{soc}(\mathcal{A})})_{h}=\overline{\left(\operatorname{soc}(\mathcal{A})_{h}\right)}$. Now the same $\operatorname{argu-}$ ment as in Theorem 3.4 (iii) implies that $d_{m}$ is continuous on $\overline{\operatorname{soc}(\mathcal{A})}$.

Corollary 3.11. If $\mathcal{A}$ is a $C^{*}$-algebra with minimal idempotents such that $\operatorname{soc}(\mathcal{A})=\mathcal{A}$, then each Jordan higher derivation from $\mathcal{A}$ into a Banach Jordan algebra $\mathcal{B}$ with continuous $d_{0}$ is continuous. In particular,
if $\mathcal{A}=\mathcal{K}(\mathcal{H})$ ), the $C^{*}$-algebra of all compact operators on a Hilbert space $\mathcal{H}$, then every Jordan higher derivation from $\mathcal{A}$ into a Banach Jordan algebra $\mathcal{B}$ with continuous $d_{0}$, is continuous.

Proof. By the hypothesis, $d_{0}$ is continuous on $\mathcal{A}$. Suppose by induction that $d_{0}, \ldots, d_{m-1}$ are continuous on $\mathcal{A}$. Then by Theorem $3.10, d_{m}$ is continuous on $\overline{\operatorname{soc}(\mathcal{A})}=\mathcal{A}$. The last assertion follows by the fact that $\operatorname{soc}(\mathcal{K}(\mathcal{H}))$ is $\mathcal{F}(\mathcal{H})$, the ideal of finite rank bounded operators on $\mathcal{H}$, which is dense in $\mathcal{K}(\mathcal{H})$.

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