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STRUCTURAL PROPERTIES OF INNER AMENABLE DISCRETE GROUPS

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ABSTRACT. A discrete group G is called inner amenable if there exists a mean m on $l^{\infty}(G)$ such that $m(xf_x) = m(f)$ whenever $f \in l^{\infty}(G), x \in G$ and $m \neq \delta_e$. The condition is considerably weaker than ordinary amenability. In this paper, among the others, we investigate the structures of inner amenable discrete groups by theoretical set theory. A sequence of characterizations of inner amenable discrete groups is given.

1. Introduction

Throughout this paper let G be a discrete topological group. Let $l^1(G)$ denote the set of all functions ϕ in B(G) (space of bounded complexvalued functions on G) such that $\sum_{x \in G} |\phi(x)| < \infty$. With pointwise addition and scalar multiplication, with convolution

$$\phi * \psi(x) := \sum_{y \in G} \phi(y) \psi(y^{-1}x) \qquad (x \in G)$$

as product and with the norm $\|\phi\|_1 := \sum_{x \in G} |\phi(x)|$, $l^1(G)$ is a Banach algebra said to be the discrete group algebra of G [4].

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A mean m on $l^{\infty}(G)$ is a positive linear functional on $l^{\infty}(G)$ satisfying m(1) = 1. For a function $f: G \to \mathbb{C}$, we put

$$_{x}f_{x}(y) = f(x^{-1}yx)$$

for any $x, y \in G$. We say that G is *inner amenable* if $l^{\infty}(G)$ admits of a non-trivial inner invariant mean m, that is, $m \neq \delta_e$ and $m(xf_x) = m(f)$ for all $x \in G$ and $f \in l^{\infty}(G)$ (see [10], [11] and [12]).

Effros [2] introduced the notion of inner amenability which gives a new classification of discrete groups. We know that the Dirac measure concentrated on $\{e\}$, denoted by δ_e , always defines a trivial inner invariant mean on $l^{\infty}(G)$, namely $\delta_e(f) = f(e)$ for $f \in l^{\infty}(G)$. Recall that Gis called non-inner amenable, following Effros [2], if δ_e is the only inner invariant mean on $l^{\infty}(G)$. For example the discrete groups $\operatorname{GL}(n, \mathbb{C})$ and $\operatorname{SL}(n, \mathbb{C})$ are not inner amenable [10]. For a locally compact group G, δ_e is an inner invariant mean on $C_b(G)$. Losert and Rindler [5] studied the possibility of the inner invariant extension of δ_e . Further results on inner amenability may be found in ([3], [5], [7] and [8]).

In this paper, we present a few results in the theory of inner amenable discrete groups. A number of equivalent conditions characterizing inner amenable discrete groups is given.

2. Main results

Recall that for subsets F and U of a group G, $C_F(U)$ denotes the conjugation class of U in F, i.e., $C_F(U) = \bigcup \{xUx^{-1}; x \in F\}$ [13]. In the following theorem, a sequence of characterizations of inner amenable discrete groups is given.

Theorem 2.1. For an infinite discrete group G, the following conditions are equivalent:

- (i) G is an inner amenable group.
- (ii) For every finite subset F of G and $\epsilon > 0$, there exists a finite non-empty subset U of G such that $e \notin U$ and

$$1 - \epsilon \le \frac{\left|\bigcap_{x \in F} x^{-1} U x\right|}{\left|C_F(U)\right|} \le 1.$$

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(iii) For every finite subset F of G and $\epsilon > 0$, there exists a finite non-empty subset U of G such that $e \notin U$ and

$$1 - \epsilon \le \frac{\left|\bigcap_{x \in F} x^{-1} U x\right|}{\left|U\right|} \le 1.$$

(iv) There exists a net $(U_{\alpha})_{\alpha \in I}$ of finite non-empty subsets in G such that $e \notin U_{\alpha}$ for all $\alpha \in I$, and

$$\lim_{\alpha} \frac{\left|\bigcap_{x \in F} x^{-1} U_{\alpha} x\right|}{\left|U_{\alpha}\right|} = 1$$

for every finite subset F of G.

(v) For every $n_1, ..., n_m \in \mathbb{N}$, $\epsilon > 0$ and all $\sum_{i=1}^{n_1} \alpha_{i1} \delta_{y_{i1}}, ..., \sum_{i=1}^{n_m} \alpha_{im} \delta_{y_{im}} \in l^1(G)$, there exists a simple function ϕ with $\phi \ge 0$, $\|\phi\|_1 = 1$, $\phi(e) = 0$ such that $\|\sum_{i=1}^{n_k} \phi_i\|_1 \le \|\sum_{i=1}^{n_k} \phi_i\|_1 \le \|\sum_{i=1}^{n_k} \phi_i\|_1 \le \|\phi\|_1 \le \|\nabla \phi\|_1$

$$\left\|\sum_{i=1}^{n_{\kappa}} \alpha_{ik} y_{ik} \phi_{y_{ik}}\right\|_{1} \leq \left|\sum_{i=1}^{n_{\kappa}} \alpha_{ik}\right| + \epsilon \quad for \ all \quad 1 \leq k \leq m.$$

Proof. Assume that G is inner amenable. Let F be a finite subset in G and $\epsilon \in (0, 1)$ be given. By ([13], Theorem 2) there exists a finite non-empty subset V of G such that $e \notin V$ and

$$1 \le \frac{|C_{F^2}(V)|}{|V|} \le \frac{1}{1-\epsilon}$$

Let $U = C_F(V)$, so $0 < |V| \le |U|$ and $e \notin U$. Since $V \subseteq \bigcap_{x \in F} x^{-1} C_F(V) x$, we have

$$1 \leq \frac{|C_F(U)|}{|\bigcap_{x \in F} x^{-1} U x|} = \frac{|C_F(C_F(V))|}{|\bigcap_{x \in F} x^{-1} C_F(V) x|} \leq \frac{|C_{F^2}(V)|}{|V|} \leq \frac{1}{1-\epsilon}.$$

Consequently

$$1 - \epsilon \le \frac{\left|\bigcap_{x \in F} x^{-1} U x\right|}{\left|C_F(U)\right|} \le 1.$$

Thus (i) implies (ii).

(ii) \Rightarrow (iii) Trivial.

(iii) \Rightarrow (iv). Let \mathcal{F} be the family of finite subsets of G. Let I be the directed set $\mathcal{F} \times \mathbb{R}^+$ with $(F_1, \epsilon_1) = \alpha_1 < \alpha_2 = (F_2, \epsilon_2)$ for $\alpha_1, \alpha_2 \in I$ if and only if $F_1 \subseteq F_2$ and $\epsilon_2 < \epsilon_1$. By (iii), to $\alpha = (F, \epsilon) \in I$ we may associate a finite subset U_{α} of G such that $e \notin U_{\alpha}$ and

$$1 - \epsilon \le \frac{\left|\bigcap_{x \in F} x^{-1} U_{\alpha} x\right|}{\left|U_{\alpha}\right|} \le 1.$$

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Clearly we obtain the desired net $\{U_{(F,\epsilon)}\}$.

(iv) \Rightarrow (v). Let $\sum_{i=1}^{n_1} \alpha_{i1} \delta_{y_{i1}}, ..., \sum_{i=1}^{n_m} \alpha_{im} \delta_{y_{im}} \in l^1(G)$ and $\epsilon > 0$ be given. Put $E = \{y_{11}, ..., y_{n_m m}\}, F = E \bigcup E^{-1} \bigcup \{e\}$ and $4(\sum_{i=1}^{n_1} |\alpha_{i1}| + ... + \sum_{i=1}^{n_m} |\alpha_{im}|)\delta = \epsilon$. By the hypothesis there exists a net $(U_{\alpha})_{\alpha \in I}$ of finite non-empty subsets in G such that $e \notin U_{\alpha}$ for all $\alpha \in I$ and

$$\lim_{\alpha} \frac{\left|\bigcap_{x \in F} x^{-1} U_{\alpha} x\right|}{\left|U_{\alpha}\right|} = 1.$$

So, $\frac{1}{1+\delta} \leq \frac{|\bigcap_{x \in F} x^{-1}U_{\alpha}x|}{|U_{\alpha}|} \leq 1$ for some $\alpha \in I$. Let $U = \bigcap_{x \in F} x^{-1}U_{\alpha}x$, so |U| > 0 and $e \notin U$. Since $C_F(U) \subseteq U_{\alpha}$, we have

$$\frac{1}{1+\delta} \le \frac{|\bigcap_{x \in F} x^{-1} U_{\alpha} x|}{|U_{\alpha}|} = \frac{|U|}{|U_{\alpha}|} \le \frac{|U|}{|C_F(U)|} \le 1.$$

Consequently

$$1 \le \frac{|C_F(U)|}{|U|} \le 1 + \delta.$$

It is easy to see that

$$\frac{|y_{ik}Uy_{ik}^{-1}\Delta U|}{|U|} \le 4\delta \quad \text{ for all } 1 \le k \le m, \ 1 \le i \le n_k$$

Let $\phi = \frac{\chi_U}{|U|}$. Thus $\|\phi\|_1 = 1$, $\phi \ge 0$ and $\phi(e) = 0$. For all $1 \le k \le m$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n_k} \alpha_{ik \ y_{ik}} \phi_{y_{ik}} - \sum_{i=1}^{n_k} \alpha_{ik} \phi \right\|_1 &= \sum_{x \in G} \left| \sum_{i=1}^{n_k} \alpha_{ik} (y_{ik} \phi_{y_{ik}}(x) - \phi(x)) \right| \\ &\leq \sum_{i=1}^{n_k} |\alpha_{ik}| \sum_{x \in G} \frac{|\chi_U(y_{ik}^{-1} x y_{ik}) - \chi_U(x)|}{|U|} \\ &= \sum_{i=1}^{n_k} |\alpha_{ik}| \frac{|y_{ik} U y_{ik}^{-1} \Delta U|}{|U|} < 4\delta \sum_{i=1}^{n_k} |\alpha_{ik}| \le \epsilon \end{aligned}$$

Consequently

$$\left\|\sum_{i=1}^{n_k} \alpha_{ik} y_{ik} \phi_{y_{ik}}\right\|_1 \le \left|\sum_{i=1}^{n_k} \alpha_{ik}\right| + \epsilon.$$

(v) \Rightarrow (i). Let a finite subset $F = \{y_1, ..., y_n\}$ of G and $\epsilon > 0$ be given. For any i = 1, ..., n, consider $\delta_{y_i} - \delta_e \in l^1(G)$. By the assumption, there exists a simple function ϕ with $\phi \ge 0$, $\phi(e) = 0$ such that

$$\|y_i \phi_{y_i} - \phi\|_1 < \epsilon \quad \text{for any} \quad 1 \le i \le n.$$

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Hence G is inner amenable by ([6], Theorem 1). \Box

We denote by Ω the family of all subsets of G and by \mathcal{F} the family of finite subsets of G and $\mathcal{F}_{\circ} = \{\mathcal{U} \in \mathcal{F}; e \notin U\}.$

Definition 2.2. For an infinite discrete group G, we define

$$\rho = \inf\{\sup\{\frac{|\bigcap_{x\in F} x^{-1}Ux|}{|U|} : U \in \mathcal{F}_{\circ}\}: \mathcal{F} \in \mathcal{F}\}$$
$$\eta = \sup\{\inf\{\frac{|C_F(U)|}{|U|}: U \in \mathcal{F}_{\circ}\}: \mathcal{F} \in \mathcal{F}\}.$$

We have $0 \le \rho \le 1$; $1 \le \eta \le \infty$.

Corollary 2.3. For an infinite discrete group G, the following properties are equivalent:

(i) G is an inner amenable group.
(ii) ρ = 1.
(iii) η = 1.

Proof. The statement is an immediate consequence of Theorem 2.1. \Box

It is easy to see that if G is not inner amenable, then $\rho = 0$. Indeed, if G is not inner amenable, by Theorem 2.1 there is a finite subset F of G and a $\epsilon \in (0, 1)$ such that

$$\frac{|\bigcap_{x\in F} x^{-1}Ux|}{|U|} < 1 - \epsilon$$

for all $U \in \mathcal{F}_{\circ}$. Now choose $U \in \mathcal{F}_{\circ}$ such that $|\bigcap_{x \in F} x^{-1}Ux| > 0$. We have

$$\frac{\left|\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}Ux)x\right|}{|U|} = \frac{\left|\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}Ux)x\right|}{|\bigcap_{x\in F} x^{-1}Ux|} \frac{\left|\bigcap_{x\in F} x^{-1}Ux\right|}{|U|} < (1-\epsilon)^2.$$

Assume, without loss of generality, that $|\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}Ux)x| > 0$ for some $U \in \mathcal{F}_{\circ}$. For every $U \in \mathcal{F}_{\circ}$ such that $|\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}Ux)x| > 0$, we have

$$\frac{|\bigcap_{x \in F} x^{-1}(\bigcap_{x \in F} x^{-1}(\bigcap_{x \in F} x^{-1}Ux)x)x|}{|U|}$$

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$$= \frac{\left|\bigcap_{x \in F} x^{-1} (\bigcap_{x \in F} x^{-1} (\bigcap_{x \in F} x^{-1} Ux) x) x\right|}{\left|\bigcap_{x \in F} x^{-1} (\bigcap_{x \in F} x^{-1} Ux) x\right|}$$
$$= \frac{\left|\bigcap_{x \in F} x^{-1} (\bigcap_{x \in F} x^{-1} Ux) x\right|}{\left|\bigcap_{x \in F} x^{-1} Ux\right|} \frac{\left|\bigcap_{x \in F} x^{-1} Ux\right|}{\left|U\right|} \le (1 - \epsilon)^{3}.$$

We conclude that, for every $U \in \mathcal{F}_{\circ}$ such that $|\bigcap_{x \in F} x^{-1}Ux| > 0$,

$$\frac{\left|\bigcap_{x\in F^2} x^{-1}Ux\right|}{|U|} = \frac{\left|\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}Ux)x\right|}{|U|} < (1-\epsilon)^2.$$

Moreover, for every $U \in \mathcal{F}_{\circ}$ such that $|\bigcap_{x \in F} x^{-1}(\bigcap_{x \in F} x^{-1}Ux)x| > 0$, we have

$$\frac{|\bigcap_{x\in F^3} x^{-1}Ux|}{|U|} = \frac{|\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}(\bigcap_{x\in F} x^{-1}Ux)x)x|}{|U|} \le (1-\epsilon)^3.$$

We construct the sequence $\frac{|\bigcap_{x \in F^n} x^{-1}Ux|}{|U|}$ inductively. As $\lim_{n \to \infty} (1-\epsilon)^n = 0$, we conclude that $\rho = 0$.

It is easy to see that if G is not inner amenable, then $\eta = \infty$. The following theorem is an analogue of Paterson ([9], p.89).

Theorem 2.4. For an infinite discrete group G, the following properties are equivalent:

- (i) G is an inner amenable group.
- (ii) $\sup \sum_{i=1}^{n} \chi_{A_i} a_i \chi_{A_i a_i} \ge 0$, whenever $A_1, ..., A_n \in \Omega$, $a_1, ..., a_n \in G$, $n \in \mathbb{N}$.

Proof. Let *m* be an inner invariant mean on $l^{\infty}(G)$. If $A_1, ..., A_n \in \Omega$ and $a_1, ..., a_n \in G$, then

$$0 = m(\sum_{i=1}^{n} \chi_{A_i} - a_i \chi_{A_i a_i}) \le \sup \sum_{i=1}^{n} \chi_{A_i} - a_i \chi_{A_i a_i}.$$

Thus, the property (ii) holds.

To prove the converse, assume that the property holds. If G is not inner amenable, by ([1], Proposition 2.8) there exist $f_1, ..., f_n \in l^{\infty}(G)$ and $a_1, ..., a_n \in G$ such that

$$\alpha = \sup \sum_{i=1}^{n} f_i - a_i f_{ia_i} < 0.$$

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Let $\beta = -\alpha^{-1}(2n+1)$. For i = 1, ..., n we define $h_i(x) = [\beta f_i(x)] \in \mathbb{Z}$, $x \in G$, [] denoting the integral part. We have

$$\sup \sum_{i=1}^{n} h_i - a_i h_{ia_i} \le \beta \sup \sum_{i=1}^{n} f_i - a_i f_{ia_i} + 2n = \beta \alpha + 2n = -1.$$
(2.1)

As all h_i (i = 1, ..., n) take their values in \mathbb{Z} , there exist $k_1, ..., k_p \in l^{\infty}(G)$ [resp. $l_1, ..., l_q \in l^{\infty}(G)$] taking their values in $\{0, 1\}$ [resp. $\{0, -1\}$] and $b_1, ..., b_p, c_1, ..., c_q \in \{a_1, ..., a_n\}$ such that

$$\sum_{j=1}^{p} k_j - b_j k_{jb_j} + \sum_{j=1}^{q} l_j - c_j l_{jc_j} = \sum_{i=1}^{n} h_i - a_i h_{ia_i}.$$

Let $B_j = k_j^{-1}(\{1\})$ for j = 1, ..., p and $C_j = l_j^{-1}(\{-1\})$ for j = 1, ..., q. Then

$$\sum_{j=1}^{p} \chi_{B_j} - {}_{b_j} \chi_{B_j} - \sum_{j=1}^{q} \chi_{C_j} - {}_{c_j} \chi_{C_j} = \sum_{i=1}^{n} h_i - {}_{a_i} h_{ia_i}.$$
(2.2)

On the other hand,

$$-\sum_{j=1}^{q} \chi_{C_j} - c_j \chi_{C_j} = \sum_{j=1}^{q} \chi_{c_j C_j c_j^{-1}} - c_j^{-1} (\chi_{c_j C_j c_j^{-1}})_{c_j^{-1}}$$
(2.3)

So (2.1), (2.2) and (2.3) imply that,

$$\sum_{j=1}^{p} \chi_{B_j} - {}_{b_j} \chi_{B_j} + \sum_{j=1}^{q} \chi_{c_j C_j c_j^{-1}} - {}_{c_j^{-1}} (\chi_{c_j C_j c_j^{-1}})_{c_j^{-1}} \le -1,$$

therefore the condition (ii) dose not hold.

Theorem 2.5. An infinite discrete group G is inner amenable if and only if for every $1 \le p < \infty$ and every finite subset F of G and $\epsilon > 0$, there exists a function $\phi \in l^p(G)$ with $\phi \ge 0$, $\|\phi\|_p = 1$, $\phi(e) = 0$ such that

$$\left|1 - \sum_{x \in G} \phi^{p-1}(x)\phi(y^{-1}xy)\right| < \epsilon$$

whenever $y \in F$.

Proof. Before we go on, we state an elementary fact: If $0 \le a \le b$ and $1 \le s < \infty$, we have $(b-a)^s \le b^s - a^s$.

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Assume that G is inner amenable. Let F be a finite subset of G and $\epsilon > 0$. By Theorem in [2], there exists a function $\psi \in l^1(G)$ with $\psi \ge 0$, $\|\psi\|_1 = 1$, $\psi(e) = 0$ such that $\|_y \psi_y - \psi\|_1 < \epsilon^{\frac{1}{p}}$ for all $y \in F$. Let $\phi = \psi^{\frac{1}{p}}$. For any $y \in F$,

$$\begin{aligned} \|y\phi_y - \phi\|_p &= \left(\sum_{x \in G} \left|\phi(y^{-1}xy) - \phi(x)\right|^p\right)^{\frac{1}{p}} \\ &\leq \left(\sum_{x \in G} \left|\phi^p(y^{-1}xy) - \phi^p(x)\right|\right)^{\frac{1}{p}} \\ &= \left(\sum_{x \in G} \left|\psi(y^{-1}xy) - \psi(x)\right|\right)^{\frac{1}{p}} \\ &\leq \|y\phi_y - \phi\|_1^p < \epsilon. \end{aligned}$$

On the other hand, $\phi^{p-1} \in l^q(G)$ (here $p^{-1} + q^{-1} = 1$) and $\sum_{x \in G} \phi^{p-1}(x)\phi(x) = 1$. For any $y \in F$, by Holder's inequality,

$$\begin{aligned} \left| 1 - \sum_{x \in G} \phi^{p-1}(x)\phi(y^{-1}xy) \right| &= \left| \sum_{x \in G} \phi^{p-1}(x)\phi(x) - \sum_{x \in G} \phi^{p-1}(x)\phi(y^{-1}xy) \right| \\ &= \left| \sum_{x \in G} \phi^{p-1}(x)(\phi(x) - \phi(y^{-1}xy)) \right| \\ &\leq \left\| \phi^{p-1} \right\|_q \left(\sum_{x \in G} \left| \phi(x) - \phi(y^{-1}xy) \right|^q \right)^{\frac{1}{q}} \\ &= \left\| \phi - y\phi_y \right\|_p < \epsilon. \end{aligned}$$

To prove the converse, consider the finite subset F in G and $\epsilon > 0$. By the assumption there exists a function $\phi \in l^2(G)$ with $\phi \ge 0$, $\|\phi\|_2 = 1$, $\phi(e) = 0$ such that

$$\left|1 - \sum_{x \in G} \phi^{p-1}(x)\phi(y^{-1}xy)\right| < \frac{\epsilon}{2}$$

whenever $y \in F$. For every $y \in F$, we have

$$\begin{split} \|\phi - {}_{y}\phi_{y}\|_{2}^{2} &= \sum_{x \in G} |\phi(x) - \phi(y^{-1}xy)|^{2} \\ &= \sum_{x \in G} \phi(x)^{2} + \sum_{x \in G} \phi^{2}(y^{-1}xy) - 2\sum_{x \in G} \phi(x)\phi(y^{-1}xy) \\ &= 2 - 2\sum_{x \in G} \phi(x)\phi(y^{-1}xy) < \epsilon. \end{split}$$

This shows that $\|\phi - {}_{y}\phi_{y}\|_{2} < \epsilon$ for all $y \in F$. Now let $\theta = \phi^{2} \in l^{1}(G)$; for any $y \in F$, by the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} \|\psi - {}_{y}\psi_{y}\| &= \sum_{x \in G} \left|\phi^{2}(x) - {}_{y}\phi^{2}_{y}(x)\right| \\ &= \sum_{x \in G} \left|\phi(x) - {}_{y}\phi_{y}(x)\right| \left|\phi(x) + {}_{y}\phi_{y}(x)\right| \\ &\leq \|\psi - {}_{y}\psi_{y}\|_{2} \left(\|\phi\|_{2} + \|_{y}\phi_{y}\|_{2}\right) < \epsilon. \end{aligned}$$

By Theorem in [2], G is inner amenable.

For every $\phi \in l^1(G)$ and $f \in l^p(G)$ the *-convolution $\phi \star f$

$$\phi \star f(x) = \sum_{x \in G} \phi(x) f(x^{-1}yx) \quad (x \in G)$$

exists and represents an element of $l^p(G)$ of norm $\|\phi \star f\|_p \leq \|\phi\|_1 \|f\|_p$ (see [1] for the details).

Theorem 2.6. Let G be an inner amenable group and $1 \le p < \infty$. Then for every $\phi \in l^1(G)$ with $\phi \ge 0$ and $\|\phi\|_1 = 1$,

$$\sup\{\|\phi \star f\|_p; \ f \in l^p(G), \ f \ge 0, \ \|f\|_p \le 1, \ f(e) = 0\} = 1.$$

Proof. Let G be an inner amenable group and $\phi \in l^1(G) \|\phi\|_1 = 1$, $\phi \geq 0$. Without loss of generality, we may assume that $F = \text{supp } \phi$ is finite. Let $\epsilon > 0$ be given. By ([13], Theorem 1) there exists a non-empty finite subset U in G such that $e \notin U$ and $\frac{|yUy^{-1}\Delta U|}{|U|} < \epsilon$ for all $y \in F$.

For $f = \frac{\chi_U}{|U|}$, we have

$$\begin{split} |\phi \star f - f||_p^p &= \left\| \phi \star \frac{\chi_U}{|U|} - \frac{\chi_U}{|U|} \right\|_p^p = \sum_{x \in G} \left| \phi \star \frac{\chi_U}{|U|}(x) - \frac{\chi_U}{|U|}(x) \right|^p \\ &= \sum_{x \in G} \left| \sum_{y \in F} \phi(y) \left(\frac{\chi_U(y^{-1}xy)}{|U|} - \frac{\chi_U(x)}{|U|} \right) \right|^p \\ &\leq \frac{1}{|U|^p} \left(\sum_{y \in F} \phi(y) \sum_{x \in G} \left| \chi_U(y^{-1}xy) - \chi_U(x) \right| \right)^p \\ &= \left(\sum_{y \in F} \phi(y) \frac{|yUy^{-1}\Delta U|}{|U|} \right)^p < \epsilon^p. \end{split}$$

It follows that $\|\phi \star f - f\|_p < \epsilon$, and so $\|f\|_p - \epsilon \le \|\phi \star f\|_p$. This shows that

$$\sup\{\|\phi \star f\|_p; \ f \in l^p(G), \ f \ge 0, \ \|f\|_p \le 1, \ f(e) = 0\} = 1.$$

This completes the proof.

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