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Author(s):

F. Mirzaei and R. Nekooei

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ON RADICAL FORMULA AND PRÜFER DOMAINS

F. MIRZAEI AND R. NEKOOEI*

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Abstract. In this paper we characterize the radical of an arbitrary submodule N of a finitely generated free module F over a commutatitve ring R with identity. Also we study submodules of F which satisfy the radical formula. Finally we derive necessary and sufficient conditions for R to be a Prüfer domain, in terms of the radical of a cyclic submodule in $R \bigoplus R$.
Keywords: Prime submodules, Radical of a submodule, Radical formula, Prüfer domains, Dedekind domains.
MSC(2010): Primary: 13A99; Secondary: 13C99, 13F05.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. A proper submodule P of an R-module M is called a p-prime submodule, if $rm \in P$ for $r \in R$ and $m \in M$ implies $m \in P$ or $r \in p = (P : M)$, where $(P : M) = \{r \in R \mid rM \subseteqq P\}$. Let I be an ideal of R. The radical, $\sqrt{I}$, is defined to be the intersection of all prime ideals of R containing I. We denote the radical of I by $\sqrt{I}$. Let X be a subset of an R-module M. We denote the submodule of M that X generates, by $< X >$ or RX. The prime radical, $Rad_M T$, of a submodule T in an R-module M is defined to be the intersection of all prime submodules of M containing T. If there is no prime submodule containing T, then $Rad_M T = M$. In particular $Rad_M M = M$. We use the notation $R^{(n)}$ for $\underbrace{R \oplus \cdots \oplus R}_{n-times}$ and $I^{(n)}$ for $\underbrace{I \oplus \cdots \oplus I}_{n-times}$, where I is an ideal of R.

Let M be an R-module and T be a submodule of M. The envelope of T in M is defined to be the set

$$E_M(T) = \{rm \mid r \in R, m \in M; r^n m \in T, \text{ for some } n \in \mathbf{Z}^+\}.$$

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*Corresponding author.

We say that the submodule T of an R-module M satisfies the radical formula in M (T s.t.r.f. in M) if $Rad_M T = \langle E_M(T)\rangle$. An R-module M s.t.r.f. if for every submodule T of M, the prime radical of T is the submodule generated by its envelope, i.e. $Rad_M T = \langle E_M(T)\rangle$. A ring R s.t.r.f. provided that for every R module M, M s.t.r.f. The question of what kind of rings and modules s.t.r.f. has studied by many authors, see [1, 3, 6, 7, 10].

In [1], Azizi has shown that every arithmetical ring with $dimR \leq 1$ satisfies the radical formula. In [9], Parkash proved that every arithmetical ring satisfies the radical formula and Buyruk and Pusat Yilmaz in [2], proved that if R is a Prüfer domain, then the free R-module $R^{(2)}$ satisfies the radical formula.

In [11] Pusat-Yilmaz and Smith have described $Rad_F(T)$, where T is a finitely generated submodule of a free R-module $F = R^{(n)}$. In this paper we generalize this characterization for an arbitrary submodule N of F and we characterize some submodules of F satisfying the radical formula. Finally we apply this characterization on the radical of a cyclic submodule of $R^{(2)}$ to give necessary and sufficient conditions for an integral domain R to be a Prüfer domain.

2. **Radical of a submodule and radical formula**

Let $X_i = (x_{i1}, \dots, x_{in}) \in F = R^{(n)}$, for some $x_{ij} \in R$, $1 \leq i \leq m$, $1 \leq j \leq n$, $m \leq n$. We put

$$B_{m\times n} = [X_1 \dots X_m] = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \ddots & & & \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} \in M_{m\times n}(R).$$

Thus the jth row of the matrix $[X_1 \dots X_m]$ consists of the components of element X_j in F. We use $B(j_1, \dots, j_k) \in M_{m\times k}(R)$ to denote the submatrix of B consisting of the columns $j_1, \dots, j_k \in \{1, \dots, n\}$ and

$$[X_1 \dots X_m]_m = \sum_{j_1,\dots,j_m \in \{1,\dots,n\}} RdetB(j_1, \dots, j_m)$$

the ideal generated by $\{\det B(j_1, \dots, j_m) \mid j_1, \dots, j_m \in \{1, \dots, n\}\}$. We use N to be a non-zero submodule of F generated by the set $\Psi = \{X_i = (x_{i1}, \dots, x_{in}) \in F \mid i \in \Omega\}$. We put $\Re_t = \sum_{i_1,\dots,i_t \in \Omega} R[X_{i_1} \dots X_{i_t}]_t$, $1 \leq t \leq n$. Note that $\Re_1 \supseteq \Re_2 \supseteq \cdots \supseteq \Re_n = \Re$.

We first state two useful results.

Lemma 2.1. *Let F be the free R-module $R^{(n)}$. Then $\Re \subseteq (N : F) \subseteq \sqrt{\Re}$.*

Proof. [8], Lemma 1.1. □

The following lemma is proved in [8], Lemma 1.5. But we give the proof of part (ii) of this lemma, because we use this proof in Proposition 2.5.

Lemma 2.2. *Let F be the free R-module $R^{(n)}$, p be a prime ideal of R and $B = [X_1 \dots X_k] \in M_{k\times n}(R)$ for some $X_i \in F$, $1 \le i \le k$ and positive integer $k < n$. Put*
$T_p(B) = \{X = (x_1, \dots, x_n) \in F \mid det\beta(i_1, \dots, i_{k+1}) \in p, for\ every\ i_1, \dots, i_{k+1} \in \{1, \dots, n\}\}$, where $\beta = [X\ X_1 \dots X_k] \in M_{k+1\times n}(R)$. Then
i) $T_p(B)$ is a submodule of F.
ii) If $X = (x_1, \dots, x_n) \in T_p(B)$, then $det(B(i_1, \dots, i_k))X \in pF + \langle B\rangle$ for all submatrices $B(i_1, \dots, i_k)$ of B, where $\langle B\rangle$ is the R-submodule of F generated by the rows of B. (Note that in this part, the ideal p is not necessarily prime.)
iii) If the determinant of every submatrix $k \times k$ of B is in p, then $T_p(B) = F$.
iv) If there exists a submatrix $B(j_1, \dots, j_k) \in M_{k\times k}(R)$ of B such that $det(B(j_1, \dots, j_k)) \notin p$, then $T_p(B)$ is a p-prime submodule of F.

Proof. ii) Let $X = (x_1, \dots, x_n) \in T_p(B)$ and $B(j_1, \dots, j_k) \in M_{k\times k}(R)$ be a submatrix of B. Without loss of generality, assume that $j_1 < j_2 < \dots < j_k$. Since $det\beta(i_1, \dots, i_{k+1}) \in p$ for every $i_1, \dots, i_{k+1} \in \{1, \dots, n\}$, there exists $p_t \in p$ such that $x_t detB(j_1, \dots, j_k) = p_t + \sum_{i=1}^{k} (-1)^{i+1} x_{j_i} detB(t, j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k)$ for every $1 \le t \le n$, $t \ne j_i$, $1 \le i \le k$. It follows that $det(B(j_1, \dots, j_k))(x_1, \dots, x_n) = X_p + \sum_{i=1}^{k} Y_i$, for some $X_p \in p^{(n)}$ and $Y_i = (y_{i1}, \dots, y_{in}) \in F$, $1 \le i \le k$. We fix $1 \le i \le k$. Then $y_{it} = (-1)^{i+1} x_{j_i} detB(t, j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k)$, $1 \le t \le n$, $t \ne j_1, \dots, j_k$ and $y_{ij_i} = x_{j_i} detB(j_1, \dots, j_k)$ and $y_{ij_s} = 0$, $1 \le s \le k$, $s \ne i$. Therefore $y_{it} = \sum_{m=1}^{k} (-1)^{m+i} x_{mt} x_{j_i} det[B(t, j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k)]_{m1}$, $1 \le t \le n$, $t \ne j_1, \dots, j_k$ and $y_{ij_i} = \sum_{m=1}^{k} (-1)^{m+i} x_{mj_i} x_{j_i} det[B(j_1, \dots, j_k)]_{mi}$. Also $y_{ij_s} = \sum_{m=1}^{k} (-1)^{m+i} x_{mj_s} x_{j_i} det[B(j_1, \dots, j_{i-1}, j_s, j_{i+1}, \dots, j_k)]_{mi} = 0$, $1 \le s \le k$, $s \ne i$. So $Y_i = \sum_{m=1}^{k} x_{j_i} (-1)^{m+i} det[B(j_1, \dots, j_k)]_{mi} X_m$ and hence $Y_i \in \langle B\rangle$, $1 \le i \le k$. Thus $detB(j_1, \dots, j_k)(x_1, \dots, x_n) \in pF + \langle B\rangle$. □

Let M be an R-module, p be a prime ideal of R and T be a submodule of M. In [11] Pusat-Yilmaz and Smith defined the submodule $K(T, p) = \{m \in M \mid cm \in T + pM, \text{ for } c \in R\backslash p\}$. They showed that this is the smallest p-prime submodule of M containing T and so $Rad_M T = \cap\{K(T, p) : p \text{ is a prime ideal of } R\ \}$.

Lemma 2.3. *Let F be the free R-module $R^{(n)}$ and p be a prime ideal of R. Then*
i) If $(N : F) \nsubseteq p$, then $K(N, p) = F$.
ii) If $\Re_1 \subseteq p$, then $K(N, p) = p^{(n)}$.
iii) If $\Re_1 \nsubseteq p$, then there exists a positive integer $k < n$ and a matrix $B_{k\times n} = [X_1 \dots X_k] \in M_{k\times n}(R)$, $X_i \in \Psi$, $1 \le i \le k$ such that $K(N, p) = T_p(B)$, where $T_p(B)$ is the p-prime submodule in Lemma 2.2.

Proof. i) Let p be a prime ideal of R. Assume $(N : F)$ is not contained in p and $c \in (N : F)\backslash p$. Then $cF \subseteq N$ and so $F \subseteq K(N, p)$.

ii) Let $\Re_1 \subseteq p$. Then pF contains N and since pF is a p-prime submodule of F, we get that $K(N, p) = p^{(n)}$.

iii) Let $\Re_1$ is not contained in p. Suppose that ξ is the set of all positive integers m such that there exists a matrix $B_{m\times n} = [X_1 \dots X_m] \in M_{m\times n}(R)$, for some $X_i \in \Psi$ $(1 \le i \le m)$ and a submatrix $B(j_1, \dots, j_m)$ such that $detB(j_1, \dots, j_m) \notin p$, for some $j_1, \dots, j_m \in \{1, \dots, n\}$. Since $\Psi \not\subset p^{(n)}$, hence $1 \in \xi \ne \emptyset$. Let $k = \max(\xi)$, by Lemma 2.1, we have $k < n$.
Let $B_{k\times n} = [X_1 \dots X_k] \in M_{k\times n}(R)$ such that $detB(j_1, \dots, j_k) \notin p$, for some $j_1, \dots, j_k \in \{1, \dots, n\}$. Then by Lemma 2.2(iv), we have $T_p(B)$ is a p-prime submodule of F. It is clear that $N \subseteq T_p(B)$ and by Lemma 2.2(ii), $T_p(B) \subseteq K(N, p)$. □

The Theorem 2.4, is a generalization of Theorem 1.5 in [11].

Theorem 2.4. *Let F be the free R-module $R^{(n)}$ and $N = \langle\Psi\rangle$. Then $Rad_F N = \{X = (x_1, \dots, x_n) \in \sqrt{\Re_1}F \mid [X\ X_{i_1} \dots X_{i_{k-1}}]_k \subseteq \sqrt{\Re_k}$, for every $i_1, \dots, i_{k-1} \in \Omega$, $2 \le k \le n\}$, where $\Re_k = \sum_{i_1,\dots,i_k\in\Omega} R[X_{i_1} \dots X_{i_k}]_k$ and $[X\ X_{i_1} \dots X_{i_{k-1}}]_k = \sum_{j_1,\dots,j_k\in\{1,\dots,n\}} RdetB(j_1, \dots, j_k)$ with*
$B = [X\ X_{i_1} \dots X_{i_{k-1}}]$.

Proof. Let ξ be the set of prime ideals of R containing $(N : F)$. Then by Lemma 2.3 (ii), $\sqrt{\Re_1}F = \bigcap_{\Re_1\subset p\in\xi} K(N, p)$ and so we get $Rad_F N = \bigcap_{p\in\xi} K(N, p) = \sqrt{\Re_1}F \cap [\bigcap_{\Re_1\not\subset p\in\xi} K(N, p)]$.
Let $\Delta = \{X = (x_1, \dots, x_n) \in \sqrt{\Re_1}F \mid [X\ X_{i_1} \dots X_{i_{k-1}}]_k \subseteq \sqrt{\Re_k}$, for every $i_1, \dots, i_{k-1} \in \Omega, 2 \le k \le n\}$. We show that $Rad_F N = \Delta$. Suppose that $X = (x_1, \dots, x_n) \in Rad_F N$ where $x_i \in R$, $1 \le i \le n$. Then $X \in \sqrt{\Re_1}F \cap [\bigcap_{\Re_1\not\subset p\in\xi} K(N, p)]$. Let p be any prime ideal of R containing $\Re_k$ $(2 \le k \le n)$. If

$\Re_{k-1} \subseteq p$, then $[X\ X_{i_1} \dots X_{i_{k-1}}]_k \subseteq p$, for all $i_1, \dots, i_{k-1} \in \Omega$. If $\Re_{k-1} \not\subseteq p$, then $\Re_1 \not\subseteq p$ and so by Lemma 2.3 (iii), there exists a matrix $B_{k-1\times n} = [X_1 \dots X_{k-1}] \in M_{k-1\times n}(R)$, for some $X_i \in \Psi(1 \leq i \leq k-1)$ with a submatrix $B(i_1, \dots, i_{k-1})$ such that $detB(i_1, \dots, i_{k-1}) \notin p$ and $K(N,p) = T_p(B)$. By [8], Proposition 1.7, $K(N,p) = \{Y = (y_1, \dots, y_n) \in F \mid [Y\ X_{i_1} \dots X_{i_{k-1}}]_k \subseteq P$ for every $i_1, \dots, i_{k-1} \in \Omega\}$. Since $X \in K(N,p)$, then $[X\ X_{i_1} \dots X_{i_{k-1}}]_k \subseteq p$, for every $i_1, \dots, i_{k-1} \in \Omega$. It follows that $[X\ X_{i_1} \dots X_{i_{k-1}}]_k \subseteq \sqrt{\Re_k}$, for every $i_1, \dots, i_{k-1} \in \Omega$ and hence $X \in \Delta$. So $Rad_F N \subseteq \Delta$. Now let $X = (x_1, \dots, x_n) \in \Delta$ and p be any prime ideal in ξ such that $\Re_1 \not\subseteq p$. Then by Lemma 2.3 (iii), there exists a positive integer $m < n$ and a matrix $B_{m\times n} = [X_1 \dots X_m] \in M_{m\times n}(R)$, for some $X_i \in \Psi(1 \leq i \leq m)$ with a submatrix $B(j_1, \dots, j_m)$ such that $detB(j_1, \dots, j_m) \notin p$, for some $j_1, \dots, j_m \in \{1, \dots, n\}$ and $K(N,p) = T_p(B)$. It is clear that $X \in K(N,p)$ and so $X \in Rad_F N$. Thus $\Delta = Rad_F N$. □

Proposition 2.5. *Let $F = R^{(n)}$ be a free R-module and $N = \langle\Psi\rangle$. If there exist $1 \leq j \leq n-1$ and $B = [X_1 \dots X_j] \in M_{j\times n}(R)$, for some $X_1, \dots, X_j \in \Psi$ such that B contains an $j \times j$ submatrix whose determinant is a unit in R and $\sqrt{\Re_{j+1}} = \sqrt{(N:F)}$, then N s.t.r.f in F.*

Proof. Suppose there exists a matrix $B = [X_1 \dots X_j] \in M_{j\times n}(R)$, for some $X_1, \dots, X_j \in \Psi$ with a submatrix $B(i_1, \dots, i_j) \in M_{j\times j}(R)$, for some $i_1, \dots, i_j \in \{1, \dots, n\}$ such that $detB(i_1, \dots, i_j)$ is unit. Let $X \in Rad_F N$. Then $[XX_1 \dots X_j]_{j+1} \subseteq \sqrt{\Re_{j+1}} = \sqrt{(N:F)}$. If we replace the ideal p in Lemma 2.2(ii) with $\sqrt{(N:F)}$, then $detB(i_1, \dots, i_j)X \in \sqrt{(N:F)}F + N$. It follows that $X \in \sqrt{(N:F)}F + N$ and hence $Rad_F N = \sqrt{(N:F)}F + N = \langle E_F(N)\rangle$. □

Corollary 2.6. *Let (R, m) be a local ring with m as maximal ideal. Let F be the free R-module $R^{(n)}$ and $N = \langle\Psi\rangle$. If $\Re_j = R$ and $\sqrt{\Re_{j+1}} = \sqrt{(N:F)}$, for some $1 \leq j \leq n-1$, then N s.t.r.f in F.*

Proof. Let $\Re_j = \sum_{i_1, \dots, i_j \in \Omega} R[X_{i_1} \dots X_{i_j}]_j = R$, for some $1 \leq j \leq n-1$ and $\sqrt{\Re_{j+1}} = \sqrt{(N:F)}$. Since R is a local ring, then there exists a matrix $B = [X_1 \dots X_j] \in M_{j\times n}(R)$, for some $X_1, \dots, X_j \in \Psi$ with a submatrix $B(i_1, \dots, i_j) \in M_{j\times j}(R)$, for some $i_1, \dots, i_j \in \{1, \dots, n\}$ such that $detB(i_1, \dots, i_j)$ is unit. Then by Proposition 2.5, N s.t.r.f in F. □

Proposition 2.7. *Let R be a commutative ring with identity. Let F be the free R-module $R^{(n)}$ and $N = \langle\Psi\rangle$. If $\sqrt{\Re_1} = \sqrt{\Re_2} = \cdots = \sqrt{\Re_{n-1}} = \sqrt{(N:F)}$, then $Rad_F N = \sqrt{(N:F)}F = \langle E_F(N)\rangle$.*

Proof. Let N be a submodule of F such that $\sqrt{\Re_1} = \sqrt{\Re_2} = \cdots = \sqrt{\Re_{n-1}} = \sqrt{(N:F)}$. Then by Theorem 2.4, $Rad_F N = \{X = (x_1, \dots, x_n) \in \sqrt{(N:F)}F |$

$[X\ X_{i_1}\dots X_{i_{k-1}}]_k \subseteq \sqrt{(N:F)}$, for every $i_1,\dots,i_{k-1}\in\Omega$ and $2\le k\le n\}$. Since $X_i\in\sqrt{(N:F)}F$, for every $X_i\in\Psi$, we get that $Rad_F N=\sqrt{(N:F)}F=\langle E_F(N)\rangle$. □

Theorem 2.8 is a generalization of Theorem 1.9 in [11].

Theorem 2.8. *Let $F=R^{(n)}$ be a free R-module and $N=\langle\Psi\rangle$, where $\Psi=\{X_i=(x_{i1},\dots,x_{in})\in F\mid i\in\Omega\}$.*
Let I be an ideal of R and $T=N+IF$. Then $Rad_F T=\{X=(x_1,\dots,x_n)\in\sqrt{\Re_1+I}F\mid [X\ X_{i_1}\dots X_{i_{k-1}}]_k\subseteq\sqrt{\Re_k+I}$, for every $i_1,\dots,i_{k-1}\in\Omega$, $2\le k\le n\}$, where $\Re_k=\sum_{i_1,\dots,i_k\in\Omega}R[X_{i_1}\dots X_{i_k}]_k$, $1\le k\le n$.

Proof. Let $\Psi'=\{Y_i=(y_{i1},\dots,y_{in})\in IF\mid i\in\Omega'\}$ be a subset of IF such that $IF=\langle\Psi'\rangle$. Then $T=\langle\Psi\cup\Psi'\rangle$ and so by Theorem 2.4, $Rad_F T=\{X=(x_1,\dots,x_n)\in\sqrt{\Re'_1}F\mid [X\ Z_{i_1}\dots Z_{i_{k-1}}]_k\subseteq\sqrt{\Re'_k},\ Z_{i_1},\dots,Z_{i_{k-1}}\in\Psi\cup\Psi'$, for every $i_1,\dots,i_{k-1}\in\Omega\cup\Omega'$, $2\le k\le n\}$, where $\Re'_k=\sum_{i_1,\dots,i_k\in\Omega\cup\Omega'}R[Z_{i_1}\dots Z_{i_k}]_k$, $1\le k\le n$. But it is easy to see that $\sqrt{\Re'_i}=\sqrt{\Re_i+I}$, $1\le i\le n$. Also if $X\in F$ then $[X\ Z_{i_1}\dots Z_{i_{k-1}}]_k\subseteq\sqrt{\Re_k+I}$, for every $i_1,\dots,i_{k-1}\in\Omega\cup\Omega'$ if and only if $[X\ X_{i_1}\dots X_{i_{k-1}}]_k\subseteq\sqrt{\Re_k+I}$, for every $i_1,\dots,i_{k-1}\in\Omega$. □

3. Prüfer domains

There are many equivalent conditions for an integral domain R to be a Prüfer domain [5], Theorem 24.3. In what follows we give another equivalent condition in terms of radical of a cyclic submodules of $R^{(2)}$.

Let R be an integral domain and K its field of fractions. R is said to be integrally closed if for every $a\in K$, $f(a)=0$ for some monic polynomial $f\in R[x]$, then $a\in R$. Furthermore, R is integrally closed if and only if $(I:_K I)=R$, for every finitely generated ideal I of R [4], Theorem 3.7.I, where $(I:_K I)=\{x\in K\mid xI\subseteq I\}$.

In Theorem 3.1 we give necessary and sufficient condition for an integral domain to be integrally closed, by radical of a cyclic submodules in $R^{(n)}$.

Theorem 3.1. *Let R be an integral domain with quotient field K and let F be the free R-module $R^{(n)}$. Then R is integrally closed if and only if $Rad_F(R(a_1,\dots,$
$a_n))$
$\cap\,(I_n)^{(n)}=R(a_1,\dots,a_n)$, for every $(a_1,\dots,a_n)\in F$ and $n\ge 1$, where $I_n=\langle a_1,\dots,a_n\rangle$ is a finitely generated ideal of R.*

Proof. Let R be an integrally closed domain. If $n=1$, then the proof is clear. Let $n\ge 2$ and $(x_1,\dots,x_n)\in Rad_F(R(a_1,\dots,a_n))\cap(I_n)^{(n)}$, for some $(x_1,\dots,x_n),(a_1,\dots,a_n)\in F$. We can assume that there exists $1\le t\le n$, such that $a_t\ne 0$. Since $R(a_1,\dots,a_n)$ is a cyclic submodule of F and $n\geqslant 2$

then by [8], Proposition 1.2, $(R(a_1,\dots,a_n):F)=\langle 0\rangle$. Since $(x_1,\dots,x_n)\in Rad_F(R(a_1,\dots,a_n))$ and $(R(a_1,\dots,a_n):F)=\langle 0\rangle$, by Theorem 2.4, $x_ia_t=a_ix_t$ for all i; $1\le i\le n$, $i\ne t$, and hence $a_t(x_1,\dots,x_n)=x_t(a_1,\dots,a_n)$. It follows that $\frac{x_t}{a_t}\in(I_n:_K I_n)$. Since R is integrally closed then $x_t=ra_t$, for some $r\in R$ and hence $(x_1,\dots,x_n)=r(a_1,\dots,a_n)\in R(a_1,\dots,a_n)$. Conversely, let $I_n=\langle a_1,\dots,a_n\rangle$ $(n\ge 1)$ be a finitely generated ideal of R and $\frac{f}{s}\in(I_n:_K I_n)$ for some $0\ne s$, $f\in R$. Then there exist $x_i\in I_n$, $1\le i\le n$, such that $fa_i=sx_i$. By Theorem 2.4, $(x_1,\dots,x_n)\in Rad_{R^{(n)}}(R(a_1,\dots,a_n))\cap(I_n)^{(n)}$. Then $(x_1,\dots,x_n)=r(a_1,\dots,a_n)$, for some $r\in R$. Since $s(x_1,\dots,x_n)=f(a_1,\dots,a_n)$, $f=rs$ and so $\frac{f}{s}\in R$. □

Theorem 3.2. *Let R be an integral domain. Then R is a Prüfer domain if and only if for all $a,b\in R$, $(aR+bR)^2=a^2R+b^2R$ and $I^{(2)}\cap Rad_F(R(a,b))=R(a,b)$, where $I=\langle a,b\rangle$.*

Proof. Let R be a Prüfer domain. Then R is integrally closed and by [5], Theorem 24.3, $(aR+bR)^2=a^2R+b^2R$ for all $a,b\in R$. Hence by Theorem 3.1, $I^{(2)}\cap Rad_{R^{(2)}}(R(a,b))=R(a,b)$ and $(aR+bR)^2=a^2R+b^2R$ for all $a,b\in R$. Conversely, let m be a maximal ideal of R. It is enough to show that R_m is a valuation ring. We assume $\frac{a}{s_1},\frac{b}{s_2}\in R_m$, for some $a,b\in R$, $s_1,s_2\in R-m$. If $a\notin m$ or $b\notin m$ then $bR_m\subseteq aR_m$ or $aR_m\subseteq bR_m$. Now let a,b be non-zero element of m. Since $(aR+bR)^2=a^2R+b^2R$, hence $ab=ra^2+sb^2$ for some $r,s\in R$ and so $a(b-ra)=sb^2$. Therefore by Theorem 2.4, $(sb,b-ra)\in Rad_{R^{(2)}}(R(a,b))\cap I^{(2)}$. It follows that $(sb,b-ra)=t(a,b)$ for some $t\in R$. Then $sb=ta$ and $(1-t)b=ra$ and so we have $aR_m\subseteq bR_m$ or $bR_m\subseteq aR_m$. □

A Noetherian valuation domain is called a discrete rank one valuation. Furthermore, a domain R is said to be almost Dedekind provided that, for each maximal ideal m of R, the localization R_m is a discrete rank one valuation [4], page 119. It is clear that every almost Dedekind domain is a Prüfer domain. In [4], Theorem 7.1, Chapter III, it is proved that a domain R which is not a field, is an almost Dedekind domain if and only if R is a Prüfer domain of Krull dimension one and $\{0\}$ is the only idempotent prime ideal of R. In Theorem 3.4, we give a necessary and sufficient condition for a one dimensional domain R with $\{0\}$ as only idempotent prime ideal to be an almost Dedekind domain.

Lemma 3.3. *Let R be a one dimensional local domain with maximal ideal m such that $\bigcap_{n=1}^{\infty} m^n=0$. Then R is a valuation ring if and only if $Rad_{R^{(2)}}(R(a,b))=E_{R^{(2)}}(R(a,b))$ and $(aR+bR)^2=a^2R+b^2R$, for all $a,b\in R$.*

Proof. Let R be a valuation ring. It is clear that $(aR+bR)^2 = a^2R+b^2R$, for all $a, b \in R$. Now let (a, b) be a non-zero element of $R^{(2)}$ and $(0,0) \neq (c, d) \in Rad_{R^{(2)}}(R(a, b))$. We assume that $c = rd$ and $a = sb$, for some $r, s \in R$. Then we have $(c, d) = d(r, 1)$ and $(a, b) = b(s, 1)$. It follows by Theorem 2.4, that $db(r-s) = 0$ and $d^k = tb$, for some $k \in \mathbf{N}$ and $t \in R$. Therefore $r = s$ and we have $(c, d) = d(r, 1) = d(s, 1)$, $d^k(r, 1) = d^k(s, 1) = tb(s, 1) = t(a, b)$. Hence $(c, d) \in E_{R^{(2)}}(R(a, b))$. Now let $a = sb$ and $d = rc$ for some $r, s \in R$. Then we have $(c, d) = c(1, r)$ and $(a, b) = b(s, 1)$. Now by Theorem 2.4, we have $bc(sr-1) = 0$ and $c^k = tb$, for some natural number k and $t \in R$. Therefore $sr = 1$ and we have $(c, d) = c(1, r)$ and $c^k(1, r) = c^k r(s, 1) = trb(s, 1) = tr(a, b)$. Hence $(c, d) \in E_{R^{(2)}}(R(a, b))$. Conversely let a, b be non-zero elements of R. It is enough to show that $a \in Rb$ or $b \in Ra$. Since $(aR+bR)^2 = a^2R+b^2R$, hence $ab = ra^2 + sb^2$, for some $r, s \in R$ and so $a(b-ra) = sb^2$. By Theorem 2.4, we have $(sb, b-ra) \in Rad_{R^{(2)}}(R(a, b))$. Now we assume that $a \notin Rb$, $b \notin Ra$ and we show that $sb, b-ra \in \bigcap_{n=1}^{\infty} m^n$. Hence $sb = 0$, $b - ra = 0$. Therefore $b = ra$, which is a contradiction. Since $(sb, b-ra) \in Rad_{R^{(2)}}(R(a, b)) = E_{R^{(2)}}(R(a, b))$, then sb, $b-ra \in m$ and $(sb, b-ra) = r_0(x_0, y_0)$, for some $0 \neq r_0$, $x_0, y_0 \in R$ such that $r_0^{n_0}(x_0, y_0) = t_0(a, b)$, for some $n_0 \in N$ and $t_0 \in R$. If r_0 is unit in R, then $(x_0, y_0) = \ell(a, b)$, for some $\ell \in R$ and so $(sb, b-ra) = r_0\ell(a, b)$. It follows that $sb = r_0\ell a$ and $b(1-r_0\ell) = ra$. Since R is a local ring, $r_0\ell$ or $1 - r_0\ell$ is unit and so $a \in Rb$ or $b \in Ra$, which is a contradiction. Therefore $r_0 \in m$. If x_0 or y_0 is unit, because $r_0^{n_0}ay_0 = r_0^{n_0}bx_0$ hence $ay_0 = bx_0$, then we have $b \in Ra$ or $a \in bR$. Hence $0 \neq r_0$, $x_0, y_0 \in m$ and so sb, $b-ra \in m^2$. By induction, let $(sb, b-ra) = r_0r_1 \dots r_{k-1}(x_{k-1}, y_{k-1})$, for some $0 \neq r_i$, $x_i, y_i \in m$, $0 \leq i \leq k-1$ such that $r_i^{n_i}(x_i, y_i) = t_i(a, b)$, for some $n_i \in N$ and $t_i \in R$, $0 \leq i \leq k-1$. Since $x_{k-1}, y_{k-1} \in m$ and $bx_{k-1} = ay_{k-1}$, hence by Theorem 2.4, we have $(x_{k-1}, y_{k-1}) \in Rad_{R^{(2)}}(R(a, b))$. So $(x_{k-1}, y_{k-1}) = r_k(x_k, y_k)$, for some $0 \neq r_k$, $y_k, x_k \in R$ such that $r_k^{n_k}(x_k, y_k) = t_k(a, b)$, for some $n_k \in \mathbf{N}$ and $t_k \in R$. Similarly for the case $k = 0$, we have $0 \neq r_k$, $x_k, y_k \in m$ and hence $(sb, b-ra) = r_0r_1 \dots r_k(x_k, y_k) \in (m^{k+2})^{(2)}$. □

Theorem 3.4. *Let R be a one dimensional domain such that $\bigcap_{n=1}^{\infty} m^n = 0$, for all maximal ideals m of R. Then R is almost Dedekind if and only if $(aR+bR)^2 = a^2R+b^2R$ and $Rad_{R_m^{(2)}}(R_m(a, b)) = E_{R_m^{(2)}}(R_m(a, b))$, for all maximal ideals m of R and $a, b \in R$.*

Proof. Let R be almost Dedekind domain. Then R is a Prüfer domain and hence by [5], $(aR+bR)^2 = a^2R+b^2R$, for all $a, b \in R$. So by [2], Theorem 2.4, $R^{(2)}$ s.t.r.f. as an R-module. Now let $a, b \in R$. Then $(Rad_{R^{(2)}}(R(a, b)))_m =$

$\langle E_{R^{(2)}}(R(a,b))\rangle_m = \langle E_{R_m^{(2)}}(R_m(a,b))\rangle$, for all $m \in \max(R)$. Since R_m is a valuation ring, hence by Lemma 3.3, $Rad_{R_m^{(2)}}(R_m(a,b)) = E_{R_m^{(2)}}(R_m(a,b))$. $\square$

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References

[1] A. Azizi, Radical formula and prime submodules, *J. Algebra* **307** (2007), no. 1, 454–460.

[2] D. Buyruk and D. P. Yilmaz, Modules over Prüfer domains which satisfy the radical formula, *Glasg. Math. J.* **49** (2007), no. 1, 127–131.

[3] S. Ceken and M. Alkan, On radical formula over free modules with two generators, *Numerical Analysis and Applied Mathematics ICNAAM* (2011) 333–336.

[4] L. Fuchs and L. Salce, Modules over non-Noetherian domains, Mathematical Surveys and Monographs, American Mathematical Society, Providence, 2001.

[5] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, Inc., New York, 1972.

[6] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (1991), no. 12, 3593–3602.

[7] K. H. Leung and S. H. Man, On commutative Noetherian rings which satisfy the radical formula, *Glasgow Math. J.* **39** (1997), no. 3, 285–293.

[8] F. Mirzaei and R. Nekooei, On prime submodules of a finitely generated free module over a commutative ring, *Communications In Algebra*, Accepted.

[9] A. Parkash, Arithmetical rings satisfy the radical formula, *J. Commutative Algebra* (2013), no. 2, 293–296.

[10] H. Sharif, Y. Sharifi and S. Namazi, Rings satisfying the radical formula, *Acta Math. Hungar.* **71** (1996), no. 1-2, 103–108.

[11] D. P. Yilmaz and P. F. Smith, Radical of submodules of free modules, *Comm. Algebra* **27** (1999), no. 5, 2253–2266.

(F. Mirzaei) Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, P.O. Box 76169133, Kerman, Iran.

E-mail address: `mirzaee0269@yahoo.com`

(R. Nekooei) Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, P.O. Box 76169133, Kerman, Iran.

E-mail address: `rnekooei@uk.ac.ir`