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ON RADICAL FORMULA AND PRÜFER DOMAINS

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(Communicated by Jost-Hinrich Eschenburg)

Abstract. In this paper we characterize the radical of an arbitrary submodule $N$ of a finitely generated free module $F$ over a commutative ring $R$ with identity. Also we study submodules of $F$ which satisfy the radical formula. Finally we derive necessary and sufficient conditions for $R$ to be a Prüfer domain, in terms of the radical of a cyclic submodule in $R \oplus R$.

Keywords: Prime submodules, Radical of a submodule, Radical formula, Prüfer domains, Dedekind domains.


1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. A proper submodule $P$ of an $R$-module $M$ is called a $p$-prime submodule, if $rm \in P$ for $r \in R$ and $m \in M$ implies $m \in P$ or $r \in p = (P : M)$, where $(P : M) = \{r \in R \mid rM \subseteq P\}$. Let $I$ be an ideal of $R$. The radical, $\sqrt{I}$, is defined to be the intersection of all prime ideals of $R$ containing $I$. We denote the radical of $I$ by $\sqrt{I}$. Let $X$ be a subset of an $R$-module $M$. We denote the submodule of $M$ that $X$ generates, by $< X >$ or $RX$. The prime radical, $\text{Rad}_M T$, of a submodule $T$ in an $R$-module $M$ is defined to be the intersection of all prime submodules of $M$ containing $T$. If there is no prime submodule containing $T$, then $\text{Rad}_M T = M$. In particular $\text{Rad}_M M = M$. We use the notation $R^{(n)}$ for $R \oplus \cdots \oplus R$ and $I^{(n)}$ for $I \oplus \cdots \oplus I$, where $I$ is an ideal of $R$.

Let $M$ be an $R$-module and $T$ be a submodule of $M$. The envelope of $T$ in $M$ is defined to be the set

$$E_M(T) = \{rm \mid r \in R, m \in M; r^n m \in T, \text{ for some } n \in \mathbb{Z}^+\}.$$
We say that the submodule $T$ of an $R$-module $M$ satisfies the radical formula in $M$ ($T$ s.t.r.f. in $M$) if $\text{Rad}_M T = \langle E_M(T) \rangle$. An $R$-module $M$ s.t.r.f. if for every submodule $T$ of $M$, the prime radical of $T$ is the submodule generated by its envelope, i.e. $\text{Rad}_M T = \langle E_M(T) \rangle$. A ring $R$ s.t.r.f. provided that for every $R$ module $M$, $M$ s.t.r.f. The question of what kind of rings and modules s.t.r.f. has studied by many authors, see [1,3,6,7,10].

In [1], Azizi has shown that every arithmetical ring with $\dim R \leq 1$ satisfies the radical formula. In [9], Parkash proved that every arithmetical ring satisfies the radical formula and Buyruk and Pusat Yilmaz in [2], proved that if $R$ is a Prüfer domain, then the free $R$-module $R^{(2)}$ satisfies the radical formula.

In [11] Pusat-Yilmaz and Smith have described $\text{Rad}_F(T)$, where $T$ is a finitely generated submodule of a free $R$-module $F = R^{(n)}$. In this paper we generalize this characterization for an arbitrary submodule $N$ of $F$ and we characterize some submodules of $F$ satisfying the radical formula. Finally we apply this characterization on the radical of a cyclic submodule of $R^{(2)}$ to give necessary and sufficient conditions for an integral domain $R$ to be a Prüfer domain.

2. Radical of a submodule and radical formula

Let $X_i = (x_{i1}, \ldots, x_{in}) \in F = R^{(n)}$, for some $x_{ij} \in R$, $1 \leq i \leq m$, $1 \leq j \leq n$, $m \leq n$. We put

$$B_{m \times n} = [X_1 \ldots X_m] = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \in M_{m \times n}(R).$$

Thus the $j$th row of the matrix $[X_1 \ldots X_m]$ consists of the components of element $X_j$ in $F$. We use $B(j_1, \ldots, j_k) \in M_{m \times k}(R)$ to denote the submatrix of $B$ consisting of the columns $j_1, \ldots, j_k \in \{1, \ldots, n\}$ and

$$[X_1 \ldots X_m]_{m} = \sum_{j_1, \ldots, j_m \in \{1, \ldots, n\}} \text{Rdet}(B(j_1, \ldots, j_m))$$

the ideal generated by $\{\det B(j_1, \ldots, j_m) \mid j_1, \ldots, j_m \in \{1, \ldots, n\}\}$. We use $N$ to be a non-zero submodule of $F$ generated by the set $\Psi = \{X_i = (x_{i1}, \ldots, x_{in}) \in F \mid i \in \Omega\}$. We put $\mathcal{R}_t = \sum_{i_1, \ldots, i_t \in \Omega} R[X_{i_1} \ldots X_{i_t}], 1 \leq t \leq n$. Note that $\mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \cdots \supseteq \mathcal{R}_n = \mathcal{R}$.

We first state two useful results.

**Lemma 2.1.** Let $F$ be the free $R$-module $R^{(n)}$. Then $\mathcal{R} \subseteq (N : F) \subseteq \sqrt{\mathcal{R}}$.

**Proof.** [8], Lemma 1.1. 

\[\square\]
The following lemma is proved in [8], Lemma 1.5. But we give the proof of part (ii) of this lemma, because we use this proof in Proposition 2.5.

**Lemma 2.2.** Let \( F \) be the free \( R \)-module \( R^{(n)} \), \( p \) be a prime ideal of \( R \) and \( B = [X_1 \ldots X_k] \in M_{k \times n}(R) \) for some \( X_i \in F, \ 1 \leq i \leq k \) and positive integer \( k < n \). Let

\[
T_p(B) = \{X = (x_1, \ldots, x_n) \in F \mid \det \beta(i_1, \ldots, i_{k+1}) \in p, \text{ for every } i_1, \ldots, i_{k+1} \in \{1, \ldots, n\}\}, \text{ where } \beta = [X X_1 \ldots X_k] \in M_{k+1 \times n}(R). \text{ Then}
\]

i) \( T_p(B) \) is a submodule of \( F \).

ii) If \( X = (x_1, \ldots, x_n) \in T_p(B) \), then \( \det(B(i_1, \ldots, i_k))X \in pF + \langle B \rangle \) for all submatrices \( B(i_1, \ldots, i_k) \) of \( B \), where \( \langle B \rangle \) is the \( R \)-submodule of \( F \) generated by the rows of \( B \). (Note that in this part, the ideal \( p \) is not necessarily prime.)

iii) If the determinant of every submatrix \( k \times k \) of \( B \) is in \( p \), then \( T_p(B) = F \).

Proof. ii) Let \( X = (x_1, \ldots, x_n) \in T_p(B) \) and \( B(j_1, \ldots, j_k) \in M_{k \times k}(R) \) be a submatrix of \( B \). Without loss of generality, assume that \( j_1 < j_2 < \ldots < j_k \). Since \( \det\beta(i_1, \ldots, i_{k+1}) \in p \) for every \( i_1, \ldots, i_{k+1} \in \{1, \ldots, n\} \), there exists \( p_t \in p \) such that \( x_idet(B(j_1, \ldots, j_k)) = p_t + \sum_{t=1}^{k} (-1)^{i+t} x_{j_t} det(B(t, j_1, \ldots, j_{t-1}, j_{t+1}, \ldots, j_k)) \)

for every \( 1 \leq t \leq n, t \neq j_t, 1 \leq i \leq k \). It follows that

\[
det(B(j_1, \ldots, j_k))(x_1, \ldots, x_n) = X_p + \sum_{i=1}^{k} \sum_{t=1}^{n} Y_{ijt}, \text{ for some } X_p \in p^{(n)} \text{ and } Y_{ijt} = (y_{i1}, \ldots, y_{in}) \in F, \ 1 \leq i \leq k.
\]

We fix \( 1 \leq i \leq k \). Then \( y_{it} = (-1)^{i+t} x_{j_t} det(B(t, j_1, \ldots, j_{t-1}, j_{t+1}, \ldots, j_k)) \), \( 1 \leq t \leq n, t \neq j_1, \ldots, j_k \) and \( y_{ijt} = x_{j_t} det(B(j_1, \ldots, j_k)) \) and \( y_{ijt} = 0, 1 \leq s \leq k, s \neq i \). Therefore

\[
y_{it} = \sum_{m=1}^{n} (-1)^{m+t} x_{mt} x_{j_t} det(B(t, j_1, \ldots, j_{t-1}, j_{t+1}, \ldots, j_k))_{m1}, 1 \leq t \leq n, t \neq j_1, \ldots, j_k \text{ and } y_{ijt} = \sum_{m=1}^{k} (-1)^{m+i} x_{mj} x_{j_t} det(B(j_1, \ldots, j_k))_{mi}.
\]

Also \( y_{ijt} = \sum_{m=1}^{k} (-1)^{m+i} x_{mj} x_{j_t} det(B(j_1, \ldots, j_{t-1}, j_s, j_{t+1}, \ldots, j_k))_{mi} = 0, 1 \leq s \leq k, s \neq i \). So \( Y_{i} = \sum_{m=1}^{n} x_{jm_i} (-1)^{m+i} det(B(j_1, \ldots, j_k))_{mi} X_m \) and hence \( Y_{i} \in \langle B \rangle, 1 \leq i \leq k \). Thus \( det(B(j_1, \ldots, j_k))(x_1, \ldots, x_n) \in pF + \langle B \rangle \). \( \square \)

Let \( M \) be an \( R \)-module, \( p \) be a prime ideal of \( R \) and \( T \) be a submodule of \( M \). In [11] Pusat-Yilmaz and Smith defined the submodule \( K(T, p) = \{m \in M \mid cm \in T + pM, \text{ for } c \in R \backslash \{p\} \} \). They showed that this is the smallest \( p \)-prime submodule of \( M \) containing \( T \) and so \( Rad_M T = \cap \{K(T, p) : p \text{ is a prime ideal of } R \} \).
Lemma 2.3. Let \( F \) be the free \( R \)-module \( R^{(n)} \) and \( p \) be a prime ideal of \( R \). Then

i) If \((N : F) \not\subseteq p\), then \( K(N, p) = F \).

ii) If \( R_k \subseteq p \), then \( K(N, p) = p^{(n)} \).

iii) If \( R_k \not\subseteq p \), there exists a positive integer \( k < n \) and a matrix \( B_{k \times n} = [X_1 \ldots X_k] \in M_{k \times n}(R) \), \( X_i \in \Psi \), \( 1 \leq i \leq k \) such that \( K(N, p) = T_p(B) \), where \( T_p(B) \) is the \( p \)-prime submodule in Lemma 2.2.

Proof. i) Let \( p \) be a prime ideal of \( R \). Assume \((N : F) \) is not contained in \( p \) and \( c \in (N : F) \) \( \setminus p \). Then \( cF \subseteq N \) and so \( F \subseteq K(N, p) \).

ii) Let \( R_k \subseteq p \). Then \( pF \) contains \( N \) and since \( pF \) is a \( p \)-prime submodule of \( F \), we get that \( K(N, p) = p^{(n)} \).

iii) Let \( R_k \) is not contained in \( p \). Suppose that \( \xi \) is the set of all positive integers \( m \) such that there exists a matrix \( B_{m \times n} = [X_1 \ldots X_m] \in M_{m \times n}(R) \), for some \( X_i \in \Psi \) \( (1 \leq i \leq m) \) and a submatrix \( B(j_1, \ldots, j_m) \) such that \( \text{det} B(j_1, \ldots, j_m) \not\subseteq p \), for some \( j_1, \ldots, j_m \in \{1, \ldots, n\} \). Since \( \Psi \not\subseteq p^{(n)} \), hence \( 1 \in \xi \not\subseteq \emptyset \). Let \( k = \text{max}(\xi) \), by Lemma 2.1, we have \( k < n \).

Let \( B_{k \times n} = [X_1 \ldots X_k] \in M_{k \times n}(R) \) such that \( \text{det} B(j_1, \ldots, j_k) \not\subseteq p \), for some \( j_1, \ldots, j_k \in \{1, \ldots, n\} \). Then by Lemma 2.2(iv), we have \( T_p(B) \) is a \( p \)-prime submodule of \( F \). It is clear that \( N \subseteq T_p(B) \) and by Lemma 2.2(ii), \( T_p(B) \subseteq K(N, p) \).

The Theorem 2.4, is a generalization of Theorem 1.5 in [11].

Theorem 2.4. Let \( F \) be the free \( R \)-module \( R^{(n)} \) and \( N = (\Psi) \). Then \( \text{Rad}_F N = \{X = (x_1, \ldots, x_n) \in \sqrt{R_1 F} \mid [X X_{i_1} \ldots X_{i_{k-1}}]_k \subseteq \sqrt{R_k}, \text{ for every } i_1, \ldots, i_{k-1} \in \Omega, 2 \leq k \leq n \} \), where \( R_k = \sum_{i_1, \ldots, i_k \in \Omega} R[X_{i_1} \ldots X_{i_k}] \) and \( [X X_{i_1} \ldots X_{i_{k-1}}]_k = \sum_{j_1, \ldots, j_k \in \{1, \ldots, n\}} \text{Rad}_B(j_1, \ldots, j_k) \) with \( B = [X X_{i_1} \ldots X_{i_{k-1}}] \).

Proof. Let \( \xi \) be the set of prime ideals of \( R \) containing \((N : F) \). Then by Lemma 2.3 (ii), \( \sqrt{R_1 F} = \bigcap_{p \in \xi} K(N, p) \) and so we get \( \text{Rad}_F N = \bigcap_{p \in \xi} K(N, p) = \sqrt{R_1 F} \cap \bigcap_{p \in \xi} K(N, p) \).

Let \( \Delta = \{X = (x_1, \ldots, x_n) \in \sqrt{R_1 F} \mid [X X_{i_1} \ldots X_{i_{k-1}}]_k \subseteq \sqrt{R_k}, \text{ for every } i_1, \ldots, i_{k-1} \in \Omega, 2 \leq k \leq n \} \). We show that \( \text{Rad}_F N = \Delta \). Suppose that \( X = (x_1, \ldots, x_n) \in \text{Rad}_F N \) where \( x_i \in R, 1 \leq i \leq n \). Then \( X \in \sqrt{R_1 F} \cap \bigcap_{p \in \xi} K(N, p) \). Let \( p \) be any prime ideal of \( R \) containing \( R_k \) \( (2 \leq k \leq n) \). If
Proposition 2.5. Let $F = R^{(n)}$ be a free $R$-module and $N = \langle \Omega \rangle$. If there exist $1 \leq j \leq n - 1$ and $B = [X_1 \ldots X_j] \in M_{j \times n}(R)$, for some $X_1, \ldots, X_j \in \Psi$ such that $B$ contains an $j \times j$ submatrix whose determinant is a unit in $R$ and $\sqrt{R_{j+1}} = \sqrt{(N : F)}$, then $N$ s.t.r.f in $F$.

Proof. Suppose there exists a matrix $B = [X_1 \ldots X_j] \in M_{j \times n}(R)$, for some $X_1, \ldots, X_j \in \Psi$ with a submatrix $B(i_1, \ldots, i_j) \in M_{j \times j}(R)$, for some $i_1, \ldots, i_j \in \{1, \ldots, n\}$ such that $detB(i_1, \ldots, i_j)$ is unit. Let $X \in Rad_F N$. Then $[X X_1 \ldots X_j]_{j+1} \subseteq \sqrt{R_{j+1}} = \sqrt{(N : F)}$. If we replace the ideal $p$ in Lemma 2.2(ii) with $\sqrt{(N : F)}$, then $detB(i_1, \ldots, i_j)X \in \sqrt{(N : F)}F + N$. It follows that $X \in \sqrt{(N : F)}F + N$ and hence $Rad_F N = \sqrt{(N : F)}F + N = \langle E_F(N) \rangle$.

Corollary 2.6. Let $(R, m)$ be a local ring with $m$ as maximal ideal, let $F$ be the free $R$-module $R^{(n)}$ and $N = \langle \Omega \rangle$. If $R_j = R$ and $\sqrt{R_{j+1}} = \sqrt{(N : F)}$, for some $1 \leq j \leq n - 1$, then $N$ s.t.r.f in $F$.

Proof. Let $\mathfrak{R}_j = \sum_{i_1, \ldots, i_j \in \Omega} R[X_{i_1} \ldots X_{i_j}] = R$, for some $1 \leq j \leq n - 1$ and $\sqrt{R_{j+1}} = \sqrt{(N : F)}$. Since $R$ is a local ring, then there exists a matrix $B = [X_1 \ldots X_j] \in M_{j \times n}(R)$, for some $X_1, \ldots, X_j \in \Psi$ with a submatrix $B(i_1, \ldots, i_j) \in M_{j \times j}(R)$, for some $i_1, \ldots, i_j \in \{1, \ldots, n\}$ such that $detB(i_1, \ldots, i_j)$ is unit. Then by Proposition 2.5, $N$ s.t.r.f in $F$.

Proposition 2.7. Let $R$ be a commutative ring with identity, let $F$ be the free $R$-module $R^{(n)}$ and $N = \langle \Omega \rangle$. If $\sqrt{R_1} = \sqrt{R_2} = \cdots = \sqrt{R_{n-1}} = \sqrt{(N : F)}$, then $Rad_F N = \sqrt{(N : F)}F = \langle E_F(N) \rangle$.

Proof. Let $N$ be a submodule of $F$ such that $\sqrt{R_1} = \sqrt{R_2} = \cdots = \sqrt{R_{n-1}} = \sqrt{(N : F)}$. Then by Theorem 2.4, $Rad_F N = \{X = (x_1, \ldots, x_n) \in \sqrt{(N : F)}F$.
Let $F$ be an integrally closed domain. If $a \in F$, then $a \in \sqrt{(N:F)}$, for every $i, j \in \Omega$.

Since $X_i \in \sqrt{(N:F)}$, for every $X_i \in \Psi$, we get that $\text{Rad}_F N = \sqrt{(N:F)}F = \langle E_F(N) \rangle$.

Theorem 2.8 is a generalization of Theorem 1.9 in [11].

**Theorem 2.8.** Let $F = R^{(n)}$ be a free $R$-module and $N = \langle \Psi \rangle$, where $\Psi = \{X_i = (x_{i1}, \ldots, x_{in}) \in F \mid i \in \Omega \}$.

Let $I$ be an ideal of $R$ and $T = N + IF$. Then $\text{Rad}_F T = \{X = (x_1, \ldots, x_n) \in \sqrt{\mathfrak{N}_1 + IF} \mid [X X_i \ldots X_{i-1}]_k \subseteq \sqrt{\mathfrak{N}_k + I}, \text{ for every } i_1, \ldots, i_k-1 \in \Omega, 2 \leq k \leq n\}$, where $\mathfrak{N}_k = \sum_{i_1, \ldots, i_k \in \Omega} R[X_{i_1} \ldots X_{i_k}]$, $1 \leq k \leq n$.

Proof. Let $\Psi' = \{Y_i = (y_{i1}, \ldots, y_{in}) \in IF \mid i \in \Omega'\}$ be a subset of $IF$ such that $IF = \langle \Psi' \rangle$. Then $T = \langle \Psi' \cup \Psi \rangle$, and so by Theorem 2.4, $\text{Rad}_F T \subseteq \{X = (x_1, \ldots, x_n) \in \sqrt{\mathfrak{N}_1 + IF} \mid [X Z_{i_1} \ldots Z_{i_k-1}]_k \subseteq \sqrt{\mathfrak{N}_k + I}, Z_1, \ldots, Z_{i_k-1} \in \Psi \cup \Psi'\}$, for every $i_1, \ldots, i_k-1 \in \Omega \cup \Omega'$, $2 \leq k \leq n$, where $\mathfrak{N}_k = \sum_{i_1, \ldots, i_k \in \Omega \cup \Omega'} R[Z_{i_1} \ldots Z_{i_k}]$, $1 \leq k \leq n$. But it is easy to see that $\sqrt{\mathfrak{N}_i} = \sqrt{\mathfrak{N}_0 + I}$, $1 \leq i \leq n$. Also if $X \in F$ then $[X Z_{i_1} \ldots Z_{i_k-1}]_k \subseteq \sqrt{\mathfrak{N}_k + I}$, for every $i_1, \ldots, i_k-1 \in \Omega \cup \Omega'$ if and only if $[X X_{i_1} \ldots X_{i_k-1}]_k \subseteq \sqrt{\mathfrak{N}_k + I}$, for every $i_1, \ldots, i_k-1 \in \Omega$.

3. Prüfer domains

There are many equivalent conditions for an integral domain $R$ to be a Prüfer domain [5], Theorem 24.3. In what follows we give another equivalent condition in terms of radical of a cyclic submodules of $R^{(2)}$.

Let $R$ be an integral domain and $K$ its field of fractions. $R$ is said to be integrally closed if for every $a \in K$, $f(a) = 0$ for some monic polynomial $f \in R[x]$, then $a \in R$. Furthermore, $R$ is integrally closed if and only if $(I : K I) = R$, for every finitely generated ideal $I$ of $R$ [4], Theorem 3.7.1, where $(I : K I) = \{x \in K \mid xI \subseteq I\}$.

In Theorem 3.1 we give necessary and sufficient condition for an integral domain to be integrally closed, by radical of a cyclic submodules in $R^{(n)}$.

**Theorem 3.1.** Let $R$ be an integral domain with quotient field $K$ and let $F$ be the free $R$-module $R^{(n)}$. Then $R$ is integrally closed if and only if $\text{Rad}_F(R(a_1, \ldots, a_n)) \cap (I_n)^{(n)} = R(a_1, \ldots, a_n)$, for every $(a_1, \ldots, a_n) \in F$ and $n \geq 1$, where $I_n = \langle a_1, \ldots, a_n \rangle$ is a finitely generated ideal of $R$.

Proof. Let $R$ be an integrally closed domain. If $n = 1$, then the proof is clear. Let $n \geq 2$ and $(x_1, \ldots, x_n) \in \text{Rad}_F(R(a_1, \ldots, a_n)) \cap (I_n)^{(n)}$, for some $(x_1, \ldots, x_n), (a_1, \ldots, a_n) \in F$. We can assume that there exists $1 \leq t \leq n$, such that $a_t \neq 0$. Since $R(a_1, \ldots, a_n)$ is a cyclic submodule of $F$ and $n \geq 2$
then by [8], Proposition 1.2, \((R(a_1, \ldots, a_n) : F) = \langle 0 \rangle\). Since \((x_1, \ldots, x_n) \in \text{Rad}_F(R(a_1, \ldots, a_n))\) and \((R(a_1, \ldots, a_n) : F) = \langle 0 \rangle\), by Theorem 2.4, \(x_ia_i = a_ix_i\) for all \(i; 1 \leq i \leq n; i \neq t\), and hence \(a_i(x_1, \ldots, x_n) = x_i(a_1, \ldots, a_n)\). It follows that \(\frac{x_i}{a_i} \in (I_n : K I_n)\). Since \(R\) is integrally closed then \(x_i = ra_i\), for some \(r \in R\) and hence \((x_1, \ldots, x_n) = (a_1, \ldots, a_n)\). Conversely, let \(I_n = \langle a_1, \ldots, a_n \rangle (n \geq 1)\) be a finitely generated ideal of \(R\) and \(\frac{f}{s} \in (I_n : K I_n)\) for some \(0 \neq s, f \in R\). Then there exist \(x_i \in I_n, 1 \leq i \leq n\), such that \(f_{ai} = s_{xi}\). By Theorem 2.4, \((x_1, \ldots, x_n) \in \text{Rad}_R(R(a_1, \ldots, a_n)) \cap (I_n)^{(n)}\), Then \((x_1, \ldots, x_n) = ra_i(a_1, \ldots, a_n)\), for some \(r \in R\). Since \(s(x_1, \ldots, x_n) = f(a_1, \ldots, a_n)\), \(f = rs\) and so \(\frac{f}{s} \in R\) \(\Box\).

**Theorem 3.2.** Let \(R\) be an integral domain. Then \(R\) is a Prüfer domain if and only if for all \(a, b \in R\), \((aR + bR)^2 = a^2R + b^2R\) and \(I^{(2)} \cap \text{Rad}_F(R(a, b)) = R(a, b)\), where \(I = \langle a, b \rangle\).

**Proof.** Let \(R\) be a Prüfer domain. Then \(R\) is integrally closed and by [5], Theorem 24.3, \((aR + bR)^2 = a^2R + b^2R\) for all \(a, b \in R\). Hence by Theorem 3.1, \(I^{(2)} \cap \text{Rad}_R(R(a, b)) = R(a, b)\) and \((aR + bR)^2 = a^2R + b^2R\) for all \(a, b \in R\). Conversely, let \(m\) be a maximal ideal of \(R\). It is enough to show that \(R_m\) is a valuation ring. We assume \(\frac{a}{s_1}, \frac{b}{s_2} \in R_m\), for some \(a, b \in R, s_1, s_2 \in R - m\). If \(a \notin m\) or \(b \notin m\) then \(bR_m \subseteq aR_m\) or \(aR_m \subseteq bR_m\). Now let \(a, b\) be a non-zero element of \(m\). Since \((aR + bR)^2 = a^2R + b^2R\), hence \(ab = ra^2 + sb^2\) for some \(r, s \in R\) and so \(a(b - ra) = sb^2\). Therefore by Theorem 2.4, \((sb, b - ra) \in \text{Rad}_R(R(a, b)) \cap I^{(2)}\). It follows that \((sb, b - ra) = t(a, b)\) for some \(t \in R\). Then \(sb = ta\) and \((1 - t)b = ra\) and so we have \(aR_m \subseteq bR_m\) or \(bR_m \subseteq aR_m\). \(\Box\)

A Noetherian valuation domain is called a discrete rank one valuation. Furthermore, a domain \(R\) is said to be almost Dedekind provided that, for each maximal ideal \(m\) of \(R\), the localization \(R_m\) is a discrete rank one valuation [4], page 119. It is clear that every almost Dedekind domain is a Prüfer domain. In [4], Theorem 7.1, Chapter III, it is proved that a domain \(R\) which is not a field, is an almost Dedekind domain if and only if \(R\) is a Prüfer domain of Krull dimension one and \(\{0\}\) is the only idempotent prime ideal of \(R\). In Theorem 3.4, we give a necessary and sufficient condition for a one dimensional domain \(R\) with \(\{0\}\) as only idempotent prime ideal to be an almost Dedekind domain.

**Lemma 3.3.** Let \(R\) be a one dimensional local domain with maximal ideal \(m\) such that \(\bigcap_{n=1}^\infty m^n = 0\). Then \(R\) is a valuation ring if and only if \(\text{Rad}_{R^{(2)}}(R(a, b)) = E_{R^{(2)}}(R(a, b))\) and \((aR + bR)^2 = a^2R + b^2R\), for all \(a, b \in R\).
Proof. Let \( R \) be a valuation ring. It is clear that \((aR + bR)^2 = a^2R + b^2R\), for all \( a, b \in R \). Now let \((a, b)\) be a non-zero element of \( R^{(2)} \) and \((0, 0) \neq (c, d) \in \text{Rad}_{R^{(2)}}(R(a, b))\). We assume that \( c = rd \) and \( a = sb \), for some \( r, s \in R \). Then we have \((c, d) = d(r, 1) = (s, 1)\). It follows by Theorem 2.4, that \( db(r - s) = 0 \) and \( d^k = tb \), for some \( k \in \mathbb{N} \) and \( t \in R \). Therefore \( r = s \) and we have \((c, d) = d(r, 1) = d(s, 1)\). Hence \((c, d) \in \mathbb{E}_{R^{(2)}}(R(a, b))\). Now let \( a = sb \) and \( d = rc \) for some \( r, s \in R \). Then we have \((c, d) = c(1, r)\) and \((a, b) = b(s, 1)\). Now by Theorem 2.4, we have \( bct(sr - 1) = 0 \) and \( c^k = tb \), for some natural number \( k \) and \( t \in R \). Therefore \( sr = 1 \) and we have \((c, d) = c(1, r)\) and \( c^k(1, r) = c^k r(s, 1) = trb(s, 1) = tr(a, b)\). Hence \((c, d) \in \mathbb{E}_{R^{(2)}}(R(a, b))\). Conversely let \((a, b)\) be non-zero elements of \( R \). It is enough to show that \( a \in Rb \) or \( b \in Ra \). Since \((aR + bR)^2 = a^2R + b^2R\), hence \( ab = ra^2 + sb^2 \), for some \( r, s \in R \) and \( a(b - ra) = sb^2 \). By Theorem 2.4, we have \((sb, b - ra) \in \text{Rad}_{R^{(2)}}(R(a, b))\). Now we assume that \( a \notin Rb \), \( b \notin Ra \) and we show that \( sb, b - ra \in \bigcap_{n=1}^{\infty} m^n \). Hence \( sb = 0 \), \( b - ra = 0 \). Therefore \( b = ra \), which is a contradiction. Since \((sb, b - ra) \in \text{Rad}_{R^{(2)}}(R(a, b)) = \mathbb{E}_{R^{(2)}}(R(a, b))\), then \((sb, b - ra) \in m \) and \((sb, b - ra) = r_0(x_0, y_0)\), for some \( 0 \neq r_0, x_0, y_0 \in R \) such that \( r_0^2(x_0, y_0) = t_0(a, b) \), for some \( n_0 \in \mathbb{N} \) and \( t_0 \in R \). If \( r_0 \) is unit in \( R \), then \((x_0, y_0) = \ell(a, b)\), for some \( \ell \in R \) and so \((sb, b - ra) = r_0\ell(a, b)\). It follows that \( sb = r_0\ell a \) and \( b(1 - r_0\ell) = ra \). Since \( R \) is a local ring, \( r_0\ell \) or \( 1 - r_0\ell \) is unit and so \( a \in Rb \) or \( b \in Ra \), which is a contradiction. Therefore \( r_0 \in m \). If \( x_0 \) or \( y_0 \) is unit, because \( r_0^{m_0}a_0y_0 = r_0^{m_0}b_0x_0 \) hence \( a_0y_0 = b_0x_0 \), then we have \( b \in Ra \) or \( a \in bR \). Hence \( 0 \neq r_0, x_0, y_0 \in m \) and so \( sb, b - ra \in m^2 \). By induction, let \((sb, b - ra) = r_0r_1 \ldots r_{k-1}(x_{k-1}, y_{k-1})\), for some \( 0 \neq r_i, x_i, y_i \in m, 0 \leq i \leq k - 1 \) such that \( r_i^{m_i}(x_i, y_i) = t_i(a, b) \), for some \( n_i \in \mathbb{N} \) and \( t_i \in R, 0 \leq i \leq k - 1 \). Since \( x_{k-1}, y_{k-1} \in m \) and \( bx_{k-1} = ay_{k-1} \), hence by Theorem 2.4, we have \((x_{k-1}, y_{k-1}) \in \text{Rad}_{R^{(2)}}(R(a, b))\). So \((x_{k-1}, y_{k-1}) = r_k(x_k, y_k)\), for some \( 0 \neq r_k, x_k, y_k \in R \) such that \( r_k^{m_k}(x_k, y_k) = t_k(a, b) \), for some \( n_k \in \mathbb{N} \) and \( t_k \in R \). Similarly for the case \( k = 0 \), we have \( 0 \neq r_k, x_k, y_k \in m \) and hence \((sb, b - ra) = r_0r_1 \ldots r_k(x_k, y_k) \in (m^k + 2)^{(2)}\). \( \square \)

**Theorem 3.4.** Let \( R \) be a one dimensional domain such that \( \bigcap_{n=1}^{\infty} m^n = 0 \), for all maximal ideals \( m \) of \( R \). Then \( R \) is almost Dedekind if and only if \((aR + bR)^2 = a^2R + b^2R \) and \( \text{Rad}_{R^{(2)}}(R_m(a, b)) = E_{R^{(2)}}(R_m(a, b)) \), for all maximal ideals \( m \) of \( R \) and \( a, b \in R \).

Proof. Let \( R \) be almost Dedekind domain. Then \( R \) is a Prüfer domain and hence by [5], \((aR + bR)^2 = a^2R + b^2R \), for all \( a, b \in R \). So by [2], Theorem 2.4, \( R^{(2)} \) s.t.r.f. as an \( R \)-module. Now let \( a, b \in R \). Then \((\text{Rad}_{R^{(2)}}(R(a, b)))_m = \)
\[ (E_{R_m}^{(2)}(R(a, b)))_m = (E_{R_m}^{(2)}(R_m(a, b))) \text{, for all } m \in \text{max}(R). \] Since \( R_m \) is a valuation ring, hence by Lemma 3.3, \( \text{Rad}_{R_m}^{(2)}(R_m(a, b)) = E_{R_m}^{(2)}(R_m(a, b)) \). \( \square \)

Acknowledgments
The authors would like to thank the referee for his/her useful suggestions that improved the presentation of this paper.

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