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ON COHOMOGENEITY ONE NONSIMPLY CONNECTED 7-MANIFOLDS OF CONSTANT POSITIVE CURVATURE

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Abstract. In this paper, we give a classification of non-simply connected 7-dimensional Riemannian manifolds of constant positive curvature which admit irreducible cohomogeneity- one actions. We characterize the acting groups and describe the orbits. Also their 1st and 2nd homotopy groups have been presented.
Keywords: Positively curved manifold, irreducible representation, cohomogeneity one action.
MSC(2010): Primary: 53C20; Secondary: 57S25.

1. Introduction

In this paper, we deal with non-simply connected 7-dimensional Riemannian manifolds of constant positive curvature which admit irreducible cohomogeneity one (C1) actions, that is, isometric actions whose orbit spaces are one-dimensional or equivalently the principal orbits have codimension one. C1 Riemannian manifolds have been studied by several authors. W. D. Neumann and J. Parker studied C1 Riemannian manifolds in dimensions 3, 4 ([10], [12]). C. Hoelscher in [7] gave a classifiaction of simply connected C1 Riemannian manifolds in dimensions 5, 6, and 7. C. Searle in [15] has classified simply connected Riemannian C1 manifolds with positive sectional curevature in dimensions less than seven. According to her classification, these manifolds are, up to diffeomorphism, spheres or complex projective spaces. Simply connected 7-dimensional case was treated in [13] by F. Podesta and L. Verdiani when the semisimple part of the acting group has dimension greater than six. They got only the 7-sphere up to diffeomorphism. Verdiani has also provided a classification of simply connected C1 manifolds of positive curvature in even

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dimensions; he obtained just rank one symmetric spaces, up to equivariant diffeomorphism [18,19]. The nonsimply connected ones have only been classified in dimension 4, so it is interesting to consider other dimensions. In this paper, we take the dimension to be 7 and hope to study dimension 5 in a forthcoming paper. Clearly, the nonsimply connected C1 Riemannian manifolds are covered by those which have been classified. Therefore, one need to find finite groups which commute with the actions on simply connected C1 Riemannian manifolds and act on them freely to complete the classification. In [1], Abedi, Kashani studied non-simply connected C1 manifolds of constant positive curvature. They characterized the orbits of reducible C1 actions on positive space form S^n/Γ. For the irreducible actions, they proved that if Γ is a group of type I introduced in [20, Theorem 6.1.11], the C1 manifold S^n/Γ is homogeneous. For the groups of other types listed in [20, Theorems 6.1.11, 6.3.1], they made a conjecture, saying that if the space form S^n/Γ is C1, then it is homogeneous. On the other hand, presentation of fundamental groups of closed manifolds of positive sectional curvature is an interesting problem in Riemannian geometry. By Bonnet theorem, $\pi_1(M)$ is finite, and if M is even dimensional and orientable, then $\pi_1(M)$ is trivial by Synge theorem. Hence one should study just odd dimensions. Fang, Rong, and Shankar in [4,14,16] obtained some nice results on the fundamental groups of positively curved Riemannian manifolds. Therefore, a trend in this context is to find fundamental groups of C1 closed Riemmanian manifolds with positive sectional curvature which leads to the classification of nonsimply connected cases.

This paper which is a continuation of [1] classifies up to isometry, 7-dimensional non-simply connected Riemannian manifolds of constant positive curvature which admit irreducible C1 actions. We should emphasize that, since the actions in this paper are irreducible, our results and techniques are quite interesting and different from those of [1]. As a byproduct of our results, we give a positive answer to the conjecture made by Abedi, Kashani in [1] in dimension 7. To obtain our classification, we benefit from a classification of Riemannian manifolds of constant positive curvature attributed to J. Wolf which states that every complete connected Riemannian manifold of constant positive curvature is isometric to $\frac{S^n}{(\sigma_1\oplus\ldots\oplus\sigma_r)(G)}$ where G is a finite group and σ_i's are fixed point free irreducible orthogonal representations of G [20]. We also use the auxiliary Proposition 3.2 (see section 2) which provides us with the necessary and sufficient conditions for a quotient manifold to admit a C1 action. By the proposition, we just need to know C1 actions on S^7, which have been provided by E. Straume in [17].

The paper is organized as follows. In the first section, we give the preliminaries. The second section is devoted to the classification. Our main results are Theorems 3.3, 3.7, and Propositions 3.6, 3.10, 3.12.

2. **Preliminaries**

In this section we review the classification of Riemannian manifolds of constant positive curvature and recall some basic facts about representation theory.

First we briefly outline the classification of 7-dimensional spherical space forms due to J. A. Wolf (see [20, Chapters 6, 7]). Beforehand, we need the following lemma which describes the *induced representation*:

Lemma 2.1. *[20] If σ is a representation on a vector space W of a subgroup H of finite index n in a group G, then there is a well defined induced representation σ^G of G on $V = \underbrace{W \oplus \ldots \oplus W}_{n\text{-times}}$ given by $\sigma^G(g) = (\sigma(b_j^{-1} g b_i))$ where $G = \bigcup b_i H$, $b_1 \in H$, $\sigma(c) = 0$ for $c \notin H$.*

Now we introduce the finite groups and their fixed point free representations which appear in the Wolf's classification of 7-dimensional spherical space forms.

Type I. G has representation $A^m = B^n = 1$, $ABA^{-1} = A^r$, where $m \geq 1, n \geq 1, r^n \equiv 1 \pmod{m}, (n(r-1), m) = 1$, and every prime divisor of the order d of r in the multiplicative group of residues modulo m of integers prime to m divides $n' = n/d$. Then the complex irreducible fixed point free representations of the subgroup $\langle A, B^d \rangle$ are given by

$$\sigma_k \otimes \sigma_l : A^u B^{vd} \mapsto e^{2\pi iku/m}.e^{2\pi ilv/n'}$$

where $(k, m) = 1 = (l, n)$. Define $\pi_{k,l} = (\sigma_k \otimes \sigma_l)^G$, which is a complex irreducible fixed point free representation of G by Lemma 2.1.

Type II. $G = \langle A, B, R \rangle$ where $\langle A, B \rangle$ is of type I, and R normalizes both $\langle A \rangle$ and $\langle B \rangle$, $n_1 \equiv 0 \pmod{4}$. Then the two following cases happen:

case 1. $n_1 = 4$, $d \equiv 2 \pmod{4}$, and for some s, $RB^s.A = A.RB^s$. Then the complex irreducible fixed point free representations of G are given by $\beta_{k,l}$, where $\beta_{k,l} \oplus \beta_{k,l} = \alpha_{k,l} = \pi_{k,l}^G$, and $\pi_{k,l}$ is as above.

case 2. At least one of the conditions of case 1 fails. Then the representations of G are given by $\alpha_{k,l} = \pi_{k,l}^G$.

Type IV. $G = \langle A, B, P, Q, R \rangle$ where $\langle A, B \rangle$ is a group of odd order of type I, $P^4 = 1, P^2 = Q^2, PQP^{-1} = Q^{-1}, AP = PA, AQ = QA, BPB^{-1} = Q, BQB^{-1} = PQ, P^2 = R^2, RP = QPR, RQR^{-1} = Q^{-1}$, and R normalizes both $\langle A \rangle$ and $\langle B \rangle$. Then G admits the following complex irreducible fixed point free representations:

$$\gamma_{k,l} = (\pi_{k,l} \otimes \tau)^G,$$

where $\pi_{k,l}$ is the representation of $\langle A, B^3 \rangle$, and τ is the representation of binary tetrahedral group T^*, (cf. [20, Lemma 7.1.3] for more details).

$$\xi_{k,l,j} = \pi_{k,l} \otimes o_j,$$

where $\pi_{k,l}$ is the representation of $\langle A, B^{3^v}\rangle$, $n = 3^v n''$, $(3, v) = 1$, and o_j is a representation of O_v^*, (cf. [20, Lemma 7.1.5] for more details).

$$\gamma_{k,l,j} = (\pi_{k,l} \otimes \tau_j)^G,$$

where $\pi_{k,l}$ is the representation of $\langle A, B^{3^v}\rangle$, $n = 3^v n''$, $(3, v) = 1$, and τ_j is the representation of T_v^*, (cf. [20, Lemma 7.1.3] for more details).

$$\eta_{k,l} = (\mu_{k,l})^G,$$

where $\mu_{k,l}$ is the representation of $\langle A, B, P, Q\rangle$ induced by $\sigma_k \otimes \sigma_l \otimes \alpha$ for σ_k a representation of $\langle A\rangle$, σ_l a representation of $\langle B^d\rangle$, and α a representation of **Q**8 (cf. [20, Lemma 5.6.2] for more details).

Type VI. $G = \langle K \times SL(2,5), S\rangle$ where K is of type I with order prime to 30, $SL(2,5)$, is the multiplicative group of 2×2 matrices of determinant 1 with coefficients in $\mathbb{Z}_3$, S^2 is the element of order 2 in $SL(2,5)$, S normalizes $SL(2,5)$, K, $\langle A\rangle$ and $\langle B\rangle$ of K, and conjugation of $SL(2,5)$ by S is the conjugation by the matrix $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$. Then G admits the follwing complex irreducible fixed point free representations:

$$\kappa_{k,l,j} = (\pi_{k,l} \otimes \iota_j)^G,$$

where $\pi_{k,l}$ is the representation of K and ι_j is the representation of $SL(2,5)$, (cf. [20, Lemma 7.1.7] for more details).

Now we quote the classification of 7-dimensional spherical space forms, Theorem 2.2, and the classifications of n-dimensional homogeneous spherical space forms, 2.3, from [20]. Recall that for a complex representation π on a complex vector space V we have a real representation on $V_{\mathbb{R}}$, where we forget the complex structure. In Theorem 2.2 $\hat{\pi}$ refers to the real representation corresponding to the complex representations described above. We use these results in the proofs of Theorem 3.3 and Corollaries 3.4 and 3.5.

Theorem 2.2. *[20] A 7-dimensional complete connected Riemannian manifold of constant positive curvature is isometric to one of the following:*

1. S^7/Γ, Γ *generated by* $diag(R(\frac{1}{n}), R(\frac{a}{n}), R(\frac{b}{n}), R(\frac{c}{n}))$ *where* $R(\theta) = \begin{bmatrix} cos\,2\pi\theta & -sin\,2\pi\theta \\ sin\,2\pi\theta & cos\,2\pi\theta \end{bmatrix}$;
2. $\frac{S^7}{\{(\hat{\pi}_{k_1,l_1} \oplus \hat{\pi}_{k_2,l_2})(G)\}}$, *$G$ of type I with $d = 2$;*
3. $\frac{S^7}{\hat{\pi}_{k,l}(G)}$, *$G$ of type I with $d = 4$;*
4. $\frac{S^7}{\{(\hat{\beta}_{k_1,l_1} \oplus \hat{\beta}_{k_2,l_2})(G)\}}$, *$G$ of type II with $d = 2$;*
5. $\frac{S^7}{\{(\hat{\alpha}_{k_1,l_1} \oplus \hat{\alpha}_{k_2,l_2})(G)\}}$, *$G$ of type II with $d = 1$;*
6. $\frac{S^7}{\hat{\alpha}_{k,l}(G)}$, *$G$ of type II with $d = 2$;*

7. $\frac{S^7}{\hat{\gamma}_{k,l}(G)}$, $\frac{S^7}{\hat{\xi}_{k,l,j}(G)}$, $\frac{S^7}{\hat{\gamma}_{k,l,j}(G)}$ *and* $\frac{S^7}{\hat{\eta}_{k,l}(G)}$, *$G$ of type IV with $d=1$;*
8. $\frac{S^7}{\hat{\kappa}_{k,l,j}(G)}$, *$G$ of type VI with $d=1$;*

Theorem 2.3. *[20] Let M^n be a connected homogeneous Riemannian manifold of constant positive curvature K, then M^n is isometric to a manifold S^n/Γ where (i) $\mathbb{F}$ is a field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, (ii) S^n is the sphere $\| x \| = K^{\frac{-1}{2}}$ in a left hermitian vector space V over $\mathbb{F}$ where V has real dimension $n+1$, (iii) Γ is a finite multiplicative group of elements of norm 1 in $\mathbb{F}$ which is not contained in a proper subfield $\mathbb{F}_1$, $\mathbb{R} \subset \mathbb{F}_1 \subsetneqq \mathbb{F}$, of $\mathbb{F}$, and (iv) Γ acts on S^n by $\mathbb{F}$-scalar multiplication of vectors.*

conversely, all the manifolds listed are n-dimensional Riemannian homogeneous manifolds of constant positive curvature K.

Definition 2.4. [1] An isometric action of a Lie group G on a sphere S^n is called reducible if the corresponding action on $\mathbb{R}^{n+1}$ is reducible, i.e. there are two proper subspaces of $\mathbb{R}^{n+1}$ invariant under the G-action; otherewise the action is called irreducible.

The folowing theorem of E. Straume describes irreducible C1 actions on S^7.

Theorem 2.5. *[17] Let G be a compact connected Lie subgroup of $Iso(S^7)$ and Φ be a irreducible C1 action of G on S^7. Then G and Φ can be one of the following cases.*

Table 2.1. irreducible C1 actions on 7-sphere

n	G	Φ
1	$SU(3)$	*Adjoint*
2	$SO(4)$	$\nu_1 \otimes S^3\nu_3$
3	$U(1) \times SO(4)$	$\rho_2 \otimes \rho_4$
4	$U(2) \times SU(2)$	$[\mu_2 \otimes_{\mathbb{C}} \mu_2]_{\mathbb{R}}$

For the sake of completeness, in the sequel, we take a look at some necessary concepts of representation theory.

Notation 2.6. *[21] Let V be endowed with a real inner product in the case of real representation π or a hermitian inner product in the case of complex representaion π. If we denote the adjoint of a linear map L by L^*, then π^* is defined by $\pi^*(g)v = (\pi(g^{-1}))^*v$ on the group level and by $\pi^*(X)v = -(\pi(X))^*v$ on the Lie algebra level.*

Definition 2.7. Let π_1, π_2 be two representations of a Lie group G on vector spaces V_1 and V_2, respectively. π_1 and π_2 are said to be equivalent, denoted by $\pi_1 \simeq \pi_2$, if there is an isomorphism $L : V_1 \to V_2$ such that

$L(\pi_1(g)v) = \pi_2(g)L(v)$. Such an isomorphism L is called an intertwining map.

Definition 2.8. [21] Let π be a complex representation of the complex lie algebra $\mathfrak{g}$ on a vector space V.
a) π is called orthogonal if there exists a non-degenerate symmetric bilinear form on V invariant under π.
b) π is called symplectic if there exists a non-degenerate skew-symmetric bilinear form on V invariant under π.
c) π is called complex if it is neither orthogonal nor symplectic.
d) π is called self dual if $\pi \simeq \pi^*$.

Proposition 2.9. *[21] Let π be a complex irreducible representation of a compact semisimple Lie algebra $\mathfrak{k}$ on V. Then there exists a symmetric bilinear form on V invariant under π if and only if there exists a conjugate linear intertwining map τ with $\tau^2 = Id$.*

Proposition 2.10. *[21] (a) Let π be a complex representation. π is orthogonal or symplectic if and only if π is self dual.*
(b) Let π_1 and π_2 be two complex representation. $\pi_1 \otimes \pi_2$ is self dual if and only if both representations are self dual.

Notation 2.11. *For a complex representation π on a complex vector space V we denote by $\pi_{\mathbb{R}}$ the real representation on $V_{\mathbb{R}}$, where we forget the complex structure.*
If σ is a real representation on W, $\sigma_{\mathbb{C}} = \sigma \otimes \mathbb{C}$ is a complex representation on $W_{\mathbb{C}} = W \otimes \mathbb{C}$.

Lemma 2.12. *[21] If π is a complex repreesntation of a compact Lie algebra $\mathfrak{k}$, then $\pi_{\mathbb{R}} \otimes \mathbb{C} \simeq \pi \oplus \pi^*$.*

Proposition 2.13. *[21] Let σ be a real irreducible representation of $\mathfrak{k}$. Then one and only one of the following holds:*
(a) $\sigma \otimes \mathbb{C} \simeq \pi$ with π an orthogonal irreducible representation.
(b) $\sigma \otimes \mathbb{C} \simeq \pi \oplus \pi^*$ with π an irreducible complex representation.
(c) $\sigma \otimes \mathbb{C} \simeq \pi \oplus \pi$ with π an irreducible symplectic representation.

We say that the real representation σ is of real type in case (a), of complex type in case (b) and quaternionic type in case (c).

Proposition 2.14. *[21] Let σ be a real irreducible representation of $\mathfrak{k}$, and I_σ the algebra of intertwining operators.*

(a)If σ is of real type, then $I_\sigma \simeq \mathbb{R}$.
(b)If σ is of complex type, then $I_\sigma \simeq \mathbb{C}$.
(c) If σ is of quaternionic type, then $I_\sigma \simeq \mathbb{H}$.

Proposition 2.15. *[3] An irreducible real representation of a compact abelian Lie group is either one-dimensional and of real type or two-dimensional and of complex type.*

3. Main results

In this section we deal with our classifiaction problem. The first step is to find conditions under which a quotient manifold admits a C1 action. The following proposition states the necessary and sufficient conditions.

Definition 3.1. Let (M, g) be a Riemannian manifold and $G \subseteq Iso_g(M)$ be a closed Lie subgroup. The action of G on M is C1 if there exists an orbit of codimension one, or equivalently, if the orbit space is a 1-dimensional topological space.

Proposition 3.2. *Let $\widetilde{G}$ be a compact connected Lie subgroup of isometries of a Riemannian manifold M such that the action of $\widetilde{G}$ on M is C1, let Γ be a properly discontinuous subgroup of isometries of M, $\pi : M \to \frac{M}{\Gamma}$ be the natural projection map $\pi(x) = O_\Gamma(x)$, and let $\rho : \widetilde{G} \to G$ be a covering homomorphism. Then the action of $\widetilde{G}$ descends to an action of G on M/Γ if and only if the following conditions hold:*
1. for all $\tilde{g} \in \widetilde{G}, \varphi \in \Gamma, x \in M, \quad \tilde{g}.\varphi(x) = \varphi(\tilde{g}.x)$,
2. $\{\Phi^{\tilde{g}} : \quad \tilde{g} \in ker\rho\} \subseteq \Gamma$, where

$$\begin{aligned} \Phi^{\tilde{g}} : M &\to M \\ x &\mapsto \tilde{g}.x \end{aligned}$$

Further the action is C1.

Proof. The "only if" statement has been proved in [2, Theorem I. 9. 1]. Now we prove the inverse.

Let $\rho : \widetilde{G} \to G$ be a covering homomorphism so that the two conditions hold. Define the action of G on $\frac{M}{\Gamma}$ as follows:

$$\begin{aligned} \Psi : G \times \frac{M}{\Gamma} &\to \frac{M}{\Gamma} \\ (g, [x]) &\mapsto [\tilde{g}.x], \end{aligned}$$

for some $\tilde{g} \in \widetilde{G}$ such that $\rho(\tilde{g}) = g$. First, we show that the definition of Ψ does not depend on $\tilde{g}$. let $\rho(\tilde{g}_1) = \rho(\tilde{g}_2) = g$, so $\rho(\tilde{g}_2^{-1}\tilde{g}_1) = e$, i.e, $\tilde{g}_2^{-1}\tilde{g}_1 \in ker\rho$.

Since $ker\rho$ is in the center of $\widetilde{G}$, we have $\tilde{g}_2\tilde{g}_2^{-1}\tilde{g}_1 = \tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_2$. Thus

$$\begin{aligned}\tilde{g}_1.x &= \tilde{g}_2\tilde{g}_2^{-1}\tilde{g}_1.x\\ &= \tilde{g}_2^{-1}\tilde{g}_1\tilde{g}_2.x\\ &= \tilde{g}_2^{-1}\tilde{g}_1.(\tilde{g}_2.x)\\ &= \Phi^{\tilde{g}_2^{-1}\tilde{g}_1}(\tilde{g}_2.x)\\ &\Longrightarrow [\tilde{g}_1.x] = [\tilde{g}_2.x].\end{aligned}$$

Now suppose $x, y \in M$, $[x] = [y]$, so $y = \varphi(x)$, for some $\varphi \in \Gamma$. Let $\tilde{g} \in \widetilde{G}$, we have

$$\begin{aligned}\tilde{g}.y &= \tilde{g}.\phi(x)\\ &= \phi(\tilde{g}.x)\\ &\Longrightarrow [\tilde{g}.x] = [\tilde{g}.y].\end{aligned}$$

It means that Ψ is well-defined.

It can be easily seen that Ψ is a C1 action. Since $\pi : M \to \frac{M}{\Gamma}$ is a Riemannain covering, $\pi_{*x} : T_xM \to T_{[x]}\frac{M}{\Gamma}$ is a linear isomorphism, and on the other hand $\pi(o(x)) = o(\pi(x))$, we have $dim\, o(x) = dim\, o(\pi(x))$. It shows that the action of G on $\frac{M}{\Gamma}$ is a C1 action. □

Now we state the classification which is summarized in Table 3.1 on page 577.

Theorem 3.3. *Let M^7 be a complete connected non-simply connected, compact Riemannian manifoldof constant positive curvature admiting an irreducible C1 action. Then, up to isometry, one and only one of the cases of Table 3.1 can happen. Note that in Table 3.1,* $A_k = \langle \begin{bmatrix} cos\frac{2\pi}{k} & -sin\frac{2\pi}{k}\\ sin\frac{2\pi}{k} & cos\frac{2\pi}{k}\end{bmatrix}\rangle$ *and* $B_k = \langle e^{\frac{2\pi i}{k}}\rangle$.

Proof. By Theorem 2.2, we know that M is isometric to an spherical space form S^7/Γ, where Γ is a properly discontinuous subgroup of $O(8)$. To find 7-dimensional spherical space forms which admit irreducible C1 actions, we investigate properly discontinuous subgroups of $O(8)$ which satisfy conditions of Proposition 3.2, for the actions determined in Theorem 2.5. By Theorem 2.5, there are four Lie groups $\widetilde{G}$ which act on S^7 irreducibly and of cohomogeneity one, namely $SU(3)$, $SO(4)$, $SO(2) \times SO(4)$ and $U(2) \times SU(2)$. We explore each group and its action on S^7 to obtain the division algebra of intertwining maps of the action. This leads us to get the properly discontinuous subgroups of $O(8)$, say Γ, which satisfy condition 1 of Proposition 3.2. To obtain the groups Γ which satisfy condition 2 as well, we consider a covering homomorphism from $\widetilde{G}$ and study its kernel as a discrete normal subgroup of $Z(\widetilde{G})$. Then we examine condition 2 and find all those Γ, which satisfy conditions 1, 2 of Proposition 3.2. Notice that the groups acting on S^7/Γ have the form $\widetilde{G}/H$, where $\widetilde{G}$ is one of the groups introduced in Theorem 2.5 and H is a discrete normal subgroup of $Z(\widetilde{G})$. In the following, by the aid of representation theory, we give the details of our investigation of the acting groups, case by case according to Theorem 2.5, to see if Proposition 3.2 is satisfied.

a) $\widetilde{\mathbf{G}} = \mathbf{SU(3)}$. Consider the inner product space $V = (\mathfrak{su}(3), \langle , \rangle)$, where $\langle X, Y\rangle = -traceXY$ and define the action of $SU(3)$ on V as follows:

$$\begin{aligned} Ad : SU(3) \times su(3) &\longrightarrow su(3) \\ (A, X) &\longmapsto AXA^{-1}. \end{aligned}$$

According to [17], this action is irreducible and of cohomogeneity one, $d(Ad) = ad$ is a real irreducible reprsentation of $su(3)$. By [6] $ad\otimes\mathbb{C}$ is irreducible as well, so Propositions 2.13 and 2.14 yeild $I_\sigma \simeq \mathbb{R}$. Hence condition 1 of Proposition 3.2 holds only for $\Gamma = \mathbb{Z}_2$ as the only finite subgroup of $(\mathbb{R}^*, .)$ is $\mathbb{Z}_2$.

Now we consider codition 2 of Proposition 3.2. Let $\rho : SU(3) \to G$ be a covering homomorphism. Since $ker\rho$ is a discrete normal subgroup of $Z(SU(3)) = \mathbb{Z}_3$, we have two cases:
1. $ker\rho = Id$, so condition 2 of Proposition 3.2 holds trivially, hence it gives a C1 action of $SU(3)$ on $\mathbb{RP}^7 = \frac{S^7}{\mathbb{Z}_2}$.
2. $ker\rho = \mathbb{Z}_3$. Let $g = \alpha I \in \mathbb{Z}_3 = \{Id, e^{\frac{2\pi i}{3}} Id, e^{\frac{4\pi i}{3}} Id\}$,

$$\begin{aligned} \Phi^g(x) &= g.x \\ &= gxg^{-1} \\ &= (\alpha I)x(\alpha I)^{-1} \\ &= (\alpha I)x(\alpha^{-1} I) \\ &= (\alpha\alpha^{-1})IxI \\ &= x. \end{aligned}$$

That is, $\Phi^g = I \in \Gamma = \mathbb{Z}_2$, so condition 2 of Proposition 3.2 holds which gives rise to a C1 action of $\frac{SU(3)}{\mathbb{Z}_3}$ on $\mathbb{RP}^7 = \frac{S^7}{\mathbb{Z}_2}$.

b) $\widetilde{G} = \mathbf{SO(4)}$. Here we follow [22] to describe the action of $SO(4)$ on S^7. Let V_k be the vector space of homogeneous polynomials of degree k in two complex variables z,w, then $SU(2)$ acting on vectors (z,w) via matrix multiplication induces an irreducible representation π on V_k of complex dimension $k + 1$ and preserves the inner product, which makes $z^m w^n$ into an orthogonal basis with $|z^m w^n|^2 = m!n!$. The map $(z, w) \mapsto (w, -z)$ extended to be a complex antilinear map $J_k : V_k \to V_k$, satisfies $J_k^2 = (-1)^k Id$. Now consider $V_1 \otimes V_3$ and $J_1 \otimes J_3$, which satisfies $(J_1 \otimes J_3)^2 = Id$. Thus $J_1 \otimes J_3$ induces a real structure on $\mathbb{C}^8$, hence its $+1$ eigenspace W is invariant under the action of

$G = SU(2) \times SU(2)$ and is spanned by:

$$\begin{array}{llll} xz^3 + yw^3, & i(xz^3 - yw^3), & xzw^2 + ywz^2, & i(xzw^2 - ywz^2), \\ yz^3 - xw^3, & i(yz^3 + xw^3), & xz^2w - yw^2z, & i(xz^2w + yw^2z). \end{array}$$

Since the kernel of the 2-fold covering $SU(2) \times SU(2) \to SO(4)$ is $\mathbb{Z}_2$, the action of $SU(2) \times SU(2)$ on W, say σ, induces an irreducible C1 action of $SO(4)$ on W. In fact, $\sigma = \pi|_W$ with $\sigma \otimes \mathbb{C} = \pi$. By Proposition 2.9, π is orthogonal, so according to Proposition 2.13, σ is of real type, hence $I_\sigma \simeq \mathbb{R}$ by Proposition 2.14. As the result, condition 1 holds only for $\mathbb{Z}_2$.

As $Z(SO(4)) = \mathbb{Z}_2$, $ker\rho \subseteq \mathbb{Z}_2$. Straightforward calculation shows that $\Phi^{(-Id)} = -Id \in \mathbb{Z}_2$, thus $SO(4)$ and $\frac{SO(4)}{\mathbb{Z}_2}$ act on $\mathbb{RP}^7$ with cohomogeneity one.

c) $\widetilde{G} = \mathbf{SO(2)} \times \mathbf{SO(4)}$. According to [17], irreducible C1 action of $SO(2) \times SO(4)$ on $\mathbb{R}^2 \otimes \mathbb{R}^4$ is defined by $\rho_2 \otimes \rho_4$, where ρ_n is the standard action of $SO(n)$ on $\mathbb{R}^n$. The vector subspace $span_{\mathbb{C}}\{-e_1 \otimes e_1 \otimes 1 - e_2 \otimes e_1 \otimes i, -e_1 \otimes e_2 \otimes 1 - e_2 \otimes e_2 \otimes i, -e_1 \otimes e_3 \otimes 1 - e_2 \otimes e_3 \otimes i, -e_1 \otimes e_4 \otimes 1 - e_2 \otimes e_4 \otimes i\}$ of the vector space $\mathbb{R}^2 \otimes \mathbb{R}^4 \otimes \mathbb{C}$ is invariant under the complex representation $\rho_2 \otimes \rho_4 \otimes \mathbb{C}$, thus $\rho_2 \otimes \rho_4 \otimes \mathbb{C}$ is reducible. By the proof of Proposition 2.13, ([21], p:126), $\rho_2 \otimes \rho_4 \otimes \mathbb{C} \simeq \pi \oplus \pi^*$ with π irreducible and $\pi \not\simeq \pi^*$, so $\sigma = \rho_2 \otimes \rho_4$ is of complex type, hence $I_\sigma \simeq \mathbb{C}$. Thus condition 1 of Proposition 3.2 holds for $\mathbb{Z}_m = \langle cos\frac{2\pi}{m} Id + sin\frac{2\pi}{m} I \rangle$, where I is the complex structure on $\mathbb{R}^2 \otimes \mathbb{R}^4$, since the only finite subgroups of $(\mathbb{C}^*, .)$ are $\mathbb{Z}_m = \langle e^{\frac{2\pi i}{m}} \rangle$.

Now we consider condition 2. Let H be a discrete normal subgroup of $SO(2) \times \mathbb{Z}_2$, which is the center of $SO(2) \times SO(4)$. The next two possibilities can occur:

1) $(Id, -Id) \in H$. One can easily see that $H = \mathbb{Z}_k \times \mathbb{Z}_2$, for some k, as it is well-known that the discrete normal subgroup of $SO(2)$ is finite cyclic. We show that condition 2 of Proposition 3.2 holds if and only if m is even and $k|m$.

Suppose that condition 2 holds, then $-Id = \varphi^{(Id,-Id)} \in \mathbb{Z}_m$, so m has to be even. Now assume that $A = \begin{bmatrix} cos\frac{2\pi}{k} & -sin\frac{2\pi}{k} \\ sin\frac{2\pi}{k} & cos\frac{2\pi}{k} \end{bmatrix}$. As $\varphi^{(A,-Id)} \in \mathbb{Z}_m$, we have

$$\begin{aligned} \varphi^{(A,Id)}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) &= \begin{bmatrix} \cos\frac{2\pi}{k} & 0 & 0 & 0 \\ \sin\frac{2\pi}{k} & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\frac{2\pi q}{m} & 0 & 0 & 0 \\ \sin\frac{2\pi q}{m} & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

for some $q \leq m$. Thus

$$\frac{2\pi q}{m} = 2\pi l + \frac{2\pi}{k} \Longrightarrow kq = m(kl + 1),$$

which shows that $k|m$.

Conversely, let m be even and $k|m$, i.e. $m = kq$, then we have

$$\begin{aligned}\varphi(A^l, Id)\Big(\begin{bmatrix}1&0&0&0\\0&0&0&0\end{bmatrix}\Big) &= \begin{bmatrix}\cos\frac{2\pi l}{k} & -\sin\frac{2\pi l}{k}\\ \sin\frac{2\pi l}{k} & \cos\frac{2\pi l}{k}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\end{bmatrix}\\ &= \begin{bmatrix}\cos\frac{2\pi l}{k} & 0&0&0\\ \sin\frac{2\pi l}{k} & 0&0&0\end{bmatrix}\\ &= \begin{bmatrix}\cos\frac{2\pi lq}{m} & 0&0&0\\ \sin\frac{2\pi lq}{m} & 0&0&0\end{bmatrix}\\ &= (\cos\frac{2\pi lq}{m}Id + \sin\frac{2\pi lq}{m}I)\Big(\begin{bmatrix}1&0&0&0\\0&0&0&0\end{bmatrix}\Big)\end{aligned}$$

$$\begin{aligned}\varphi(A^l, -Id)\Big(\begin{bmatrix}1&0&0&0\\0&0&0&0\end{bmatrix}\Big) &= \begin{bmatrix}\cos\frac{2\pi l}{k} & -\sin\frac{2\pi l}{k}\\ \sin\frac{2\pi q}{m} & \cos\frac{2\pi l}{k}\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&0&0&0\end{bmatrix}\\ &= \begin{bmatrix}-\cos\frac{2\pi l}{k} & 0&0&0\\ -\sin\frac{2\pi l}{k} & 0&0&0\end{bmatrix}\\ &= \begin{bmatrix}-\cos\frac{2\pi lq}{m} & 0&0&0\\ -\sin\frac{2\pi lq}{m} & 0&0&0\end{bmatrix}\\ &= -(\cos\frac{2\pi lq}{m}Id + \sin\frac{2\pi lq}{m}I)\Big(\begin{bmatrix}1&0&0&0\\0&0&0&0\end{bmatrix}\Big)\end{aligned}$$

Since $J = \cos\frac{2\pi lq}{m}Id + \sin\frac{2\pi lq}{m}I \in \mathbb{Z}_m$ and m is even, $-J \in \mathbb{Z}_m$.

2) $(Id, -Id) \notin H$. In this case H can be embedded in $SO(2)$ as a discrete subgroup, and as stated before, it is cyclic. If $H = \langle (A, Id)\rangle$, for some $A \in So(2)$, then $H =< A > \times Id$, where $A = \begin{bmatrix}\cos\frac{2\pi}{k} & -\sin\frac{2\pi}{k}\\ \sin\frac{2\pi}{k} & \cos\frac{2\pi}{k}\end{bmatrix}$. The same discussion as above shows that condition 2 holds if and only if $k|m$.

Now suppose that $H = \langle (A, -Id)\rangle$. Note that in this case the order of H, namely k, is even, since otherwise $(Id, -Id) \in H$. First, let $k = 2^r q$, where $r \geq 2$ and $(q, 2) = 1$. We show that condition 2 holds if and only if $k|m$. Suppose it holds, then we have

$$\begin{aligned}\varphi(A, -Id)\Big(\begin{bmatrix}1&0&0&0\\0&0&0&0\end{bmatrix}\Big) &= \begin{bmatrix}\cos\frac{2\pi l}{k} & -\sin\frac{2\pi l}{k}\\ \sin\frac{2\pi q}{m} & \cos\frac{2\pi l}{k}\end{bmatrix}\begin{bmatrix}-1&0&0&0\\0&0&0&0\end{bmatrix}\\ &= \begin{bmatrix}-\cos\frac{2\pi l}{k} & 0&0&0\\ -\sin\frac{2\pi l}{k} & 0&0&0\end{bmatrix}\\ &= \begin{bmatrix}\cos\frac{2\pi lq'}{m} & 0&0&0\\ \sin\frac{2\pi lq'}{m} & 0&0&0\end{bmatrix}.\end{aligned}$$

Therefore

$$\frac{2\pi q^{'}}{m} = 2\pi l + \pi + \frac{2\pi}{k} \Longrightarrow 2kq^{'} = m(2kl + k + 2)$$
$$\Longrightarrow kq^{'} = m(kl + \frac{k}{2} + 1)$$
$$\Longrightarrow k|m. \qquad (k, lk + \frac{k}{2} + 1) = 1.$$

Conversely, if $k|m$, the condition holds similarly.

Now assume that $k = 2q$ and $(q, 2) = 1$. Condition 2 holds if and only if $k|2m$. the "only if" statement is obvious, so we prove the "if" statement. Let $k|2m$, i. e., $2m = kq^{'}$, for some $q^{'}$.

$$2m = kq^{'} \Longrightarrow m = qq^{'}$$

$$\varphi(A^l, (-1)^l Id)\big(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \big) = \begin{bmatrix} \cos\frac{2\pi l}{k} & -\sin\frac{2\pi l}{k} \\ \sin\frac{2\pi q}{m} & \cos\frac{2\pi l}{k} \end{bmatrix} \begin{bmatrix} (-1)^l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (-1)^l \cos\frac{2\pi l}{k} & 0 & 0 & 0 \\ (-1)^l \sin\frac{2\pi l}{k} & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (-1)^l \cos\frac{2\pi l q^{'}}{2m}, & 0 & 0 & 0 \\ (-1)^l \sin\frac{2\pi l q^{'}}{2m} & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (-1)^l \cos\frac{\pi l q^{'}}{m}, & 0 & 0 & 0 \\ (-1)^l \sin\frac{\pi l q^{'}}{m} & 0 & 0 & 0 \end{bmatrix}$$

If l is even, then $\varphi^{(A^l,(-1)^l Id)} = \cos\frac{2\pi q^{'}\frac{l}{2}}{m} Id + \sin\frac{2\pi q^{'}\frac{l}{2}}{m} I \in \mathbb{Z}_m$ and if l is odd , we have $\varphi^{(A^l,(-1)^l Id)} = \cos\frac{\pi q^{'}(q+l)}{m} Id + \sin\frac{\pi q^{'}(q+l)}{m} I = \cos\frac{\pi q^{'}\frac{q+l}{2}}{m} Id + \sin\frac{\pi q^{'}\frac{q+l}{2}}{m} I \in \mathbb{Z}_m$ as q is odd.

d) $\widetilde{G} = \mathbf{U(2)} \times \mathbf{SU(2)}$. According to Theorem 2.5, $\sigma = (\nu_2 \otimes_{\mathbb{C}} \mu_2)_{\mathbb{R}}$ is an irreducible C1 reprsentation of $U(2) \times SU(2)$ on $(\mathbb{C}^2 \times \mathbb{C}^2)_{\mathbb{R}}$, where μ_2 is the standard representation of $SU(2)$ on $\mathbb{C}^2$ and $\nu_2 = \delta \otimes \mu_2$ with δ the representation of S^1 on $\mathbb{C}$. It is well-known that μ_2 is of quaternionic type, and by Proposition 2.15, δ is of complex type. therefore $\nu_2 \otimes \mu_2$ is of complex type by Proposition 2.10. On the other hand, according to Proposition 2.12 $[\nu_2 \otimes_{\mathbb{C}} \mu_2]_{\mathbb{R}} \otimes \mathbb{C} \simeq \pi \oplus \pi^*$, where $\pi = \nu_2 \otimes_{\mathbb{C}} \mu_2$. Hence σ is of complex type resulting in $I_\sigma \simeq \mathbb{C}$. Thus condition 1 of Proposition 3.2 holds for $\Gamma = \mathbb{Z}_m$. Investigating condition 2 is the same as in case (c). □

We collect the admissible data for our classification in the following table.

Corollary 3.4. *Among all manifolds introduced in Theorem 2.2 only the manifolds in part 1 with* $\Gamma = \langle diag(R(\frac{1}{n}), R(\frac{1}{n}), R(\frac{1}{n}), R(\frac{1}{n})\rangle$ *admit an irreducible C1 action.*

Proof. By a straightforward computation, one can see that just the group in part 1 of Theorem 2.2 is cyclic. On the other hand, by the proof of Theorem 3.3, we see that all Γ's obained in the theorem are cyclic, so the result is obtained. □

Corollary 3.5. *A compact connected non-simply connected 7-dimensional Riemannian manifold of constant positive curvature admits an irreducible C1 action only if it is homogeneous.*

Proof. Theorem 2.3 and the proof of Theorem 3.3 gives the corollary. □

The readers should note that the corollary is the positive answer to the conjecture presented in [1] in dimension 7.

In the following, we search for the isotropy subgroups of these actions to describe the orbits.

Proposition 3.6. *Let* G *be a Lie group acting on a nonsimply connectd manifold* $\frac{M}{\Gamma}$, $\widetilde{G}$ *be its covering group which acts on the manifold* M *and* $\widetilde{G}_x$ *be an isotropy subgroup of* $\widetilde{G}$. *Then the isotropy subgroup of* G *at* $\pi(x)$ *is isomorphic to* $\frac{\widetilde{H}}{ker\rho}$, *where* $\widetilde{H} = \{\tilde{g} \in \widetilde{G} : \tilde{g}.x = \varphi(x) \text{ for some } \varphi \in \Gamma\}$, $\rho : \widetilde{G} \to G$ *is the covering homomorphism, and* $\pi : M \to \frac{M}{\Gamma}$ *is the natural projection.*

Table 3.1. 7-dimensional spherical space forms admitting irreducible C1 actions

$\widetilde{G}$	$Z(\widetilde{G})$	$H \leq_{discrete} Z(\widetilde{G})$	Γ	G	M^7
$SU(3)$	$\mathbb{Z}_3$	$Id, \mathbb{Z}_3$	$\mathbb{Z}_2$	$\frac{SU(3)}{H}$	$\mathbb{RP}^7$
$SO(4)$	$\mathbb{Z}_2$	$Id, \mathbb{Z}_2$	$\mathbb{Z}_2$	$\frac{SO(4)}{H}$	$\mathbb{RP}^7$
$SO(2) \times SO(4)$	$SO(2) \times \mathbb{Z}_2$	$\mathbb{Z}_k \times \mathbb{Z}_2$	$\mathbb{Z}_m, m$ even, $k\|m$	$\frac{SO(2)\times SO(4)}{H}$	$\frac{S^7}{\mathbb{Z}_m}$
$SO(2) \times SO(4)$	$SO(2) \times \mathbb{Z}_2$	$\mathbb{Z}_k \times Id$, $\langle A_k, -Id\rangle$, $k = 2^r q, r \geq 2$	$\mathbb{Z}_m$, $k\|m$	$\frac{SO(2)\times SO(4)}{H}$	$\frac{S^7}{\mathbb{Z}_m}$
$SO(2) \times SO(4)$	$SO(2) \times \mathbb{Z}_2$	$\langle A_k, -Id\rangle$, $k = 2q$	$\mathbb{Z}_m$, $k\|2m$	$\frac{SO(2)\times SO(4)}{H}$	$\frac{S^7}{\mathbb{Z}_m}$
$U(2) \times SU(2)$	$S^1 \times \mathbb{Z}_2$	$\mathbb{Z}_k \times \mathbb{Z}_2$	$\mathbb{Z}_m, m$ even, $k\|m$	$\frac{SO(2)\times SO(4)}{H}$	$\frac{S^7}{\mathbb{Z}_m}$
$U(2) \times SU(2)$	$S^1 \times \mathbb{Z}_2$	$\mathbb{Z}_k \times Id$, $\langle B_k, -Id\rangle$, $k = 2^r q, r \geq 2$	$\mathbb{Z}_m$, $k\|m$	$\frac{SO(2)\times SO(4)}{H}$	$\frac{S^7}{\mathbb{Z}_m}$
$U(2) \times SU(2)$	$S^1 \times \mathbb{Z}_2$	$\langle B_k, -Id\rangle$, $k = 2q$	$\mathbb{Z}_m$, $k\|2m$	$\frac{SO(2)\times SO(4)}{H}$	$\frac{S^7}{\mathbb{Z}_m}$

Proof. First, notice that $ker\rho \subseteq \widetilde{H}$, since for $\tilde{g} \in ker\rho$ we have

$$\begin{aligned}\pi(\tilde{g}.x) &= \rho(\tilde{g})\pi(x)\\ &= e\pi(x)\\ &= \pi(x)\\ &\Longrightarrow \tilde{g}.x = \varphi(x),\ \text{for some}\, \varphi \in \Gamma.\end{aligned}$$

For $x \in M$, the isotropy subgroup of G at $\pi(x)$ is $G_{\pi(x)} = \{\rho(\tilde{g}) : \tilde{g} \in \widetilde{G},\ \tilde{g}.x = \varphi(x)$ for some $\varphi \in \Gamma\}$, so the result is readily obtained. ☐

According to the classification given in Theorem 3.3, we explore the principal and singular isotropy subgroups of the irreducible C1 actions on S^7/Γ case by case. To specify the notation, we use $\widetilde{K}_{\pm}$, $\widetilde{K}_0$ instead of $\widetilde{H}$ in Proposition 3.6 and call them the subgroups corresponding to the singular and principal isotropy subgroups, respectively.

Theorem 3.7. *The singular and principal orbits of irreducible C1 actions on S^7/Γ are those listed in the following table.*

TABLE 3.2. singular and principal orbits of C1 actions on S^7/Γ

$\widetilde{G}$	*singular orbits*		*principal orbit*
$SU(3)$	$\frac{SU(3)}{U(2)} \cong \mathbb{CP}^2$	$\frac{SU(3)}{T^2\times\mathbb{Z}_2}$	$\frac{SU(3)}{T^2}$
$SO(4)$	$\frac{SO(4)}{O(2)\times\mathbb{Z}_2}$	$\frac{SO(4)}{O(2)\times\mathbb{Z}_2}$	$\frac{SO(4)}{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}$
$U(1)\times SO(4)$	$\frac{U(1)\times SO(4)}{S(\mathbb{Z}_2\times O(3))\times\Gamma}$	$\frac{U(1)\times SO(4)}{SO(2)\times SO(2)\times\Gamma}$	$\frac{U(1)\times SO(4)}{\mathbb{Z}_2\times SO(2)\times\Gamma}$
$U(2)\times SU(2)$	$\frac{U(2)\times SU(2)}{S^1\times S^1\times\Gamma}$	$\frac{U(2)\times SU(2)}{\widetilde{G}_{X_-}\times\Gamma}$	$\frac{U(2)\times SU(2)}{S^1\times\Gamma}$

Here $\widetilde{G}_{X_-} = \Big\{(1, \begin{bmatrix} a & b\\ -\bar{b} & \bar{a}\end{bmatrix}, \begin{bmatrix} a & b\\ -\bar{b} & \bar{a}\end{bmatrix}), (-1, \begin{bmatrix} a & b\\ -\bar{b} & \bar{a}\end{bmatrix}, \begin{bmatrix} -a & -b\\ \bar{b} & -\bar{a}\end{bmatrix}) : a, b \in \mathbb{C},\ a\bar{a} + b\bar{b} = 1\Big\}.$

Proof. Case 1: $\widetilde{G} = SU(3)$. The two singular points of the C1 action of $SU(3)$ on S^7/Γ are $x_+ = \begin{bmatrix} \frac{\sqrt{6}}{6}i & 0 & 0\\ 0 & \frac{\sqrt{6}}{6}i & 0\\ 0 & 0 & \frac{-\sqrt{6}}{3}i\end{bmatrix}$ and $x_- = -x_+$ whose isotropy subgroups are isomorphic to $U(2)$. According to [1], $O(\pi(x_+)) = O(\pi(x_-)) = \frac{SU(3)}{U(2)} \cong \mathbb{CP}^2$ and the other singular orbit is exceptoinal. To find the exceptional and principal orbits, let $\rho : SU(3) \to G$ be either the identity homomorphism to $SU(3)$ or the natural projection to $\frac{SU(3)}{\mathbb{Z}_3}$, and $G_{\pi(x)} = \frac{\widetilde{K}}{\ker\rho}$, $G_{\pi(y)} = \frac{\widetilde{H}}{\ker\rho}$ be the isotropy subgroups of the exceptional and principal orbits of the G-action on $\mathbb{RP}^7$, respectively, where $\widetilde{K}$, $\widetilde{H}$ are as in Proposition 3.6. Since x, y are regular points of $SU(3)$-action on S^7, and T^2 is the isotropy subgroups of regular points, it is isomorphic to the identity components of $\widetilde{K}$, $\widetilde{H}$. As in this case, $\Gamma = \mathbb{Z}_2$, we get that $\widetilde{K}$, $\widetilde{H}$ have at most two components. Therefore

$$\frac{G_{\pi(x)}}{G_{\pi(y)}} = \frac{(\frac{\widetilde{K}}{\ker\rho})}{(\frac{\widetilde{H}}{\ker\rho})} = \frac{\widetilde{K}}{\widetilde{H}} = \frac{(\frac{\widetilde{K}}{T^2})}{(\frac{\widetilde{H}}{T^2})} = \frac{\pi_0(\widetilde{K})}{\pi_0(\widetilde{H})},$$

On the other hand, $\frac{\widetilde{K}}{\widetilde{H}} = \frac{\pi_0(\widetilde{K})}{\pi_0(\widetilde{H})} = \frac{G_{\pi(x)}}{G_{\pi(y)}}$ is finite and nontrivial by definition (cf. [2], p. 181), hence $\widetilde{K} = T^2 \times \mathbb{Z}_2$, $\widetilde{H} = T^2$. As the result, $\frac{SU(3)}{T^2\times\mathbb{Z}_2}$, $\frac{SU(3)}{T^2}$ are the exceptional and principal orbits, respectively.

Case 2: $\widetilde{G} = SO(4)$. Let $a = \frac{xz^3+yw^3}{2\sqrt{3}}$ and $b = \frac{xzw^2+yzw^2}{2}$. By [22], $x_0 = cos(\frac{\pi}{12})a + sin(\frac{\pi}{12})b$, $x_+ = \frac{xz^3+yw^3}{4} + \frac{xzw^2+yzw^2}{4}$ and $x_- = \frac{xz^3+yw^3}{2\sqrt{3}}$ are regular and singular points, and their isotropy subgroups are $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $O(2)$, respectively. Again an easy computation shows that $\widetilde{K}_\pm \cong O(2) \times \mathbb{Z}_2$ and $\widetilde{K}_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 3: $\widetilde{G} = SO(2)\times SO(4)$. $E_+ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a singular point and its isotropy subgroup is $S(\mathbb{Z}_2\times O(3))$. For every $\varphi \in \Gamma$, $\varphi(E_1) \in O(E_1)$. Therefore, $\widetilde{K}_+ = S(\mathbb{Z}_2 \times O(3)) \times \Gamma$. The other singular point is $E_- = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$ whose isotropy subgroup is $SO(2) \times SO(2)$ and $\widetilde{K}_- = SO(2) \times SO(2) \times \Gamma$ since for every $\varphi \in \Gamma$, $\varphi(X) \in O(X)$. The regular point is $E = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ with isotropy subgroup $\mathbb{Z}_2 \times SO(2)$, and similarly $\widetilde{K}_0 = \mathbb{Z}_2 \times SO(2) \times \Gamma$.

Case 4: $\widetilde{G} = U(2)\times SU(2)$. $X_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a singular point with $S^1\times S^1$ as its isotropy subgroup. Since $\varphi(X_+) \in O(X_+)$, for all $\varphi \in \Gamma$, $\widetilde{K}_+ = S^1 \times S^1 \times \Gamma$. We also have $X_- = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the other singular point and

$$\widetilde{G}_{X_-} = \left\{(1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}), (-1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a} \end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1\right\}$$

as the isotropy subgroup. We can easily see that for all $\varphi \in \Gamma$, $\varphi(X_-) \in O(X_-)$ and as the result $\widetilde{K}_- = \widetilde{G}_{X_-} \times \Gamma$. Finally, the principal isotropy subgroup is S^1 and similarly $\widetilde{K}_0 = S^1 \times \Gamma$. □

We now take a look at some of the properties of the orbits. First, we study their fundamental groups and then we discuss some of their geometric properties.

The two following lemmas are needed to investigate the fundamental groups.

Lemma 3.8. *[11] For a coset manifold $M = G/H$ there is an exact sequence of groups and homomorphisms*

$$0 \to \pi_2(M) \to \pi_1(H) \to \pi_1(G) \to \pi_1(M) \to \pi_0(H) \to \pi_0(G).$$

Lemma 3.9. *[9] Let G be a connected Lie group and K be a closed subgroup of G; we denote by $p : G \to \frac{G}{K}$ the quotient map. Let $q : \widetilde{G} \to G$ be the universal covering group of G, $\widetilde{K} = q^{-1}(K)$ and $\widetilde{K}_0$ be the connected component of the neutral element $1 \in G$. Then, the fundamental group $\pi_1(\frac{G}{K})$ is isomorphic to the quotient $\frac{\widetilde{K}}{\widetilde{K}_0}$.*

In the following proposition we compute the 1st, and 2nd homotopy groups of the orbits. The results are summarized in Table 3.3 on page 582.

Proposition 3.10. *The first and second homotopy groups of the orbits presented in Theorem 3.7 are listed in Table 3.3. Notice that in Table 3.3*
$\widetilde{G}_{X_-} = \Big\{(1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}), (-1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a} \end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1\Big\}$,
$G_1 = \{\pm(1,1), \pm(i,i), \pm(j,j), \pm(k,k), \pm(-1,1), \pm(-i,i), \pm(-j,j), \pm(-k,k)\}$,
$G_2 = \{\pm(1,1), \pm(i,i), \pm(j,j), \pm(k,k)\}$

Proof. Case 1: Let $G = SU(3)$. Since $\pi_1(SU(3)) = 0$, by the exact sequence given in Lemma 3.8, $\pi_1(\frac{SU(3)}{H}) \cong \pi_0(H)$, $\pi_2(\frac{SU(3)}{H}) \cong \pi_1(H)$, where $H = T^2$ or $H = T^2 \times \mathbb{Z}_2$. Now the result can be easily obtained.

Case 2: Let $G = SO(4)$, and $q : SU(2) \times SU(2) \to SO(4)$ be the universal covering map. Let $K = \mathbb{Z}_2 \times \mathbb{Z}_2$. As $\pi_1(K) = 0$, $\pi_2(\frac{SO(4)}{K}) = 0$ by the exact sequence of Lemma 3.8. Some calculations show that $\widetilde{K} = q^{-1}(K) = \{\pm(1,1), \pm(i,i), \pm(j,j), \pm(k,k)\}$. Since $K_0 = \{(1,1)\}$, $\pi_1(\frac{SO(4)}{K}) \cong \widetilde{K}$, by Lemma 3.9. If $K = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we see that $\widetilde{K} = q^{-1}(K) = \{\pm(1,1), \pm(i,i), \pm(j,j), \pm(k,k), \pm(-1,1), \pm(-i,i), \pm(-j,j), \pm(-k,k)\}$, so $\pi_1(\frac{SO(4)}{K}) \cong \widetilde{K}$. Now let $K = O(2)$. In this case $\widetilde{K} = (e^{jt}, e^{jt}) \cup (i,i)(e^{jt}, e^{jt})$, so $\pi_1(\frac{SO(4)}{K}) \cong \mathbb{Z}_2$. if we take $K = O(2) \times \mathbb{Z}_2$, $\widetilde{K} = (e^{jt}, e^{jt}) \cup (i,i)(e^{jt}, e^{jt}) \cup (e^{-jt}, e^{jt}) \cup (i,i)(e^{-jt}, e^{jt})$, and $\pi_1(\frac{SO(4)}{K}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 3: Let $G = SO(2) \times SO(4)$, and $q : \mathbb{R} \times SU(2) \times SU(2) \to SO(4)$ be the universal covering map. Let $K = S(\mathbb{Z}_2 \times O(3)) \times \Gamma$, where $\Gamma = \langle cos\frac{2\pi}{m}Id + sin\frac{2\pi}{m}I\rangle$. By some calculations we have $\widetilde{K} = \bigcup_{0\leq l\leq m-1}\{(2k\pi + \frac{2\pi l}{m}, A, A) : k \in \mathbb{Z}, A \in SU(2)\} \bigcup \bigcup_{0\leq l\leq m-1}\{((2k+1)\pi + \frac{2\pi l}{m}, A, -A) : k \in \mathbb{Z}, A \in SU(2)\}$, hence $\pi_1(\frac{SO(2)\times SO(4)}{K}) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \Gamma$. For $K = S(\mathbb{Z}_2 \times O(3))$, $\widetilde{K} = \{(2k\pi, A, A) : k \in \mathbb{Z}, A \in SU(2)\} \cup \{((2k+1)\pi, A, -A) : k \in \mathbb{Z}, A \in SU(2)\}$, and $\pi_1(\frac{SO(2)\times SO(4)}{K}) \cong \mathbb{Z}\times\mathbb{Z}_2$. Now let $K = SO(2)\times SO(2)\times\Gamma$. In this case $\widetilde{K} = \bigcup_{0\leq l\leq m-1}\{(t, \cos\frac{t+\frac{2\pi l}{m}+s}{2} - i\sin\frac{t+\frac{2\pi l}{m}+s}{2}, \cos\frac{t+\frac{2\pi l}{m}-s}{2} - i\sin\frac{t+\frac{2\pi l}{m}-s}{2}) : t, s \in \mathbb{R}\}$, and $\pi_1(\frac{SO(2)\times SO(4)}{K}) \cong \Gamma$. If we take $K = SO(2) \times SO(2)$, $\widetilde{K} = \{(t, \cos\frac{t+s}{2} - i\sin\frac{t+s}{2}, \cos\frac{t-s}{2} - i\sin\frac{t-s}{2}) : t, s \in \mathbb{R}\}$, so $\pi_1(\frac{SO(2)\times SO(4)}{K}) = 0$. For $K = \mathbb{Z}_2 \times SO(2) \times \Gamma$, $\widetilde{K} = \bigcup_{0\leq l\leq m-1}\{(2k\pi + \frac{2\pi l}{m}, \cos\frac{\frac{2\pi l}{m}+s}{2} - i\sin\frac{\frac{2\pi l}{m}+s}{2}, \cos\frac{\frac{2\pi l}{m}-s}{2} + i\sin\frac{\frac{2\pi l}{m}-s}{2}) : k \in \mathbb{Z}, s \in \mathbb{R}\} \bigcup \bigcup_{0\leq l\leq m-1}\{((2k+1)\pi + \frac{2\pi l}{m}, \cos\frac{\pi+\frac{2\pi l}{m}+s}{2} - i\sin\frac{\pi+\frac{2\pi l}{m}+s}{2}, \cos\frac{\pi\frac{2\pi l}{m}-s}{2} + i\sin\frac{\pi\frac{2\pi l}{m}-s}{2}) : k \in \mathbb{Z}, s \in \mathbb{R}\}$, hence $\pi_1(\frac{SO(2)\times SO(4)}{K}) \cong \mathbb{Z}\times\mathbb{Z}_2\times\Gamma$. If $K = \mathbb{Z}_2\times SO(2)$, $\widetilde{K} = \{(2k\pi, \cos\frac{s}{2} - i\sin\frac{s}{2}, \cos\frac{s}{2} - i\sin\frac{s}{2}) : k \in \mathbb{Z}, s \in \mathbb{R}\} \bigcup \{((2k+1)\pi, \cos\frac{\pi+s}{2} - i\sin\frac{\pi+s}{2}, \cos\frac{\pi-s}{2} + i\sin\frac{\pi-s}{2}) : k \in \mathbb{Z}, s \in \mathbb{R}\}$, so $\pi_1(\frac{SO(2)\times SO(4)}{K}) \cong \mathbb{Z} \times \mathbb{Z}_2$.

Case 4: Let $G = U(2)\times SU(2)$, and $q : \mathbb{R}\times SU(2)\times SU(2) \to S^1\times SU(2)\times SU(2)$ be the universal covering map.

Let $K = S^1\times S^1\times\Gamma$. We have $\widetilde{K} = \bigcup_{0\leq l\leq m-1}\{(t, \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is}\end{bmatrix}, \begin{bmatrix} e^{i(t+s+\frac{2\pi l}{m})} & 0 \\ 0 & e^{-i(t+s+\frac{2\pi l}{m})}\end{bmatrix}) : t, s \in \mathbb{R}\}$, so $\pi_1(\frac{U(2)\times SU(2)}{K}) \cong \Gamma$.

If we take $K = S^1 \times S^1$, $\widetilde{K} = \{(t, \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is}\end{bmatrix}, \begin{bmatrix} e^{i(t+s)} & 0 \\ 0 & e^{-i(t+s)}\end{bmatrix}) : t, s \in \mathbb{R}\}$, we have $\pi_1(\frac{U(2)\times SU(2)}{K}) = 0$. Now let $K = S^1 \times \mathbb{Z}_2 \times \Gamma$, we see that $\widetilde{K} = \bigcup_{0\leq l\leq m-1}\{(t, \begin{bmatrix} e^{i(t+\frac{2\pi l}{m})} & 0 \\ 0 & e^{-i(t+\frac{2\pi l}{m})}\end{bmatrix}, 1) : t \in \mathbb{R}\} \bigcup \bigcup_{0\leq l\leq m-1}\{(t, \begin{bmatrix} e^{i(t+\frac{2\pi l}{m})} & 0 \\ 0 & e^{-i(t+\frac{2\pi l}{m})}\end{bmatrix}, -1) : t \in \mathbb{R}\}$, and $\pi_1(\frac{U(2)\times SU(2)}{K}) \cong \mathbb{Z}_2 \times \Gamma$. But for $K = S^1 \times \mathbb{Z}_2$, $\widetilde{K} = \{(t, \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{bmatrix}, 1) : t \in \mathbb{R}\} \bigcup \{(t, \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it}\end{bmatrix}, -1) : t \in \mathbb{R}\}$, and $\pi_1(\frac{U(2)\times SU(2)}{K}) \cong \mathbb{Z}_2$. If we take $K = \widetilde{G}_{X_-} = \Big\{(1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}), (-1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}, \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a}\end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a}+b\bar{b} = 1\Big\}\times\Gamma$, then $\widetilde{K} = \bigcup_{0\leq l\leq m-1}\{(2k\pi+\frac{2\pi l}{m}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a}+b\bar{b} = 1, k \in \mathbb{Z}\} \bigcup \bigcup_{0\leq l\leq m-1}\{(\frac{2\pi l}{m}+(2k+1)\pi, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a}\end{bmatrix}, \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a}\end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1, k \in \mathbb{Z}\}$, and $\pi_1(\frac{U(2)\times SU(2)}{K}) \cong \mathbb{Z} \times \mathbb{Z}_2 \times \Gamma$. Now let $K = \Big\{(1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}), (-1, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a} \end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1\Big\}$, then $\widetilde{K} = \{(2k\pi, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1, k \in \mathbb{Z}\} \bigcup \{((2k+1)\pi, \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} -a & -b \\ \bar{b} & -\bar{a} \end{bmatrix}) : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1, k \in \mathbb{Z}\}$, so $\pi_1(\frac{U(2)\times SU(2)}{K}) \cong \mathbb{Z} \times \mathbb{Z}_2$.

The statements about second homotopy groups can be easily obtained by the above resuls and the exact sequence of Lemma 3.8. □

Table 3.3. First and second homotopy goups of the orbits

orbits of S^7/Γ	π_1	π_2	orbits of S^7	π_1	π_2
$\frac{SU(3)}{T^2\times\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}$	-	-	-
$\frac{SU(3)}{T^2}$	0	$\mathbb{Z}\times\mathbb{Z}$	$\frac{SU(3)}{T^2}$	0	$\mathbb{Z}\times\mathbb{Z}$
$\frac{SO(4)}{O(2)\times\mathbb{Z}_2}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}$	$\frac{SO(4)}{O(2)}$	$\mathbb{Z}_2$	$\mathbb{Z}$
$\frac{SO(4)}{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}$	G_1	0	$\frac{SO(4)}{\mathbb{Z}_2\times\mathbb{Z}_2}$	G_2	0
$\frac{SO(2)\times SO(4)}{S(\mathbb{Z}_2\times O(3))\times\Gamma}$	$\mathbb{Z}\times\mathbb{Z}_2\times\Gamma$	0	$\frac{U(1)\times SO(4)}{S(\mathbb{Z}_2\times O(3))}$	$\mathbb{Z}\times\mathbb{Z}_2$	0
$\frac{SO(2)\times SO(4)}{SO(2)\times SO(2)\times\Gamma}$	Γ	$\mathbb{Z}$	$\frac{U(1)\times SO(4)}{SO(2)\times SO(2)}$	0	$\mathbb{Z}$
$\frac{SO(2)\times SO(4)}{\mathbb{Z}_2\times SO(2)\times\Gamma}$	$\mathbb{Z}\times\mathbb{Z}_2\times\Gamma$	$\mathbb{Z}$	$\frac{U(1)\times SO(4)}{\mathbb{Z}_2\times SO(2)}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}$
$\frac{U(2)\times SU(2)}{S^1\times S^1\times\Gamma}$	Γ	$\mathbb{Z}$	$\frac{U(2)\times SU(2)}{S^1\times S^1}$	0	$\mathbb{Z}$
$\frac{U(2)\times SU(2)}{\widetilde{G}_{X_-}\times\Gamma}$	$\mathbb{Z}_2\times\Gamma$	0	$\frac{U(2)\times SU(2)}{\widetilde{G}_{X_-}}$	$\mathbb{Z}_2$	0
$\frac{U(2)\times SU(2)}{S^1\times\mathbb{Z}_2\times\Gamma}$	$\mathbb{Z}\times\mathbb{Z}_2\times\Gamma$	0	$\frac{U(2)\times SU(2)}{S^1\times\mathbb{Z}_2}$	$\mathbb{Z}\times\mathbb{Z}_2$	0

Theorem 3.11. *[8] Let $\pi : \widetilde{M} \to M$ be a Riemannian submersion. If $g : \widetilde{N} \to \widetilde{M}$ is a horizontal isometric immersion, then its second fundamental form is closely related to the second fundamental form of its projrction $f = \pi \circ g$. For instance, f is totally geodesic, minimal or totally umbilical if and only if g has the coresponding property.*

Proposition 3.12. *The orbits of C1 actions on S^7/Γ are not totally umbilic.*

Proof. Since the natural projection $\pi : S^7 \to S^7/\Gamma$ is a local isometry, it is a Riemannian submersion with discrete fibers, so the orbits of S^7/Γ are the image of the horinzontal submanifolds (orbits) of S^7. Thus by Theorem 3.11, the orbits of S^7/Γ are totally umbilic if and only if the orbits of S^7 are totally umbilic. Now by Proposition 3.10, we see that first and/or second homotopy groups of the orbits G/H of S^7 are not the same as those of the spheres, which follows that none of the orbits is homotopy equivalent to a sphere. Since totally umbilic submanifolds of spheres are again spheres, no orbit $\widetilde{G}/\widetilde{H}$ is totally umbilic in S^7/Γ. □

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