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INVOLUTIVENESS OF LINEAR COMBINATIONS OF A QUADRATIC OR TRIPOTENT MATRIX AND AN ARBITRARY MATRIX

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Abstract. In this article, we characterize the involutiveness of the linear combination of the form $a_1A_1 + a_2A_2$ when $a_1, a_2$ are nonzero complex numbers, $A_1$ is a quadratic or tripotent matrix, and $A_2$ is arbitrary, under certain properties imposed on $A_1$ and $A_2$.

Keywords: Quadratic matrix, involutive matrix, linear combination.


1. Introduction and Preliminary Results

The symbols $\mathbb{C}$ and $\mathbb{C}^*$ will denote the set of complex numbers and nonzero complex numbers, respectively. Let $\mathbb{C}^{n \times m}$ denote the set of all complex $n \times m$ matrices. If $A \in \mathbb{C}^{n \times m}$, then $A^*$ denotes the conjugate transpose of $A$. The identity matrix of order $n$ will be denoted by $I_n$. For a given square complex matrix $A$, the set of eigenvalues of $A$ will be denoted by $\sigma(A)$. The symbol $\oplus$ will denote the direct sum of matrices.

The inheritance of the idempotency, involutiveness, or tripotency by linear combinations of idempotents, involutive, or tripotents has very useful applications in the theory of distributions of quadratic forms in normal variables (see e.g. [20–22]). The reader may find in [21] more applications of idempotent and tripotent matrices. The sets of idempotent and involutive matrices can be dealt by a uniform approach: a quadratic matrix.

Let us define the concept of quadratic matrix and review some properties. Following [1], a matrix $A \in \mathbb{C}^{n \times n}$ is said to be quadratic if there exists a second degree polynomial $p : \mathbb{C} \rightarrow \mathbb{C}$ such that $p(A) = 0$. Thus, quadratic matrices are a wide class of matrices containing idempotent ($A^2 = A$), involutive ($A^2 = I_n$),
and several other types of matrices. The reader is referred to [9] to consult deeper properties of quadratic matrices.

In [19, Theorem 2.1], it was established a useful expression for quadratic matrices. Concretely, for \( A \in \mathbb{C}^{n \times n} \) and \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq \beta \) one has \((A - \alpha I_n)(A - \beta I_n) = 0\) if and only if there exists a nonsingular \( S \in \mathbb{C}^{n \times n} \) such that \( A = S(\alpha I_p \oplus \beta I_q)S^{-1} \), where \( p, q \in \{0, 1, \ldots, n\} \). A matrix \( A \in \mathbb{C}^{n \times n} \) is said to be an \( \{\alpha, \beta\} \)-quadratic matrix if \((A - \alpha I_n)(A - \beta I_n) = 0\). Observe that, in particular, an idempotent is a \( \{0, 1\} \)-quadratic matrix and an involutive matrix is a \( \{-1, 1\} \)-quadratic matrix.

The linear combination of the form
\[
A = a_1 A_1 + a_2 A_2, \quad A_1, A_2 \in \mathbb{C}^{n \times n}, \quad a_1, a_2 \in \mathbb{C}^*
\]
was investigated by many researchers and many useful results was obtained (see e.g. [2–4, 6, 7, 13, 15, 17, 19, 20, 22] and references therein).

The purpose of this paper is to investigate the necessary and sufficient conditions for \( A = a_1 A_1 + a_2 A_2 \) to be involutive matrix, where \( A_1 \) is a quadratic or tripotent matrix and \( A_2 \) is arbitrary under some certain conditions.

2. Main results

We begin the study of the linear combination (1.1) when \( A_1 \) is a quadratic matrix and \( A_1, A_2 \) satisfy certain condition.

**Theorem 2.1.** Let \( A_1, A_2 \in \mathbb{C}^{n \times n} \setminus \{0\} \) and \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \neq 0 \) and \( \alpha \neq \beta \). Moreover, let \( A \) be a linear combination of the form (1.1) with \( a_1, a_2 \in \mathbb{C}^* \). If \( A_1 \) is a \( \{\alpha, \beta\} \)-quadratic and \( A_1 A_2 A_1 = A_2 A_1 \), then \( A^2 = I_n \) if and only if there is a nonsingular matrix \( V \in \mathbb{C}^{n \times n} \) such that
\[
(2.1) \quad A_1 = V \left( \begin{array}{cc} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{array} \right) V^{-1}
\]
and \( A_2 \) satisfies one of the following cases.

(i) \( \beta \neq 1 \)
\[
(2.2) \quad A_2 = V \left( \begin{array}{cccc} \frac{1-a_1 \alpha}{a_2} I_q & 0 & 0 & L \\ 0 & \frac{1-a_1 \alpha}{a_2} I_{p-q} & M & 0 \\ 0 & 0 & \frac{1-a_1 \beta}{a_2} I_r & 0 \\ 0 & 0 & 0 & \frac{1-a_1 \beta}{a_2} I_{n-p-r} \end{array} \right) V^{-1},
\]
being \( L \in \mathbb{C}^{q \times (n-p-r)} \) and \( M \in \mathbb{C}^{(p-q) \times r} \) arbitrary.

(ii) \( \beta = 1, \, \alpha a_1 = 1 \).
\[
(2.3) \quad A_2 = V \left( \begin{array}{ccc} 0 & 0 & L \\ 0 & \frac{1-\alpha_2}{a_2} I_r & 0 \\ S & 0 & \frac{1+\alpha_2}{a_2} I_{n-r-p} \end{array} \right) V^{-1},
\]
being \( S \in \mathbb{C}^{(n-r-p) \times p} \) arbitrary.
(iii) $\beta = 1$, $\alpha a_1 = -1$.

\begin{equation}
A_2 = V \begin{pmatrix}
0 & 0 & 0 \\
R & \frac{a+1}{a-2} I_r & 0 \\
0 & 0 & \frac{1-a}{a-2} I_{n-r-p}
\end{pmatrix} V^{-1},
\end{equation}

being $R \in \mathbb{C}^{r \times p}$ arbitrary.

Proof. Since $A_1$ is an $\{\alpha, \beta\}$-quadratic matrix, there exist $p \in \{0, 1, \ldots, n\}$ and a nonsingular matrix $U \in \mathbb{C}^{n \times n}$ such that $A_1 = U(\alpha I_p \oplus \beta I_{n-p}) U^{-1}$. Let us write $A_2 = U \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} U^{-1}$, where $X_1 \in \mathbb{C}^{p \times p}$. Since $A_1 A_2 A_1 = A_2 A_1$ and $\alpha \neq 0$, we conclude that

\begin{equation}
\alpha X_1 = X_1, \quad \alpha \beta X_2 = \beta X_2, \quad \beta X_3 = X_3, \quad \beta^2 X_4 = X_4.
\end{equation}

It is evident that if $A_1$ is represented as in (2.1) and $A_2$ is represented as in (2.2), (2.3), or (2.4), and the scalars $a_1, \alpha, \beta$ satisfy the corresponding conditions, then $A^2 = I_n$.

Let us assume $A^2 = I_n$. We split the proof in several cases.

Case I: $\beta \neq 1$. From the third equality of (2.5), we get $X_3 = 0$. So,

\begin{equation}
A_2 = U \begin{pmatrix} X_1 & X_2 \\ 0 & X_4 \end{pmatrix} U^{-1}.
\end{equation}

Hence,

$$A = a_1 A_1 + a_2 A_2 = U \begin{pmatrix} a_1 \alpha I_p + a_2 X_1 & a_2 X_2 \\ 0 & a_1 \beta I_{n-p} + a_2 X_4 \end{pmatrix} U^{-1}$$

and

$$A^2 = U \begin{pmatrix} (a_1 \alpha I_p + a_2 X_1)^2 & a_1 a_2 (\alpha + \beta) X_2 + a_2^2 X_1 X_2 + a_2^2 X_2 X_4 \\ 0 & (a_1 \beta I_{n-p} + a_2 X_4)^2 \end{pmatrix} U^{-1}.$$

From $A^2 = I_n$, we conclude that

\begin{equation}
(a_1 \alpha I_p + a_2 X_1)^2 = I_p,
\end{equation}

\begin{equation}
(a_1 \beta I_{n-p} + a_2 X_4)^2 = I_{n-p},
\end{equation}

and

\begin{equation}
a_1 a_2 (\alpha + \beta) X_2 + a_2^2 X_1 X_2 + a_2^2 X_2 X_4 = 0.
\end{equation}

By (2.7), there exist $q \in \{0, \ldots, p\}$ and a nonsingular matrix $V_1 \in \mathbb{C}^{p \times p}$ such that

$$a_1 \alpha I_p + a_2 X_1 = V_1 \begin{pmatrix} I_q & 0 \\ 0 & -I_{p-q} \end{pmatrix} V_1^{-1},$$

which implies

\begin{equation}
X_1 = V_1 \begin{pmatrix} \frac{1-a_1 \alpha}{a_2} I_q & 0 \\ \frac{-1-a_1 \alpha}{a_2} I_{p-q} \end{pmatrix} V_1^{-1}.
\end{equation}
From (2.8), there exist \( r \in \{0, \ldots, n - p\} \) and a nonsingular matrix \( V_2 \in \mathbb{C}^{(n-p)\times(n-p)} \) such that
\[
a_1 \beta I_{n-p} + a_2 X_4 = V_2 \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-p-r} \end{pmatrix} V_2^{-1},
\]
that is,
\[
(2.11) \quad X_4 = V_2 \begin{pmatrix} \frac{1-a_2 \beta}{a_2} I_r & 0 \\ 0 & \frac{-1-a_1 \beta}{a_2} I_{n-p-r} \end{pmatrix} V_2^{-1}.
\]
Let \( X_2 \) be written as \( X_2 = V_1 \begin{pmatrix} K & L \\ M & N \end{pmatrix} V_2 \), where \( K \in \mathbb{C}^{q \times r} \). By (2.9), (2.10), and (2.11), we get
\[
a_1 a_2 (\alpha + \beta) X_2 + a_2^2 (X_1 X_2 + X_2 X_4) = V_1 \begin{pmatrix} 2a_2 K & 0 \\ 0 & -2a_2 N \end{pmatrix} V_2^{-1} = 0.
\]
Thus, \( K = 0 \) and \( N = 0 \). Now, \( X_2 \) reduces to
\[
(2.12) \quad X_2 = V_1 \begin{pmatrix} 0 & L \\ M & 0 \end{pmatrix} V_2^{-1}.
\]
Let us define \( V = U(V_1 \oplus V_2) \). We obtain
\[
A_1 = U \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_r \end{pmatrix} U^{-1} = U(V_1 \oplus V_2) \left( V_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_2^{-1} \end{pmatrix} \right) \left( \alpha I_p \begin{pmatrix} 0 \\ 0 \end{pmatrix} \beta I_{n-p} \right) (V_1^{-1} \oplus V_2^{-1}) U^{-1}
= V \begin{pmatrix} \alpha I_p & 0 \\ 0 & \beta I_{n-p} \end{pmatrix} V^{-1}.
\]
By (2.6), (2.10), (2.11), and (2.12), we obtain
\[
A_2 = U \begin{pmatrix} X_1 & X_2 \\ 0 & X_4 \end{pmatrix} U^{-1}
= U \begin{pmatrix} V_1 \left( \frac{1-a_1 \alpha}{a_2} I_q \right) & 0 \\ 0 & \frac{-1-a_1 \alpha}{a_2} I_{p-q} \end{pmatrix} V_1^{-1} \begin{pmatrix} 0 & L \\ M & 0 \end{pmatrix} V_2^{-1}
= U(V_1 \oplus V_2) \begin{pmatrix} \frac{1-a_1 \alpha}{a_2} I_q & 0 \\ 0 & \frac{-1-a_1 \alpha}{a_2} I_{p-q} \end{pmatrix} \begin{pmatrix} V_1 \left( \frac{1-a_1 \alpha}{a_2} I_r \right) & 0 \\ 0 & \frac{-1-a_1 \alpha}{a_2} I_{n-p-r} \end{pmatrix} V_2^{-1} U^{-1}.
= U(V_1 \oplus V_2) \begin{pmatrix} \frac{1-a_1 \alpha}{a_2} I_q & 0 \\ 0 & \frac{-1-a_1 \alpha}{a_2} I_{p-q} \end{pmatrix} \begin{pmatrix} V_1 \left( \frac{1-a_1 \alpha}{a_2} I_r \right) & 0 \\ 0 & \frac{-1-a_1 \alpha}{a_2} I_{n-p-r} \end{pmatrix} (V_1^{-1} \oplus V_2^{-1}) U^{-1}.
\]
Thus, \( A_2 \) can be written as in (2.2).
Case II: \( \beta = 1 \). Since \( \alpha \neq \beta \), we have \( \alpha \neq 1 \). Hence from (2.5), we get \( X_1 = 0 \) and \( X_2 = 0 \). Then
\[
A = a_1A_1 + a_2A_2 = U \begin{pmatrix} a_1I_p & 0 \\ a_2X_3 & a_1I_{n-p} + a_2X_4 \end{pmatrix} U^{-1}.
\]
Since \( A^2 = I_n \), we conclude that
\[
(2.13) \quad (\alpha a_1)^2 = 1, \quad (a_1I_{n-p} + a_2X_4)^2 = I_{n-p}, \quad (1+\alpha)a_1a_2X_3 + a_2^2X_4X_3 = 0.
\]
By the second equality of (2.13), there exist \( r \in \{0, \ldots, n-p\} \) and a nonsingular matrix \( T \in \mathbb{C}^{(n-p)\times(n-p)} \) such that \( a_1I_{n-p} + a_2X_4 = T(I_r \oplus -I_{n-p-r})T^{-1} \), a simple computation shows that
\[
(2.14) \quad X_4 = T \begin{pmatrix} \frac{1-\alpha a_2}{a_2} I_r & 0 \\ 0 & \frac{1-\alpha a_1}{a_2} I_{n-r-p} \end{pmatrix} T^{-1}.
\]
By the first equality of (2.13), we get \( a_1 = 1/\alpha \) or \( a_1 = -1/\alpha \). (we can use the first equality of (2.13) because if \( p = 0 \), then (2.1) would yield \( A_1 = \beta I_n \), which is not possible in view that \( A_1 \) is \( \{\alpha, \beta\} \)-quadratic matrix). Let us write \( X_3 = X_3 = T \begin{pmatrix} R \\ S \end{pmatrix} \), where \( R \in \mathbb{C}^{r \times p} \). Then
\[
(1 + \alpha)a_1a_2X_3 + a_2^2X_4X_3 = T \begin{pmatrix} (\alpha a_1a_2 + a_2)R \\ (\alpha a_1a_2 - a_2)S \end{pmatrix}.
\]
Thus, from the last equality of (2.13) we have that
\[
(2.15) \quad (\alpha a_1a_2 + a_2)R = 0 \quad \text{and} \quad (\alpha a_1a_2 - a_2)S = 0.
\]
Case II.a: \( \alpha a_1 = 1 \). Equalities (2.14) and (2.15) reduce to
\[
(2.16) \quad X_4 = T \begin{pmatrix} \frac{\alpha - 1}{\alpha a_2} I_r & 0 \\ 0 & \frac{\alpha + 1}{\alpha a_2} I_{n-r-p} \end{pmatrix} T^{-1} \quad \text{and} \quad R = 0.
\]
We have
\[
A_1 = U(I_p \oplus I_{n-p})U^{-1} = U(I_p \oplus T)(\alpha I_p \oplus I_{n-p})(I_p \oplus T^{-1})U^{-1}
\]
and
\[
A_2 = U \begin{pmatrix} 0 & 0 \\ X_3 & X_4 \end{pmatrix} U^{-1}
\]
\[
\quad = U \begin{pmatrix} 0 & 0 \\ T \begin{pmatrix} 0 \\ S \end{pmatrix} & T \begin{pmatrix} \frac{\alpha - 1}{\alpha a_2} I_r & 0 \\ 0 & \frac{\alpha + 1}{\alpha a_2} I_{n-r-p} \end{pmatrix} T^{-1} \end{pmatrix} U^{-1}
\]
\[
\quad = U(I_p \oplus T) \begin{pmatrix} 0 & 0 \\ \frac{\alpha - 1}{\alpha a_2} I_r & 0 \\ 0 & \frac{\alpha + 1}{\alpha a_2} I_{n-r-p} \end{pmatrix} (I_p \oplus T^{-1})U^{-1}.
\]
It is enough to define \( V = U(I_p \oplus T) \) to get the expression of this case.

Case II.b: \( \alpha a_1 = -1 \). The proof of this case is quite similar to the previous one.
Example 2.2. Let us solve in this example the following problem. Let

\[ A_1 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}. \]

Find all numbers \( a_1, a_2 \in \mathbb{C}^* \) such that \( a_1 A_1 + a_2 A_2 \) is involutive.

Observe that \( A_1 \) is a \( (2, 1) \)-quadratic matrix and \( A_1 A_2 A_1 = A_2 A_1 \). Furthermore, a simple computation shows that \( \sigma(A_2) = \{0, -1\} \). Following the notation of the previous result, we can assume \( \alpha = 2 \) and \( \beta = 1 \). By the previous result, we must consider two alternatives: \( a_1 = 1/2 \) or \( a_1 = -1/2 \). For the first alternative, by (2.3), we obtain \( \sigma(A_2) \subset \{0, 1/(2a_2), -3/(2a_2)\} \), and thus, \( -1 \in \{1/(2a_2), -3/(2a_2)\} \), so we have two possibilities for \( a_2 \), namely \( a_2 = -1/2 \) or \( a_2 = 3/2 \). For the second alternative, by (2.4), we obtain \( \sigma(A_2) \subset \{0, 3/(2a_2), -1/(2a_2)\} \), and thus, \( -1 \in \{3/(2a_2), -1/(2a_2)\} \), so we have now two possibilities for \( a_2 \), namely \( a_2 = -3/2 \) or \( a_2 = 1/2 \).

Observe that we have four possibilities:

\[ (a_1, a_2) \in \{(1/2, -1/2), (1/2, 3/2), (-1/2, 1/2), (-1/2, -3/2)\}. \]

It is enough now to check if \( a_1 A_1 + a_2 A_2 \) is involutive. None of the above four possibilities yield to the involutiveness of \( a_1 A_1 + a_2 A_2 \). Thus, we do not find \( a_1, a_2 \in \mathbb{C}^* \) such that \( a_1 A_1 + a_2 A_2 \) is involutive.

Example 2.3. Let \( A_1 \) be as in the previous example. We shall find all matrices \( A_2 \in \mathbb{C}^{3 \times 3} \) and \( a_1 \in \mathbb{C}^* \) that \( A_1 A_2 A_1 = A_2 A_1 \) and \( a_1 A_1 + a_2 \) is involutive.

By following the notation of the previous result, we can assume \( \alpha = 2, \beta = 1 \). Obviously, we have \( a_2 = 1 \). Only cases (ii) and (iii) of the previous result can be satisfied, and thus, \( a_1 = \pm 1/\alpha = \pm 1/2 \). By a diagonalization of \( A_1 \), the expression (2.1) in this example is

\[ A_1 = V \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V^{-1}, \quad V = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad p = 1. \]

If case (ii) of the previous result holds, then \( a_1 = 1/2 \). Now, \( A_2 \) must be of the form (2.3). Since \( r \in \{0, \ldots, n - p\} = \{0, 1, 2\} \), then depending on the value of \( r \), some blocks of \( A_2 \) disappear, yielding to the following possibilities for \( V^{-1} A_2 V \) (respectively for \( r = 0, 1, 2 \)):

\[
\begin{pmatrix}
0 & 0 & 0 \\
x & -3/2 & 0 \\
y & 0 & -3/2
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 1/2 & 0 \\
z & 0 & -3/2
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1/2
\end{pmatrix},
\]

being \( x, y, z \in \mathbb{C} \) arbitrary. The case (iii) can be dealt by a similar way. We omit the details.
Remark 2.4. Observe that under the hypothesis of Theorem 2.1, \( \sigma(A_2) \) is localized with no effort, because (2.2), (2.3), and (2.4) prove that \( A_2 \) is similar to certain triangular matrices.

Remark 2.5. We shall show how to manage the condition \( A_1 A_2 A_1 = A_1 A_2 \) with no effort. Let \( M \) be an \{\( \alpha, \beta \)\}-quadratic matrix. It is evident that \( M^* \) is a \{\( \pi, \beta \)\}-quadratic matrix. Thus, if \( A_1, A_2 \in \mathbb{C}^{n \times n} \) are such that \( a_1 A_1 + a_2 A_2 \) is involutive, \( A_1 \) is \{\( \alpha, \beta \)\}-quadratic matrix, and \( A_1 A_2 A_1 = A_1 A_2 \), we can apply Theorem 2.1 to \( A_1^* \) and \( A_2^* \).

As an example of the wide applicability of this result we shall prove two corollaries.

Corollary 2.6. Let \( A_1, A_2 \in \mathbb{C}^{n \times n} \) be two nonzero linearly independent idempotent matrices such that \( A_1 A_2 A_1 = A_2 A_1 \). Moreover, let \( A \) be a linear combination of the form (1.1) with \( a_1, a_2 \in \mathbb{C}^* \). Then \( A^2 = I_n \) if and only if one of the following conditions holds.

1. \( (a_1, a_2) \in \{(-1, 2), (-1, -2)\} \) and \( A_1 = I_n \).
2. \( (a_1, a_2) \in \{(2, -1), (-2, 1)\} \) and \( A_2 = I_n \).
3. \( (a_1, a_2) \in \{(1, 1), (-1, -1)\} \) and \( A_1 A_2 = A_2 A_1 = 0, A_1 + A_2 = I_n \).
4. \( (a_1, a_2) \in \{(1, -1), (-1, 1)\} \) and \( A_1 + A_2 = A_1 A_2 + I_n, A_2 A_1 = 0 \).

Proof. It is straightforward that any characteristic (i)-(iv) leads to \( A^2 = I_n \).

Assume that \( A^2 = I_n \). Since \( A_1 \) is a nonzero idempotent, we have two possibilities: \( A_1 = I_n \) or \( A_1 \) is a \{1, 0\}-quadratic matrix. For the first of the above possibilities, by writing \( A_2 = W(I_x \oplus 0)W^{-1} \), being \( x = \text{rank}(A_2) \) with \( 0 \neq x \neq n \) we easily obtain the characteristic (i) of the Theorem.

Now, we assume that \( A_1 \) is a \{1, 0\}-quadratic matrix. From Theorem 2.1, there exist a nonsingular matrix \( V \in \mathbb{C}^{n \times n}, p \in \{1, \ldots, n-1\}, q \in \{0, 1, \ldots, p\} \), and \( r \in \{0, 1, \ldots, n-p\} \) such that \( A_1 = V(I_p \oplus 0)V^{-1} \) and

\[
A_2 = V \begin{pmatrix}
\frac{1-a_1}{a_2} I_q \\
0 & 0 & 0 & L \\
0 & -\frac{1+a_1}{a_2} I_{p-q} & M & 0 \\
0 & 0 & 0 & \frac{1}{a_2} I_p \\
0 & 0 & 0 & -\frac{1}{a_2} I_{n-r-p}
\end{pmatrix}
V^{-1}.
\]

(2.17)

We shall denote \( \lambda = \frac{1-a_1}{a_2}, \mu = -\frac{1+a_1}{a_2}, \rho = 1/a_2 \) and \( \sigma = -1/a_2 \).

Since \( A_2^2 = A_2 \), we get

\[
\begin{pmatrix}
\lambda I_q & 0 & 0 & L \\
0 & \mu I_{p-q} & 0 & M \\
0 & 0 & \rho I_p & 0 \\
0 & 0 & 0 & \sigma I_{n-r-p}
\end{pmatrix}^2 = \begin{pmatrix}
\lambda I_q & 0 & 0 & L \\
0 & \mu I_{p-q} & 0 & M \\
0 & 0 & \rho I_p & 0 \\
0 & 0 & 0 & \sigma I_{n-r-p}
\end{pmatrix},
\]

We shall split the proof to the following cases according to the values of \( q \).

1. If \( q = 0 \), then \( \mu \in \{0, 1\} \). Hence \( a_1 = -1 \) or \( a_1 + a_2 = -1 \).
(II) If \( q = p \), then \( \lambda \in \{0, 1\} \). Hence \( a_1 = 1 \) or \( a_1 + a_2 = 1 \).

(III) If \( 0 < q < p \), then \( (a_1, a_2) = (-1, 2) \) or \( (a_1, a_2) = (1, -2) \).

Again we split the proof to the following cases according the value of \( r \).

(A) If \( r = 0 \), then \( \sigma^2 = \sigma \). Since \( \sigma \neq 0 \), then we obtain \( a_2 = -1 \).

(B) If \( r = n - p \), then \( \rho^2 = \rho \). Thus, \( a_2 = 1 \).

(C) If \( 0 < r < n - p \) we arrive at a contradiction.

We now combine cases (I), (II), (III) with (A), (B). The combinations (III-A) and (III-B) are clearly unfeasible.

(I-A): We have \( q = r = 0 \) and \( a_1 = a_2 = -1 \). From (2.17), we have \( A_2 = V(0 \oplus I_{n-p})V^{-1} \). This situation leads to the part of the characteristic (iii) of the theorem.

(II-A): We have \( q = p \), \( r = 0 \), and \( (a_1, a_2) \in \{(1, -1), (2, -1)\} \). From (2.17), we have

\[
A_2 = V\left( \begin{array}{cc} \lambda I_p & L \\ 0 & \sigma I_{n-p} \end{array} \right) V^{-1}.
\]

Since \( A^2_2 = A_2 \), we get \((\lambda + \sigma - 1)L = 0\). If \( (a_1, a_2) = (1, -1) \), then \( A_2 = V\left( \begin{array}{cc} 0 & L_p \\ 0 & I_{n-p} \end{array} \right) V^{-1} \) and this situation leads to the part of the characteristic (iv). If \( (a_1, a_2) = (2, -1) \), then \( \lambda = \sigma = 1 \) and, therefore, \((\lambda + \sigma - 1)L = 0\) leads to \( L = 0 \). So, we have \( A_2 = I_n \). This is the part of the characteristic (ii).

(I-B): We have \( q = 0 \), \( r = n - p \), and \( (a_1, a_2) \in \{(-1, 1), (-2, 1)\} \). From (2.17), we have \( A_2 = V\left( \begin{array}{cc} \mu I_p & M \\ 0 & \rho I_{n-p} \end{array} \right) V^{-1} \). Since \( A^2_2 = A_2 \), we get \((\mu + \rho - 1)M = 0\). If \( (a_1, a_2) = (-1, 1) \), then \( A_2 = V\left( \begin{array}{cc} 0 & M \\ 0 & I_{n-p} \end{array} \right) V^{-1} \) and this situation leads to the part of the characteristic (iv). If \( (a_1, a_2) = (-2, 1) \), then \((\mu + \rho - 1)M = 0\) leads to \( M = 0 \) and \( A_2 = I_n \), and this is the part of the characteristic (ii).

(II-B): We have \( q = p \), \( r = n - p \), and \( a_1 = a_2 = 1 \). From (2.17), we get \( A_2 = V(0 \oplus I_{n-p})V^{-1} \). This situation leads to a part of the characteristic (iii).

The proof is finished. \(\square\)

Corollary 2.7. Let \( A_1, A_2 \in \mathbb{C}^{n\times n} \setminus \{0\} \) be two linearly independent matrices such that \( A^2_1 = I_n, A^2_2 = A_2, A_1A_2A_1 = A_2A_1 \) and let \( A \) be a linear combination of the form (1.1) with \( a_1, a_2 \in \mathbb{C}^* \). Then \( A^2 = I_n \) if and only if \( (a_1, a_2) \in \{(-1, 2), (1, -2)\} \) and \( A_1A_2 = A_2A_1 = A_2 \).

Proof. It is evident that if \( A_1A_2 = A_2A_1 = A_2 \), then \((A_1 - 2A_2)^2 = (-A_1 + 2A_2)^2 = I_n \).

Now, assume that \( A^2 = I_n \). Since \( A^2 = I_n \), we have three possibilities for \( A_1 \): \( A_1 = I_n \) or \( A_1 = -I_n \), or \( A_1 \) is a \((-1, 1)\)-quadratic matrix. If \( A_1 = I_n \), then by writing \( A_2 = R(I_x \oplus 0)R^{-1} \), where \( x = \text{rank}(A_2) \in \{1, \ldots, n-1\} \), we get
\((a_1, a_2) \in \{(1, -2), (-1, 2)\}\), which is the part (i) of Corollary 2.1. If \(A_1 = -I_n\), then by \(A_1A_2A_1 = A_2A_1\), we obtain \(A_2 = 0\) which is not possible. If \(A_1\) is a \((-1, 1)\)-quadratic matrix, then by Theorem 2.1, there exist a nonsingular matrix \(V\) and \(p \in \{1, \ldots, n-1\}\) such that \(A_1 = V(-I_p \oplus I_{n-p})V^{-1}\) and \(A_2\) is written as in (2.3) or (2.4). Case (ii) of Theorem 2.1 and \(A_2^2 = A_2\) lead to \((a_1, a_2) = (-1, 2)\) and \(A_2 = V(0 \oplus I_r \oplus 0)V^{-1}\), whereas case (iii) and \(A_2^2 = A_2\) lead to \((a_1, a_2) = (1, -2)\) and \(A_2 = V(0 \oplus 0 \oplus I_{n-r-p})V^{-1}\). □

A square matrix \(A\) is called group invertible if there exists a matrix \(X\) such that \(AXA = A, XAX = X,\) and \(AX =XA\). It can be proved that this matrix \(X\) is unique (if it exists) and it is customarily written as \(A^\#\) (see [5, Section 4.4]). It is easy to see that any diagonalizable matrix is group invertible. This generalized inverse is necessary to define the sharp ordering: Let \(A, B \in \mathbb{C}^{n \times n}\) be two group invertible matrices. We write \(A \leq B\) when \(AA^\# = BA^\#\) and \(A^\#A = A^\#B\) (see [18, Chapter 4]). If \(A\) is nonsingular and \(A \leq B\) then obviously \(A = B\). Thus, if we assume in addition that \(A\) is an \((\alpha, \beta)\)-quadratic matrix and we want to deal with non-trivial linear combinations \(aA + bB\), where \(a, b \in \mathbb{C}^*\), we can assume that \(\alpha\) or \(\beta\) are zero.

**Theorem 2.8.** Let \(A_1 \in \mathbb{C}^{n \times n}\) be an \((\alpha, 0)\)-quadratic matrix, \(A_2 \in \mathbb{C}^{n \times n}\), \(\alpha \in \mathbb{C}^*\), and \(A\) be a linear combination of the form (1.1) with \(a_1, a_2 \in \mathbb{C}^*\). Assume that \(A_1\) and \(A_2\) are linearly independent matrices and \(A_1 \neq A_2\). Then

\[
A^2 = I_n \quad \Leftrightarrow \quad a_2^2(A_2 - A_1)^2 = I_n - \alpha^{-1}A_1, \quad 1 = [\alpha(a_1 + a_2)]^2.
\]

**Proof.** Since \(A_1\) is an \((\alpha, 0)\)-quadratic matrix, then there exists a nonsingular matrix \(U\) such that \(A_1 = U(\alpha I_p \oplus 0)U^{-1}\), where \(p \in \{1, \ldots, n-1\}\). Let us write \(A_2 = U(X \ Y)U^{-1}\), where \(X \in \mathbb{C}^{p \times p}\). By employing the two conditions of the sharp ordering, we get that \(A_2 = U(\alpha I_p \oplus T)U^{-1}\).

\[\Rightarrow:\] Since \(A^2 = I_n\), we get that \(1 = [\alpha(a_1 + a_2)]^2\) and \((a_2T)^2 = I_{n-p}\). Therefore,

\[
a_2^2(A_2 - A_1)^2 = a_2^2U(0 \oplus T^2)U^{-1} = U(0 \oplus I_{n-p})U^{-1} = I_n - \alpha^{-1}A_1.
\]

\[\Leftarrow:\] The condition \(a_2^2(A_2 - A_1)^2 = I_n - \alpha^{-1}A_1\) leads to \(a_2^2T^2 = I_{n-p}\). Hence \(a_1A_1 + a_2A_2 = U((a_1 + a_2)\alpha I_p \oplus a_2T)U^{-1}\) clearly is involutive. □

**Remark 2.9.** Let \(A_1, A_2 \in \mathbb{C}^{n \times n}\) satisfy the hypotheses of the former theorem. If we want check the existence of \(a_1, a_2 \in \mathbb{C}^*\) such that \(a_1A_1 + a_2A_2\) is involutive (and in this case, find such \(a_1, a_2\)), then the above result gives us a procedure. First, find the spectrum of \(A_1\), or equivalently, find \(\alpha \in \sigma(A_1) \setminus \{0\}\). Second, check if \((A_2 - A_1)^2\) is a scalar multiple of \(I_n - \alpha^{-1}A_1\). If not, the problem has not solution. If yes, from \(a_2^2(A_2 - A_1)^2 = I_n - \alpha^{-1}A_1\), we can find the feasible values of \(a_2\) and from \(1 = [\alpha(a_1 + a_2)]^2\), we can find the feasible values of \(a_1\).
Let us deal now with another condition that was appeared in [16], namely $A_2A_1^#A_1 = A_1$. We will assume that $A_1$ is singular (if otherwise, then $A_2A_1^#A_1 = A_1$ reduces to $A_1 = A_2$). Observe that $A_2A_1^#A_1 = A_1 \iff A_2A_1^# = A_1A_1^#$. Hence $A_2A_1^#A_1 = A_1$ implies $A_1 \leq A_2$.

Let us observe that if $A_1, A_2$ satisfy $A_2A_1^# = A_1A_1^#$ and $A_1A_2 = A_2A_1$, then by writing $A_1 = U(K \oplus 0)U^{-1}$, where $U$ and $K$ are nonsingular (this is possible because $A_1$ is group invertible), we have that $A_2$ can be written as $A_2 = U(K \oplus T)U^{-1}$ for some matrix $T$. Hence $A_1^#A_1 = A_2A_2^*$; which leads to $A_1^# \leq A_2$. Therefore, in next theorem we will assume the condition $A_1A_2 \neq A_2A_1$, since otherwise, we can apply Theorem 2.8.

**Theorem 2.10.** Let $A_1 \in \mathbb{C}^{n \times n}$ be $\{\alpha, 0\}$-quadratic, $A_2 \in \mathbb{C}^{n \times n}$, $\alpha \in \mathbb{C}^*$, and $A$ be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Assume that $A_1$ and $A_2$ are linearly independent matrices, $A_1A_2 \neq A_2A_1$, and $A_2A_1^#A_1 = A_1$. Then $A^2 = I_n$ if and only if there exist a nonsingular matrix $V \in \mathbb{C}^{n \times n}$, $p \in \{1, \ldots, n-1\}$, and $q \in \{0, \ldots, n-p\}$ such that $A_1 = V(\alpha I_p \oplus 0)V^{-1}$ and it is satisfied one of the following cases:

1. $\alpha(a_1 + a_2) = 1$,
   (i) $A_2 = V\left(\begin{array}{cc} \alpha I_p & \frac{1}{n} \frac{1}{2^s} \frac{1}{2^t} Y \\ 0 & -\frac{1}{n} \frac{1}{2^s} l_{n-p} \end{array}\right)V^{-1}$ being $Y \neq 0$ an arbitrary matrix in $\mathbb{C}^{n \times (n-p)}$.
   (ii) $A_2 = V\left(\begin{array}{cc} \frac{1}{n} \frac{1}{2^s} l_n & 0 \\ 0 & \frac{1}{n} \frac{1}{2^s} l_{n-p-q} \end{array}\right)V^{-1}$ being $Y \neq 0$ an arbitrary matrix in $\mathbb{C}^{n \times n}$.

2. $\alpha(a_1 + a_2) = -1$,
   (i) $A_2 = V\left(\begin{array}{cc} \alpha I_p & \frac{1}{n} \frac{1}{2^s} \frac{1}{2^t} Y \\ 0 & \frac{1}{n} \frac{1}{2^s} l_{n-p} \end{array}\right)V^{-1}$ being $Y \neq 0$ an arbitrary matrix in $\mathbb{C}^{n \times (n-p)}$.
   (ii) $A_2 = V\left(\begin{array}{cc} \frac{1}{n} \frac{1}{2^s} l_n & 0 \\ 0 & \frac{1}{n} \frac{1}{2^s} l_{n-p-q} \end{array}\right)V^{-1}$ being $Y \neq 0$ an arbitrary matrix in $\mathbb{C}^{n \times (n-p)}$.

**Proof.** The ‘if’ part is evident. We will prove the reciprocal: There exist a nonsingular matrix $U \in \mathbb{C}^{n \times n}$ and $p \in \{1, \ldots, n-1\}$ such that

$$A_1 = U(\alpha I_p \oplus 0)U^{-1}, \quad A_2 = U\left(\begin{array}{cc} X & Y \\ Z & T \end{array}\right)U^{-1}, \quad X \in \mathbb{C}^{p \times p}.$$  

By employing $A_2A_1^#A_1 = A_1$, we get $X = \alpha I_p$ and $Z = 0$. Let us observe that since $0 \neq p \neq n$, then all blocks in (2.18) for $A_1$ and $A_2$ appear. Since $A_1A_2 \neq A_2A_1$, then $Y \neq 0$. By using $A^2 = I_n$, we get

$$A_2A_1^# = A_1 \quad \alpha(a_1 + a_2) \in \{-1, 1\}, \quad \alpha(a_1 + a_2)Y + a_2YT = 0, \quad \alpha_2T^2 = I_{n-p}. $$
The last equality of (2.19) implies the existence of a nonsingular matrix $U_1 \in \mathbb{C}^{(n-p) \times (n-p)}$ such that $a_2 T = U_1 (I_q \oplus -I_{n-p-q}) U_1^{-1}$ for some $q \in \{0, \ldots, n-p\}$. Let us write $Y = (\gamma_1 \gamma_2) U_1^{-1}$ for $Y_1 \in \mathbb{C}^{p \times q}$. From the second equality of (2.19), we get

\[(2.20) \quad [\alpha (a_1 + a_2) + 1] Y_1 = 0 \quad \text{and} \quad [\alpha (a_1 + a_2) - 1] Y_2 = 0.\]

We have the following possibilities:

(a) If $q = 0$, in view of the decompositions of $T$ and $Y$, we get $T = - \frac{1}{a_2} I_{n-p}$ and $Y = Y_2 U_1^{-1}$. Since $Y \neq 0$, then $Y_2 \neq 0$, hence (2.20) leads to $\alpha (a_1 + a_2) = 1$. Setting $V = U$ allows us to prove the case (i.1).

(b) If $q = n - p$, in view of the decompositions of $T$ and $Y$, we get $T = \frac{1}{a_2} I_{n-p}$ and $Y = Y_1 U_1^{-1}$. Since $Y \neq 0$, then $Y_1 \neq 0$, hence (2.20) leads to $\alpha (a_1 + a_2) = -1$. Setting $V = U$ allows us to prove the case (ii.1).

(c) If $0 \neq q \neq n - p$, since $Y \neq 0$, then $Y_1 \neq 0$ or $Y_2 \neq 0$.

(c.i) If $Y_1 \neq 0$, then (2.20) implies $\alpha (a_1 + a_2) = -1$ and $Y_2 = 0$. Also we have

\[
A_2 = U \left( \begin{array}{cc} \alpha I_p & 0 \\ 0 & a_2^{-1} U_1 (I_q \oplus -I_{n-p-q}) U_1^{-1} \end{array} \right) U^{-1}
\]

\[
= U \left( \begin{array}{cc} I_p & 0 \\ 0 & U_1 \end{array} \right) \left( \begin{array}{cc} \alpha I_p & (Y_1 I_q) \\ 0 & a_2^{-1} (I_q \oplus -I_{n-p-q}) \end{array} \right) \left( \begin{array}{cc} I_p & 0 \\ 0 & U_1^{-1} \end{array} \right) U^{-1}.
\]

Setting $V = U (I_p \oplus U_1)$ and renaming $Y = Y_1$ permit obtain the case (ii.2).

(c.ii) If $Y_2 \neq 0$, then again (2.20) yields $\alpha (a_1 + a_2) = 1$ and $Y_1 = 0$. As before, we have

\[
A_2 = U \left( \begin{array}{cc} \alpha I_p & 0 \\ 0 & a_2^{-1} U_1 (I_q \oplus -I_{n-p-q}) U_1^{-1} \end{array} \right) U^{-1}
\]

\[
= U \left( \begin{array}{cc} I_p & 0 \\ 0 & U_1 \end{array} \right) \left( \begin{array}{cc} \alpha I_p & (0 Y_2) \\ 0 & a_2^{-1} (I_q \oplus -I_{n-p-q}) \end{array} \right) \left( \begin{array}{cc} I_p & 0 \\ 0 & U_1^{-1} \end{array} \right) U^{-1}.
\]

Setting $V = U (I_p \oplus U_1)$ and renaming $Y = Y_2$ permit obtain the case (i.2).

The proof is finished. \( \square \)

**Remark 2.11.** Observe that under the hypothesis of the above theorem, finding the spectrum of $A_2$ is simple since $A_2$ is a triangular matrix.

**Example 2.12.** Let

\[
A_1 = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right), \quad A_2 = \left( \begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array} \right).
\]

We will find all $a_1, a_2 \in \mathbb{C}^*$ such that $a_1 A_1 + a_2 A_2$ is involutive.
First of all, we check that $A_1$ and $A_2$ satisfy the hypotheses of Theorem 2.10. It is simple to see that $A_1^2 = 2A_1$; and therefore $A_1$ is a $\{2, 0\}$-quadratic matrix and $A_1^\# = \frac{1}{2}A_1$. Now, checking $A_2 A_1^\# A_1 = A_1$ and $A_1 A_2 \ne A_2 A_1$ is straightforward.

$A_1$ is a $\{2, 0\}$-quadratic matrix. Hence $a_1 + a_2 \in \{-1/2, 1/2\}$. The matrix $A_2$ is a triangular matrix; so, it is clear that $\sigma(A_2) = \{1, 2\}$. Then characteristics (i.2) and (ii.2) of Theorem 2.10 are impossible. Characteristic (i.1) leads to $-1/a_2 = 1$ and $a_1 + a_2 = 1/2$; which yields $(a_1, a_2) = (3/2, -1)$. Characteristic (ii.1) implies $(a_1, a_2) = (-3/2, 1)$. Both of these lead to $A^2 = I_2$. Observe that this reasoning is independent on the size of the involved matrices (we have chosen $2 \times 2$ matrices for the sake of the readability).

**Example 2.13.** Let $A_1$ be as in the previous example. We will find all matrices $A_2 \in \mathbb{C}^{2 \times 2}$ and $a_1 \in \mathbb{C}^*$ such that $A_2 A_1^\# A_1 = A_1$ and $a_1 A_1 + A_2$ is involutive.

We have

$$A_1 = V \text{diag}(2, 0)V^{-1}, \quad V = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

If $A_2 A_1 = A_1 A_2$, then by the proof of Theorem 2.8, we obtain $A_2 = V \text{diag}(2, t)V^{-1}$ with $t \in \{-1, 1\}$. Also, Theorem 2.8 implies $1 = [2(a_1 + 1)]^2$, that is to say $a_1 \in \{-1/2, -3/2\}$.

If $A_2 A_1 \ne A_1 A_2$, then by Theorem 2.10 there exists $y \in \mathbb{C}^*$ such that $2(a_1 + 1) = 1, A_2 = V \begin{pmatrix} 3 & y \\ 0 & -1 \end{pmatrix} V^{-1}$ or $2(a_1 + 1) = -1, A_2 = V \begin{pmatrix} 0 & y \\ 3 & 1 \end{pmatrix} V^{-1}$. I.e.,

$$a_1 = -\frac{1}{2}, \quad A_2 = \frac{1}{2} \begin{pmatrix} 1 - y & 3 + y \\ 3 - y & 1 + y \end{pmatrix},$$

or

$$a_1 = \frac{3}{2}, \quad A_2 = \frac{1}{2} \begin{pmatrix} 3 - y & 1 + y \\ 1 - y & 3 + y \end{pmatrix}.$$ 

Next results concern with linear combinations of the form (1.1), when $A_1$ is tripotent (i.e. $A_1^2 = A_1$). If $A_1$ is tripotent, then Theorem 2.1 of [6] implies the existence of a nonsingular $U \in \mathbb{C}^{n \times n}$ such that

$$A_1 = U(I_s \oplus -I_t \oplus 0)U^{-1}, \quad (2.21)$$

where $s, t \in \{0, 1, \ldots, n\}$ and $s + t = \text{rank}(A_1)$. It is evident that if $t = 0$, then $A_1$ is idempotent. Also, it would be clear that if $s = 0$, then $-A_1$ is idempotent. In next result we impose the hypothesis $A_1^2 \ne \pm A_1$ since $A_1^2 \ne \pm A_1$ were covered in Theorem 2.1.

**Theorem 2.14.** Let $A_1, A_2 \in \mathbb{C}^{n \times n}$ be two linearly independent matrices such that $A_1^2 = A_1, A_1^2 \ne \pm A_1, A_1 A_2 A_1 = A_2 A_1$ and let $A$ be a linear combination of the form (1.1) with $a_1, a_2 \in \mathbb{C}^*$. Then $A^2 = I_n$ if and only if there exists a nonsingular matrix $V \in \mathbb{C}^{n \times n}$ such that

$$A_1 = V(I_k \oplus I_l \oplus -I_m \oplus -I_j \oplus 0 \oplus 0)V^{-1}, \quad (2.22)$$

and one of the following conditions holds.
\[(2.23)\quad A_2 = V \begin{pmatrix}
0 & 0 & Y_1 & Y_2 & \frac{-a_2}{2}(Y_1Z_1 + Y_2Z_2) & W_2 \\
0 & -\frac{2}{a_2}I_l & 0 & 0 & W_3 & 0 \\
0 & 0 & 0 & 0 & Z_1 & 0 \\
0 & 0 & 0 & 0 & Z_2 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{a^2}I_e & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{a^2}I_f \\
\end{pmatrix} V^{-1},\]

where \( Y_1 \in \mathbb{C}^{k \times m}, Y_2 \in \mathbb{C}^{k \times j}, Z_1 \in \mathbb{C}^{m \times e}, Z_3 \in \mathbb{C}^{j \times e}, W_2 \in \mathbb{C}^{k \times f}, W_3 \in \mathbb{C}^{l \times e}\) and \(k, l, m, j, e, f\) are nonnegative integers;

\[(2.24)\quad A_2 = V \begin{pmatrix}
\frac{a_2}{2}I_k & 0 & 0 & 0 & 0 & W_2 \\
0 & 0 & Y_1 & Y_2 & W_3 & \frac{-a_2}{2}(Y_1Z_1 + Y_2Z_2) \\
0 & 0 & 0 & 0 & 0 & Z_1 \\
0 & 0 & 0 & 0 & 0 & Z_2 \\
0 & 0 & 0 & 0 & \frac{1}{a^2}I_e & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{a^2}I_f \\
\end{pmatrix} V^{-1},\]

where \( Y_1 \in \mathbb{C}^{l \times m}, Y_2 \in \mathbb{C}^{l \times j}, Z_1 \in \mathbb{C}^{m \times f}, Z_2 \in \mathbb{C}^{j \times f}, W_2 \in \mathbb{C}^{k \times f}, W_3 \in \mathbb{C}^{l \times e}\) and \(k, l, m, j, e, f\) are nonnegative integers.

**Proof.** Since \( A_1^3 = A_1 \) and \( A_1^2 \neq \pm A_1 \), there exist a nonsingular matrix \( U \in \mathbb{C}^{n \times n} \) and \( s, t \in \{1, \ldots, n-1\} \) such that \( A_1 \) is written as in (2.21). Let us write

\[A_2 = A_1^2 = U \begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33} \\
\end{pmatrix} U^{-1},\]

where \( X_{11} \in \mathbb{C}^{s \times s} \) and \( X_{22} \in \mathbb{C}^{t \times t} \). Since \( A_1A_2A_1 = A_2A_1 \), we get \( X_{21} = 0, X_{22} = 0, X_{31} = 0, X_{32} = 0 \) which imply

\[A_2 = U \begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
0 & 0 & X_{23} \\
0 & 0 & X_{33} \\
\end{pmatrix} U^{-1}.\]

Hence,

\[A = a_1A_1 + a_2A_2 = U \begin{pmatrix}
a_1I_s + a_2X_{11} & a_2X_{12} & a_2X_{13} \\
0 & -a_1I_t & a_2X_{23} \\
0 & 0 & a_2X_{33} \\
\end{pmatrix} U^{-1}.\]

Since \( A^2 = I_n \), we conclude that

\[(2.25)\quad (a_1I_s + a_2X_{11})^2 = I_s,\]

\[(2.26)\quad a^2I_t = I_t,\]

\[(2.27)\quad (a_2X_{33})^2 = I_{n-s-t},\]

\[(2.28)\quad X_{11}X_{12} = 0,\]

\[(2.29)\quad -a_1X_{23} + a_2X_{23}X_{33} = 0,\]
(2.30) \[ a_1X_{13} + a_2X_{11}X_{13} + a_2X_{12}X_{23} + a_2X_{13}X_{33} = 0. \]

Since \( t > 0 \), by (2.26), we have \( a_1 = 1 \) or \( a_1 = -1 \).

If \( a_1 = 1 \), by (2.25), there exists a nonsingular matrix \( V_1 \in \mathbb{C}^{s \times s} \) such that \( I_s + a_2X_{11} = V_1(I_k \oplus -I_j)V_1^{-1} \), hence,

(2.31) \[ X_{11} = V_1 \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{a_2}I_t \end{pmatrix} V_1^{-1}. \]

By (2.27), there exists a nonsingular \( V_3 \in \mathbb{C}^{(n-s-t) \times (n-s-t)} \) such that \( a_2X_{33} = V_3(I_o \oplus -I_f)V_3^{-1} \), hence,

(2.32) \[ X_{33} = V_3 \begin{pmatrix} \frac{1}{a_2}I_e & 0 \\ 0 & -\frac{1}{a_2}I_f \end{pmatrix} V_3^{-1}. \]

Let us write \( X_{12} \) and \( X_{23} \) as follows

(2.33) \[ X_{12} = V_1 \begin{pmatrix} P \\ Q \end{pmatrix} \quad \text{and} \quad X_{23} = \begin{pmatrix} R & S \end{pmatrix} V_3^{-1}, \]

where \( P \in \mathbb{C}^{k \times s} \) and \( R \in \mathbb{C}^{(n-s-t) \times e} \). By (2.28), (2.31), and the first equality of (2.33), we get \( Q = 0 \). By (2.29), (2.32), and the second equality of (2.33), we get \( S = 0 \). Hence \( X_{12} \) and \( X_{23} \) can be rewritten as

(2.34) \[ X_{12} = V_1 \begin{pmatrix} Y_1 & Y_2 \\ 0 & 0 \end{pmatrix} V_2^{-1} \quad \text{and} \quad X_{23} = V_2 \begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix} V_3^{-1}, \]

where \( Y_1 \in \mathbb{C}^{k \times m} \), \( Y_2 \in \mathbb{C}^{k \times j} \), \( Z_1 \in \mathbb{C}^{m \times e} \), \( Z_2 \in \mathbb{C}^{j \times e} \). Let us write \( X_{13} = V_1 \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} V_3^{-1} \), where \( W_1 \in \mathbb{C}^{k \times e} \). By (2.30), we have \( W_1 = -\frac{a_2}{2}(Y_1Z_1 + Y_2Z_2) \) and \( W_4 = 0 \), so

(2.35) \[ X_{13} = V_1 \begin{pmatrix} -\frac{a_2}{2}(Y_1Z_1 + Y_2Z_2) & W_2 \\ W_3 & 0 \end{pmatrix} V_3^{-1}. \]

From (2.31), (2.32), (2.34), and (2.35), it follows that \( A_1 \) can be written as in (2.22) and \( A_2 \) as in (2.23) where \( V = U(V_1 \oplus I_t \oplus V_3) \).

If \( a_1 = -1 \), using the same method, it is easy to verify that \( A_1 \) can be written as in (2.22) and \( A_2 \) as in (2.24).

Next result deal with another condition.

**Theorem 2.15.** Let \( A_1, A_2 \in \mathbb{C}^{n \times n} \) be linearly independent matrices. Moreover, let \( A \) be a linear combination of the form (1.1) with \( a_1, a_2 \in \mathbb{C}^* \). If \( A_1^2 = A_1 \) and \( A_1 \neq A_2 \), then \( A^2 = I_n \) if and only if there exist a nonsingular matrix \( V \in \mathbb{C}^{n \times n}, p, q \in \{0, \ldots, n\}, \) and \( r \in \{0, \ldots, n - p - q\} \) such that \( a_1 + a_2 \in \{-1, 1\}, \)

\[ A_1 = V(I_p \oplus -I_q \oplus 0 \oplus 0)V^{-1}, \]

and

\[ A_2 = V \left( I_p \oplus -I_q \oplus \frac{1}{a_2}I_r \oplus -\frac{1}{a_2}I_{n-r-p-q} \right) V^{-1}. \]
Proof. The ‘if’ part is evident. We shall prove the ‘only’ part: Since $A_1 \neq A_2$, then $A_1 A_1^\# = A_2 A_2^\#$ and $A_1^\# A_1 = A_2^\# A_2$, which by pre and postmultiplying by $A_2^2$, we get $A_1 A_2 = A_2 A_1 = A_1^2$. Since $A_1^2 = A_1$, there exists a unitary matrix $U$ such that $A_1 = U(P \oplus 0)U^{-1}$, where $P = I_p \oplus -I_q$ and $p, q \in \{0, \ldots, n\}$. Observe that $p + q \neq 0$, since otherwise $A_1 = 0$. By using $A_1 A_2 = A_2 A_1 = A_1^2$, we deduce the existence of $T \in \mathbb{C}^{(n-p-q) \times (n-p-q)}$ such that $A_2 = U(P \oplus T)U^{-1}$. Since $A^2 = I_n$, we get that $1 = (a_1 + a_2)^2$ and $(a_2 T)^2 = I_{n-p-q}$. The proof finishes as in the proof of Theorem 2.8. □

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