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INFINITELY MANY SOLUTIONS FOR A BI-NONLOCAL EQUATION WITH SIGN-CHANGING WEIGHT FUNCTIONS

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Abstract. In this paper, we investigate the existence of infinitely many solutions for a bi-nonlocal equation with sign-changing weight functions. We use some natural constraints and the Ljusternik-Schnirelman critical point theory on C^1-manifolds, to prove our main results.
Keywords: Infinitely many solutions, Nehari manifold, sign-changing weight function, bi-nonlocal equation.
MSC(2010): Primary: 35J25; Secondary: 35M99.

1. Introduction

In this paper, we study the existence of infinitely many solutions for the bi-nonlocal problem

$$(1.1)\qquad \begin{cases} -M(\int_\Omega |\nabla u|^p dx)\triangle_p u = \lambda f(x)|u|^{q-2}u & \\ +g(x)|u|^{\gamma-2}u\big[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\big]^{2r}, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary,

$$\triangle_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

is the p-Laplacian operator, $1<p<N$, $M(s)=s^\alpha$ with $\alpha>0$, $1<q<p^*:=\frac{Np}{N-p}$, $\max\{p(\alpha+1),q\}<\gamma(2r+1)<p^*$ and r,λ are positive parameters, and the weight functions $f,g\in C(\overline{\Omega})$ may change sign on Ω.
Nonlocal problems arise in many applications. In [16], we can find a nonlocal equation arising in population dynamic models (see also [1, 2, 11, 12] and references therein). In this field we can mention the equation

$$\Delta u^m = uf\big(x,\int_\Omega u^r\big),$$

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where f denotes the crowding effect and the nonlocal term means that the crowding effect depends on the entire population. Equations of the form

$$u_{tt} - \left(a \int_\Omega |\nabla u|^2 dx + b \right) \Delta u = f(x,u),$$

were introduced by Kirchhoff [18] to describe the transversal oscillations of a stretched string. Motivated by mathematical difficulties caused by the nonlocal term and physical applications of Kirchhoff type equations, many researchers studied these kind of equations. See, for example, [2–4, 7–9, 15, 17, 22] and the references therein. The nonlocal elliptic problem

$$-\text{div}\left[a\left(\int_\Omega u \right) \nabla u \right] = f - \lambda u, \text{ in } \Omega,$$

describes a balance of population for some species of bacteria where Ω has been considered as a container of bacteria, u is the density of bacteria within this container and f denotes the supply of beings by external sources. For more details about this problem see [10] and references cited therein.

Corrêa and Figueiredo [13] considered problem (1.1) with $\lambda = 0$ or 1, $f(x) = 1$, $g(x) = \mu g_1(x)$ where g_1 is sign-changing and μ is a positive parameter. They studied several cases: $\gamma < \frac{p(\alpha+1)}{2r+1}$, $\gamma > \frac{p(\alpha+1)}{2r+1}$, $p(\alpha+1) < q \le 2_*$ and $p-1 < q < p(\alpha+1)$. Corrêa and Figueiredo used the Krasnolselskii genus, the mountain-pass theorem and the concentration-compactness principle, to prove the existence and multiplicity of solutions. In this paper, we consider problem (1.1) with two sign-changing weight functions and under conditions different from those used in [13]. We use the Nehari manifold method and the Ljusternik-Schnirelman critical point theory on C^1-manifolds, to prove the existence and multiplicity of solutions. When $q < p(\alpha+1)$ the Nehari manifold need not be a closed manifold of class C^1, and therefore the Ljusternik-Schnirelman critical point theory does not apply. To overcome this difficulty we divide the Nehari manifold into three subsets corresponding to local minima, local maxima and points of inflexion of fibrering maps (see [5]). Then we prove that the subset related to local maxima of fibrering maps is a closed C^1-manifold.

Throughout this paper, we assume that $f^+ \neq 0$ an $g^+ \neq 0$ where

$$f^+ = \max\{f, 0\}, \ g^+ = \max\{g, 0\}.$$

Let

$$\Omega_f^+ := \{x \in \Omega : f(x) > 0\}, \ \Omega_g^+ := \{x \in \Omega : g(x) > 0\}.$$

Our main results are as follows:

Theorem 1.1. *Assume that $q < p(\alpha+1)$. Then there exists $\bar{\lambda} > 0$ such that for each $0 < \lambda < \bar{\lambda}$,*

(*i*) *problem (1.1) has at least two nontrivial solutions* (*ii*) *if* $\Omega_g^+ \setminus \Omega_f^+$ *has nonempty interior, then problem* (1.1) *has infinitely many solutions.*

Theorem 1.2. *Assume that* $p(\alpha+1) < q$ *and* $g \geq 0$ *on* Ω*. Then problem* (1.1) *has infinitely many solutions for each* $\lambda > 0$*.*

In section 2, we prove Theorem 1.1 and the proof of Theorem 1.2 is given in section 3.

2. **The case** $q < p(\alpha + 1)$

Consider the space $X := W_0^{1,p}(\Omega)$ endowed with the norm

$$||u|| := \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

and denote the norm in $L^t(\Omega)$ for $t \geq 1$ by $||u||_t$. Also, the best Sobolev constant of embedding $X \hookrightarrow L^t(\Omega)$ with $1 \leq t < p^*$ is denoted by S_t. The corresponding energy functional of problem (1.1) is defined on X by

$$J_\lambda(u) := \frac{1}{p}\hat{M}(||u||^p) - \frac{\lambda}{q}\int_\Omega f(x)|u|^q dx - \frac{1}{(2r+1)\gamma}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1},$$

where $\hat{M}(t) = \int_0^t M(s)ds$. It is easy to verify that $J_\lambda \in C^1(X, \mathbb{R})$ and critical points of J_λ are weak solutions of problem (1.1).
The Nehari manifold associated with the functional J_λ is defined as

$$N_\lambda := \{u \in X \setminus \{0\} : \langle J_\lambda'(u), u\rangle = 0\},$$

where $\langle\ ,\rangle$ denotes the usual duality pairing between X^* and X. Then $u \in X \setminus \{0\}$ is in N_λ if and only if

$$||u||^p M(||u||^p) - \lambda\int_\Omega f(x)|u|^q dx - \left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1} = 0. \tag{2.1}$$

Let $u \in X \setminus \{0\}$, then the fibering map [14] corresponding to u is defined by $\beta_{u,\lambda}(t) = J_\lambda(tu), t \in \mathbb{R}^+$. According to the definition of $\beta_{u,\lambda}$ we have

$$\begin{aligned}
\beta_{u,\lambda}(t) &= \frac{1}{p}\hat{M}(t^p||u||^p) - \frac{\lambda t^q}{q}\int_\Omega f(x)|u|^q dx \\
&\quad - \frac{t^{(2r+1)\gamma}}{(2r+1)\gamma}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}, \\
\beta_{u,\lambda}'(t) &= t^{p-1}||u||^p M(t^p||u||^p) \\
&\quad - \lambda t^{q-1}\int_\Omega f(x)|u|^q dx - t^{(2r+1)\gamma-1}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1},
\end{aligned}$$

$$\begin{aligned}\beta''_{u,\lambda}(t) &= (p-1)t^{p-2}||u||^p M(t^p||u||^p) + pt^{2(p-1)}||u||^{2p} M'(t^p||u||^p)\\ &\quad - (q-1)\lambda t^{q-2}\int_\Omega f(x)|u|^q dx\\ &\quad - ((2r+1)\gamma-1)t^{(2r+1)\gamma-2}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1},\end{aligned}$$

and for $t > 0$, $tu \in N_\lambda$ if and only if $\beta'_{u,\lambda}(t) = 0$. Now we split the Nehari manifold according to the positive critical points of fibring maps into three parts as follows:

$$\begin{aligned}N_\lambda^+ &= \{u \in N_\lambda : \beta''_{u,\lambda}(1) > 0\},\\ N_\lambda^0 &= \{u \in N_\lambda : \beta''_{u,\lambda}(1) = 0\},\\ N_\lambda^- &= \{u \in N_\lambda : \beta''_{u,\lambda}(1) < 0\},\end{aligned}$$

In the following we prove some properties of J_λ on the Nehari manifold N_λ. Arguing as in Brown and Zhang [5] we have the following result about the local minimizers on the Nehari manifold.

Lemma 2.1. *Suppose that u_0 is a local minimizer for J_λ on N_λ and that $u_0 \notin N_\lambda^0$. Then $J'_\lambda(u_0) = 0$.*

The next result is about the Palais-Smal condition. A functional $J \in C^1(E)$ is said to satisfy the Palais-Smale condition at the level $c \in \mathbb{R}$, (the $(PS)_c$ condition in short) if any sequence $\{u_n\} \subset E$ such that

$$J(u_n) \to c, \qquad J'(u_n) \to 0, \tag{2.2}$$

admits a convergent subsequence. Any sequence satisfying (2.2) is called a $(PS)_c$ sequence.

Lemma 2.2. *Each bounded $(PS)_c$ sequence for J_λ on X has a convergent subsequence.*

Proof. The proof is similar to the proof of Lemma 3.2 in [13] and we omit it. □

By Lemma 2.2 and the following lemma we get that J_λ satisfies the Palais-Smale condition on X.

Lemma 2.3. *The functional J_λ is coercive and bounded from below on N_λ.*

Proof. By (2.1) and the Sobolev inequality for any $u \in N_\lambda$ we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\hat{M}(||u||^p) - \frac{1}{(2r+1)\gamma}||u||^p M(||u||^p) \\ &\quad - \lambda(\frac{1}{q} - \frac{1}{(2r+1)\gamma})\int_\Omega f(x)|u|^q dx \\ &\geq (\frac{1}{p(\alpha+1)} - \frac{1}{(2r+1)\gamma})||u||^{p(\alpha+1)} \\ &\quad - \lambda(\frac{1}{q} - \frac{1}{(2r+1)\gamma})||f||_\infty S_q^{\frac{-q}{p}}||u||^q. \end{aligned}$$

Since $q < p(\alpha+1)$, J_λ is coercive and bounded from below on N_λ. □

Lemma 2.4. *There exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, $N_\lambda^0 = \emptyset$.*

Proof. By the definition, if $u \in N_\lambda^0$, then we have

$$\begin{aligned} \beta''_{u,\lambda}(1) &= (p-1)||u||^p M(||u||^p) + p||u||^{2p} M'(||u||^p) \\ &\quad - (q-1)\lambda\int_\Omega f(x)|u|^q dx - ((2r+1)\gamma - 1)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1} \\ &= \big(p(\alpha+1) - q\big)||u||^{p(\alpha+1)} - \big((2r+1)\gamma - q\big)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1} \\ &= \big(p(\alpha+1) - (2r+1)\gamma\big)||u||^{p(\alpha+1)} \\ &\quad + \big((2r+1)\gamma - q\big)\lambda\int_\Omega f(x)|u|^q dx \\ &= \big(p(\alpha+1) - q\big)\lambda\int_\Omega f(x)|u|^q dx \\ &\quad + \big(p(\alpha+1) - (2r+1)\gamma\big)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1} = 0. \end{aligned}$$

Consequently, for any $u \in N_\lambda^0$ we have

$$||u||^{p(\alpha+1)} = \frac{(2r+1)\gamma - q}{p(\alpha+1) - q}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}, \tag{2.3}$$

$$\lambda\int_\Omega f(x)|u|^q dx = \frac{(2r+1)\gamma - p(\alpha+1)}{p(\alpha+1) - q}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}, \tag{2.4}$$

and

$$\begin{aligned} \big((2r+1)\gamma - p(\alpha+1)\big)||u||^{p(\alpha+1)} &= \big((2r+1)\gamma - q\big)\lambda\int_\Omega f(x)|u|^q dx, \\ &\leq \big((2r+1)\gamma - q\big)\lambda||f||_\infty S_q^{\frac{-q}{p}}||u||^q. \end{aligned}$$

Hence

$$\|u\| \le \left(\frac{(2r+1)\gamma - q}{(2r+1)\gamma - p(\alpha+1)} \lambda \|f\|_\infty S_q^{\frac{-q}{p}} \right)^{\frac{1}{p(\alpha+1)-q}}. \tag{2.5}$$

By (2.3) for $u \in N_\lambda^0$ we get that $\int_\Omega g(x)|u|^\gamma dx > 0$ and by the Sobolev inequality we obtain

$$\frac{\|u\|^\gamma}{\int_\Omega g(x)|u|^\gamma dx} \ge \frac{S_\gamma^{\frac{\gamma}{p}}}{\|g\|_\infty}. \tag{2.6}$$

Let $M_\lambda := \{u \in N_\lambda : \int_\Omega g(x)|u|^\gamma dx > 0\}$ and define the function $\eta_\lambda : M_\lambda \to \mathbb{R}$ by

$$\eta_\lambda(u) = C_* \left(\frac{\|u\|^{p(\alpha+1)[\frac{\gamma(2r+1)-1}{p(\alpha+1)-1}]}}{\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}} \right)^{\frac{1}{\frac{\gamma(2r+1)-1}{p(\alpha+1)-1}-1}} - \lambda \int_\Omega f(x)|u|^q dx,$$

where

$$C_* = C(p,q,r,\gamma,\alpha) := \frac{(2r+1)\gamma - p(\alpha+1)}{p(\alpha+1)-q} \left(\frac{(2r+1)\gamma - q}{p(\alpha+1)-q} \right)^{-\frac{\gamma(2r+1)-1}{\gamma(2r+1)-p(\alpha+1)}}.$$

By (2.5) and (2.6), for $u \in N_\lambda^0$ we get

$$\begin{aligned} \eta_\lambda(u) &= C_* \left(\frac{\left(\frac{(2r+1)\gamma - q}{p(\alpha+1)-q} \left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{(2r+1)} \right)^{\frac{\gamma(2r+1)-1}{p(\alpha+1)-1}}}{\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}} \right)^{\frac{1}{\frac{\gamma(2r+1)-1}{p(\alpha+1)-1}-1}} \\ &\quad - \lambda \int_\Omega f(x)|u|^q dx \\ &= \frac{(2r+1)\gamma - p(\alpha+1)}{p(\alpha+1)-q} \left[\frac{1}{\gamma} \int_\Omega g(x)|u|^\gamma dx \right]^{2r+1} \\ &\quad - \lambda \int_\Omega f(x)|u|^q dx = 0. \end{aligned} \tag{2.7}$$

Using the Sobolev inequity, (2.5) and (2.6), for $u \in N_\lambda^0$ we obtain

$$\begin{aligned} \eta_\lambda(u) &\ge C_* \left(\frac{\|u\|^{\gamma(2r+1)}}{\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}} \right)^{\frac{1}{\frac{\gamma(2r+1)-1}{p(\alpha+1)-1}-1}} \|u\| - \lambda \|f\|_\infty S_q^{\frac{-q}{p}} \|u\|^q \\ &\ge \|u\|^q \left[C_* \left(\frac{\gamma S_\gamma^{\frac{\gamma}{p}}}{\|g\|_\infty} \right)^{\frac{(2r+1)(p(\alpha+1)-1)}{\gamma(2r+1)-p(\alpha+1)}} \frac{1}{\|u\|^{q-1}} - \lambda \|f\|_\infty S_q^{\frac{-q}{p}} \right] \\ &\ge \|u\|^q \left[\frac{C_* \left(\frac{\gamma S_\gamma^{\frac{\gamma}{p}}}{\|g\|_\infty} \right)^{\frac{(2r+1)(p(\alpha+1)-1)}{\gamma(2r+1)-p(\alpha+1)}}}{\lambda^{\frac{q-1}{p(\alpha+1)-q}} \left(\frac{(2r+1)\gamma-q}{(2r+1)\gamma-p(\alpha+1)} \|f\|_\infty S_q^{\frac{-q}{p}} \right)^{\frac{q-1}{p(\alpha+1)-q}}} - \lambda \|f\|_\infty S_q^{\frac{-q}{p}} \right]. \end{aligned}$$

Now, assume that there exists a sequence $\{\lambda_n\}$ in $(0,\infty)$ such that $\lambda_n \to 0$ and $N^0_{\lambda_n} \neq \emptyset$ for all $n \in \mathbb{N}$. Then, since $\frac{q-1}{p(\alpha+1)-q} > 0$, we should have $\eta_{\lambda_n}(u) > 0$ for all $u \in N^0_{\lambda_n}$ and n large, which contradicts (2.7). Consequently, there exists $\lambda_0 > 0$ such that $N^0_\lambda = \emptyset$ for all $\lambda \in (0,\lambda_0)$. □ □

Let

$$\lambda_1 = \left(\frac{p(\alpha+1)-q}{\left(\frac{||g||_\infty}{\gamma S_\gamma^{\frac{\gamma}{p}}}\right)^{2r+1}\big(\gamma(2r+1)-q\big)}\right)^{\frac{1}{\gamma(2r+1)-p(\alpha+1)}} \frac{\gamma(2r+1)-p(\alpha+1)}{\big(\gamma(2r+1)-q\big)||f||_\infty S_q^{\frac{-q}{p}}}.$$

Proposition 2.5. *Suppose that $0 < \lambda < \min\{\lambda_0, \lambda_1\}$ where λ_0 comes from Lemma 2.4. For $u \in X$ we have:*

(i) If $\int_\Omega g(x)|u|^\gamma dx > 0$ and $\int_\Omega f(x)|u|^q dx \leq 0$, then the equation $\beta'_{u,\lambda}(t) = 0$ has a unique positive solution t_u^- such that $t_u^- u \in N_\lambda^-$.

(ii) If $\int_\Omega g(x)|u|^\gamma dx > 0$ and $\int_\Omega f(x)|u|^q dx > 0$, then the equation $\beta'_{u,\lambda}(t) = 0$ has exactly two positive solutions t_u^- and t_u^+ with $t_u^+ < t_u^-$ such that $t_u^\pm u \in N_\lambda^\pm$.

(iii) If $\int_\Omega g(x)|u|^\gamma dx \leq 0$ and $\int_\Omega f(x)|u|^q dx > 0$, then the equation $\beta'_{u,\lambda}(t) = 0$ has exactly one positive solution t_u^+ such that $t_u^+ u \in N_\lambda^+$.

Proof. (i) Since $\int_\Omega f(x)|u|^q dx \leq 0$ and $p(\alpha+1) < \gamma(2r+1)$, there exists $t_0 > 0$ small enough such that $\beta'_{u,\lambda}(t_0) > 0$. From the condition $\int_\Omega g(x)|u|^\gamma dx > 0$ we have $\beta'_{u,\lambda}(t) \to -\infty$ as $t \to \infty$. Then $\beta'_{u,\lambda}(t) = 0$ has at least one solution in $(0,\infty)$. Now suppose that there exist $t_1, t_2 \in (0,\infty)$ such that $\beta'_{\lambda,u}(t_1) = \beta'_{\lambda,u}(t_2) = 0$. Then we have

$$(2.8)\quad \begin{cases} t_1^{p(\alpha+1)}||u||^{p(\alpha+1)} = & t_1^q \lambda \int_\Omega f(x)|u|^q dx \\ & +t_1^{(2r+1)\gamma}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}, \\ t_2^{p(\alpha+1)}||u||^{p(\alpha+1)} = & t_2^q \lambda \int_\Omega f(x)|u|^q dx \\ & +t_2^{(2r+1)\gamma}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}, \end{cases}$$

hence

$$(2.9)\quad \begin{aligned} &(t_1t_2)^{p(\alpha+1)-q}(t_2^{(2r+1)\gamma-p(\alpha+1)} - t_1^{(2r+1)\gamma-p(\alpha+1)})||u||^{p(\alpha+1)} \\ &= (t_2^{(2r+1)\gamma-q} - t_1^{(2r+1)\gamma-q})\lambda\int_\Omega f(x)|u|^q dx. \end{aligned}$$

Since $\int_\Omega f(x)|u|^q dx \leq 0$, (2.9) implies that $t_1 = t_2$. Then there is a unique $t_u^- > 0$ such that $t_u^- u \in N_\lambda$ and so

$$||t_u^- u||^{p(\alpha+1)} - \lambda\int_\Omega f(x)|t_u^- u|^q dx = \left[\frac{1}{\gamma}\int_\Omega g(x)|t_u^- u|^\gamma dx\right]^{2r+1}.$$

This implies that

$$\begin{aligned}\beta''_{t_u^- u,\lambda}(1) &= \big(p(\alpha+1)-\gamma(2r+1)\big)||t_u^- u||^{p(\alpha+1)} \\ &\quad +\big(\gamma(2r+1)-q\big)\int_\Omega f(x)|t_u^- u|^q dx < 0.\end{aligned}$$

Consequently $t_u^- u \in N_\lambda^-$.

(ii) Consider $u \in X$ such that assumptions appeared in Proposition 2.5 *(ii)* hold. Without loss of generality we can assume that $||u|| = 1$. Let

$$h_1(t) := t^{p(\alpha+1)-q} - \lambda \int_\Omega f(x)|u|^q dx - t^{\gamma(2r+1)-q}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}.$$

Then $\beta'_{u,\lambda}(t) = t^{q-1}h_1(t)$. It is easy to see that the equation $\beta'_{u,\lambda}(t) = 0$ has at most two positive solutions. Using the assumptions $q < p(\alpha+1)$ and $\int_\Omega f(x)|u|^q dx > 0$, we have $\beta'_{u,\lambda}(t) < 0$ for t small enough. Also from the conditions $p(\alpha+1) < \gamma(2r+1)$ and $\int_\Omega g(x)|u|^\gamma dx > 0$ we have $\lim_{t\to\infty}\beta'_{u,\lambda}(t) = -\infty$. Let us to show that the equation $\beta'_{u,\lambda}(t) = 0$ has at least two positive solutions. To this end, it is enough to prove that there exists $s > 0$ such that $\beta'_{u,\lambda}(s) > 0$. Define

$$h(t) := t^{p(\alpha+1)-q} - \lambda||f||_\infty S_q^{-\frac{q}{p}} - \big(\frac{||g||_\infty}{\gamma S_\gamma^{\frac{\gamma}{p}}}\big)^{2r+1} t^{\gamma(2r+1)-q}$$

Then $h(t)$ attains its maximum at

$$t_{max} = \left(\frac{p(\alpha+1)-q}{\big(\frac{||g||_\infty}{\gamma S_\gamma^{\frac{\gamma}{p}}}\big)^{2r+1}\big(\gamma(2r+1)-q\big)}\right)^{\frac{1}{\gamma(2r+1)-p(\alpha+1)}},$$

and since $\lambda < \lambda_1$, $h(t_{max}) > 0$. Thus by the Sobolev inequality we have $\beta'_{u,\lambda}(t_{max}) = t_{max}^{q-1}h_1(t_{max}) \geq t_{max}^{q-1}h(t_{max}) > 0$. Then $\beta'_{u,\lambda}(t) = 0$ has exactly two positive solutions t_u^+ and t_u^- which are respectively points of a local minimum and a local maximum for $\beta_{u,\lambda}(t)$. Since $\beta_{u,\lambda}(t)$ has only two local extrema, $\beta'_{u,\lambda}(t) < 0$ for t small enough and $\lim_{t\to\infty}\beta'_{u,\lambda}(t) = -\infty$, then $t_u^+ < t_u^-$, $t_u^- u \in N_\lambda^-$ and $t_u^+ u \in N_\lambda^+$.

(iii). Since $q < p(\alpha+1)$ and in view of assumptions in Proposition 2.5 (iii), we conclude that $\beta'_{u,\lambda}(t) < 0$ for t small enough and $\lim_{t\to\infty}\beta'_{u,\lambda}(t) = +\infty$. Then $\beta'_{u,\lambda}(t) = 0$ has at least one solution in $(0,\infty)$. Similar to the proof of Proposition 2.5 (i), we can show that there exists a unique $t_u^+ > 0$ such that $t_u^+ u \in N_\lambda^+$. □

Lemma 2.6. *Suppose that $0 < \lambda < \lambda_0$. Then*

(i) N_λ^- is a closed C^1-manifold which is bounded away from zero;

(ii) If $u \neq 0$ is a critical point of $J_\lambda|_{N_\lambda^-}$ then it is a critical point of J_λ;

(iii) $J_\lambda|_{N_\lambda^-}$ satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$.

Proof. (i) Let $u \in N_\lambda^-$. Then

$$\beta''_{u,\lambda}(1) = \big(p(\alpha+1)-q\big)||u||^{p(\alpha+1)} - \big((2r+1)\gamma - q\big)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1} < 0.$$

Using the Sobolev inequality we have

$$\big(p(\alpha+1)-q\big)||u||^{p(\alpha+1)} \le \big((2r+1)\gamma - q\big) \times \left(\frac{||g||_\infty S_\gamma^{\frac{-\gamma}{p}}}{\gamma}\right)^{2r+1} ||u||^{\gamma(2r+1)}. \tag{2.10}$$

Since $p(\alpha+1) < (2r+1)\gamma$, inequality (2.10) implies that N_λ^- is bounded away from zero. In particular, for any $u \in N_\lambda^-$ we have

$$||u|| \ge \left(\frac{\big(p(\alpha+1)-q\big)\gamma^{2r+1}}{\big((2r+1)\gamma - q\big)\big(||g||_\infty S_\gamma^{\frac{-\gamma}{p}}\big)^{2r+1}}\right)^{\frac{1}{\gamma(2r+1)-p(\alpha+1)}}. \tag{2.11}$$

Now define $\varphi_\lambda : X \to \mathbb{R}$ by

$$\varphi_\lambda(u) = \langle J'_\lambda(u), u\rangle = ||u||^p M(||u||^p) - \lambda\int_\Omega f(x)|u|^q dx - \left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}. \tag{2.12}$$

Thus $\varphi_\lambda \in C^1(X,\mathbb{R})$ and $N_\lambda^- = \varphi_\lambda^{-1}(0)\setminus[\{0\}\cup N_\lambda^+]$. Since $N_\lambda^0 = \emptyset$ and N_λ^- is bounded away from zero, we have

$$\overline{N_\lambda^-} \cap [\overline{N_\lambda^+} \cup \{0\}] = \emptyset,$$

and this implies that N_λ^- is closed. By the definition of N_λ^-,

$$\langle \varphi'_\lambda(u), u\rangle = \beta''_{u,\lambda}(1) < 0,$$

for each $u \in N_\lambda^-$. Then each point of N_λ^- is regular for φ_λ and this completes the proof of Lemma 2.6 (i).
(ii) Since $N_\lambda^0 = \emptyset$ and using Lemma 2.6 (i), N_λ^- is bounded away from $N_\lambda^+ \cup \{0\}$. Then there exists an open set $A \subset X$ such that $N_\lambda^- = \{u \in A : \varphi_\lambda(u) = 0\}$. Now, let $u \in N_\lambda^-$ be a critical point of J_λ constrained to N_λ^-. Then there exists a Lagrange multiplier $\delta \in \mathbb{R}$ such that

$$\langle J'_\lambda(u), v\rangle = \delta\langle \varphi'_\lambda(u), v\rangle, \text{ for every } v \in X.$$

Since $J'_\lambda(u)|_{\mathbb{R}u} \equiv 0$ and $\langle \varphi'_\lambda(u), u\rangle = \beta''_{u,\lambda}(1) < 0$, we deduce that $\delta = 0$. Then u is a critical point of J_λ in X.
(iii) By Lemma 2.3 every constrained $(PS)_c$ sequence for J_λ is bounded. The rest of the proof is similar to the proof of Lemma 3.2 in [13] and we omit it. □

In the next lemma we prove that J_λ has different minimum energy levels on N_λ^- and N_λ^+. First we need some notations. Define

$$c_\lambda^+ = \inf_{u\in N_\lambda^+} J_\lambda(u),\ c_\lambda^- = \inf_{u\in N_\lambda^-} J_\lambda(u),$$

$$\lambda_2 = \frac{\big((2r+1)\gamma - p(\alpha+1)\big)qS_q^{\frac{q}{p}}}{\big((2r+1)\gamma - q\big)p(\alpha+1)||f||_\infty} \times \left(\frac{\big(p(\alpha+1)-q\big)\gamma^{2r+1}}{\big((2r+1)\gamma-q\big)\big(||g||_\infty S_\gamma^{\frac{-\gamma}{p}}\big)^{2r+1}}\right)^{\frac{p(\alpha+1)-q}{\gamma(2r+1)-p(\alpha+1)}}.$$

Proposition 2.7. *Assume that* $0<\lambda<\min\{\lambda_0,\lambda_1,\lambda_2\}$*. Then we have*

(*i*) $c_\lambda^+ < 0$.

(*ii*) *There exists constant* $c_0>0$*, such that* $c_\lambda^- \geq c_0$.

Proof. (i) Let $u\in N_\lambda^+$. Then $\beta''_{u,\lambda}(1)>0$ and this implies that

$$-\big((2r+1)\gamma - p(\alpha+1)\big)||u||^{p(\alpha+1)} > -\big((2r+1)\gamma-q\big)\lambda\int_\Omega f(x)|u|^q dx.$$

Then

$$\begin{aligned} J_\lambda(u) &= \Big(\frac{1}{p(\alpha+1)} - \frac{1}{\gamma(2r+1)}\Big)||u||^{p(\alpha+1)} \\ &\quad - \Big(\frac{1}{q} - \frac{1}{\gamma(2r+1)}\Big)\lambda\int_\Omega f(x)|u|^q dx \\ &\leq -\frac{\big((2r+1)\gamma - p(\alpha+1)\big)\big(p(\alpha+1)-q\big)}{(2r+1)\gamma p(\alpha+1)q}||u||^{p(\alpha+1)} < 0. \end{aligned}$$

Consequently $c_\lambda^+<0$.

(ii) Assume that $u\in N_\lambda^-$. Then we have (2.11). By the Sobolev inequality we get

$$\begin{aligned} J_\lambda(u) &\geq \Big(\frac{1}{p(\alpha+1)} - \frac{1}{\gamma(2r+1)}\Big)||u||^{p(\alpha+1)} \\ &\quad - \Big(\frac{1}{q} - \frac{1}{\gamma(2r+1)}\Big)\lambda||f||_\infty S_q^{\frac{-q}{p}}||u||^q \\ &\geq \left(\frac{\big(p(\alpha+1)-q\big)\gamma^{2r+1}}{\big((2r+1)\gamma-q\big)\big(||g||_\infty S_\gamma^{\frac{-\gamma}{p}}\big)^{2r+1}}\right)^{\frac{q}{\gamma(2r+1)-p(\alpha+1)}} \\ &\times\Bigg[\frac{(2r+1)\gamma - p(\alpha+1)}{(2r+1)\gamma p(\alpha+1)}\left(\frac{\big(p(\alpha+1)-q\big)\gamma^{2r+1}}{\big((2r+1)\gamma-q\big)\big(||g||_\infty S_\gamma^{\frac{-\gamma}{p}}\big)^{2r+1}}\right)^{\frac{p(\alpha+1)-q}{\gamma(2r+1)-p(\alpha+1)}} \\ &\quad - \lambda\Big(\frac{(2r+1)\gamma-q}{(2r+1)\gamma q}\Big)||f||_\infty S_q^{\frac{-q}{p}}\Bigg] \end{aligned}$$

The condition $\lambda < \lambda_2$ completes the proof. □

In order to prove Theorem 1.1 (ii), we need to recall the Krasnoselskii genus [20] and a result about critical points of functionals on C^1-submanifolds [21]. Let

$$\Sigma = \{A \subset X : A \text{ is closed}, \ A = -A\}. \tag{2.13}$$

For $A \neq \emptyset$ and $A \in \Sigma$, the Krasnoselskii genus of A is defined as the least integer n such that there exists an odd function $f \in C(A, \mathbb{R}^n \setminus \{0\})$ and is denoted by $\gamma(A)$. Set $\gamma(\emptyset) = 0$ and $\gamma(A) = \infty$ if there exists no f with the above property for any n. Now we recall a consequence of Corollary 4.1 in [21].

Theorem 2.8. *Suppose that M is a closed symmetric C^1-submanifold of a real Banach space X and $0 \notin M$. Assume that $J \in C^1(M, \mathbb{R})$ is even and bounded below. Define*

$$c_j := \sup_{A \in \Gamma_j} \inf_{u \in A} J(u),$$

where

$$\Gamma_j := \{A \subset M : A = -A, \ A \text{ is compact}, \ \gamma(A) \geq j\}. \tag{2.14}$$

If $\Gamma_j \neq \emptyset$ for all $j \geq 1$ and if J satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$, then all c_j are critical values of J and $c_j \to \infty$ as $j \to \infty$.

The last part of the above theorem $c_j \to \infty$, doesn't exist in [21], but by a deformation lemma for functionals on C^1-manifolds [6] and a standard argument ([19] Proposition 9.33), one can show that $c_j \to \infty$.

Proof of Theorem 1.1. Assume that $\bar{\lambda} = \min\{\lambda_0, \lambda_1, \lambda_2\}$ and $0 < \lambda < \bar{\lambda}$.
(i) Using Lemma 2.6 and Proposition 2.7, N_λ^- is C^1 closed manifold and J_λ is bounded from below on N_λ^-. Then by the Ekeland variational principle [23], there exists a constrained Palais-Smale sequence $\{u_n\} \subset N_\lambda^-$ such that $J_\lambda(u_n) \to c_\lambda^-$.Using Lemmas 2.2 and 2.3, there exists $u^- \in X$ such that $u_n \to u^-$ up to a subsequence. Thus

$$J_\lambda(u^-) = c_\lambda^- > 0.$$

This implies that $u^- \neq 0$ and by Lemma 2.1, $J'_\lambda(u^-) = 0$.
Since N_λ^+ is not a closed submanifold, we cannot use the above method to obtain a $(PS)_{c_\lambda^+}$ minimizing sequence. We continue the proof as follows.
By Proposition 2.7, we have

$$c_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u) = \inf_{u \in N_\lambda} J_\lambda(u) < 0.$$

Using Lemma 2.3, J_λ is bounded from below on N_λ^+. Hence there exists a sequence $\{u_n\} \subset N_\lambda^+$ such that

$$\lim_{n \to \infty} J_\lambda(u_n) = c_\lambda^+,$$

By the compact embedding $X \hookrightarrow L^s(\Omega)$ for $1 \le s < p^*$ and since J_λ is coercive (cf. Lemma 2.3), $\{u_n\}$ is bounded. Then there exists $u^+ \in X$ such that $u_n \rightharpoonup u^+$ and $u_n \to u^+$ in $L^s(\Omega)$ for $1 \le s < p^*$, up to a subsequence. Now we show that $u_n \to u^+$ strongly in X. Arguing by contradiction, assume that

$$||u^+|| < \liminf_{n\to\infty} ||u_n||, \tag{2.15}$$

By this fact that $\beta''_{u_n,\lambda}(1) > 0$ for any $n \in \mathbb{N}$, we have

$$\liminf_{n\to\infty} ||u_n||^{p(\alpha+1)} \le \frac{\big((2r+1)\gamma - q\big)\lambda}{(2r+1)\gamma - p(\alpha+1)} \int_\Omega f(x)|u^+|^q dx. \tag{2.16}$$

Since $c_\lambda^+ < 0$, we conclude $\{u_n\}$ is bounded away from zero. Thus (2.16) implies that $\int_\Omega f(x)|u^+|^q dx > 0$. Hence $u^+ \ne 0$ and using Propositions 2.5 there exists $t_{u^+} > 0$ such that $t_{u^+}u^+ \in N_\lambda^+$. Then

$$\begin{aligned} J(t_{u^+}u^+) \ge c_\lambda^+ &= \lim_{n\to\infty} J_\lambda(u_n) \\ &\ge (\frac{1}{p(\alpha+1)} - \frac{1}{\gamma(2r+1)})||u^+||^{p(\alpha+1)} \\ &\quad - (\frac{1}{q} - \frac{1}{\gamma(2r+1)})\lambda \int_\Omega f(x)|u^+|^q dx, \end{aligned}$$

and consequently

$$\begin{aligned} &(\frac{1}{p(\alpha+1)} - \frac{1}{\gamma(2r+1)})(t_{u^+}^{p(\alpha+1)} - 1)||u^+||^{p(\alpha+1)} \ge \\ &(\frac{1}{q} - \frac{1}{\gamma(2r+1)})(t_{u^+}^q - 1)\lambda \int_\Omega f(x)|u^+|^q dx. \end{aligned} \tag{2.17}$$

Inequalities (2.15)-(2.17) imply that $0 < t_{u^+} \le 1$. Using Proposition 2.7, for any $n \in \mathbb{N}$, the first positive critical point of $\beta_{u_n,\lambda}(t)$ is a local minimum point. Since $\beta_{u_n,\lambda}(t)$ is decreasing on $(0,1)$, we have

$$\beta'_{u_n,\lambda}(t_u^+) \le 0,$$

for any $n \in \mathbb{N}$. Subsequently we get

$$\begin{aligned} &\liminf_{n\to\infty} \Bigg(t_{u^+}^{p(\alpha+1)-1}||u_n||^{p(\alpha+1)} - \lambda t_{u^+}^{q-1} \int_\Omega f(x)|u_n|^q dx \\ &\qquad - t_{u^+}^{(2r+1)\gamma-1}\left[\frac{1}{\gamma}\int_\Omega g(x)|u_n|^\gamma dx\right]^{2r+1}\Bigg) \\ &= \liminf_{n\to\infty} t_{u^+}^{p(\alpha+1)-1}||u_n||^{p(\alpha+1)} \\ &\quad - \left(t_{u^+}^{q-1}\lambda \int_\Omega f(x)|u|^q dx + t_{u^+}^{(2r+1)\gamma-1}\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}\right) \le 0. \end{aligned} \tag{2.18}$$

Since $\beta'_{t_{u^+}u^+,\lambda}(1) = 0$, we obtain

$$t_{u^+}^{p(\alpha+1)-1}||u^+||^{p(\alpha+1)} = t_{u^+}^{q-1}\lambda\int_\Omega f(x)|u^+|^q dx + t_{u^+}^{(2r+1)\gamma-1}\left[\frac{1}{\gamma}\int_\Omega g(x)|u^+|^\gamma dx\right]^{2r+1}. \tag{2.19}$$

Finally (2.18) and (2.19) imply that

$$\liminf_{n\to\infty} ||u_n||^{p(\alpha+1)} \le ||u^+||^{p(\alpha+1)},$$

and it contradicts (2.15). Therefor, $u_n \to u^+$ strongly and $t_{u^+} = 1$. Subsequently, $J_\lambda(u^+) = c_\lambda^+$ and by Lemma 2.1, $J'(u^+) = 0$. Since $N_\lambda^+ \cap N_\lambda^- = \emptyset$, then u^+ and u^- are two distinct nontrivial solutions.
(ii) By Lemmas 2.2, 2.3 and 2.6, all conditions appeared in Theorem 2.8 are satisfied if we show that $\Gamma_j \neq \emptyset$ for $j \geq 1$ where

$$\Gamma_j := \{A \subset N_\lambda^- : A = -A,\ A \text{ is compact},\ \gamma(A) \geq j\}.$$

Let X_j be a subspace spanned by j linearly independent functions $v_k \in C_0^\infty(\Omega)$ such that supp $v_k \subset \Omega_g^+ \setminus \Omega_f^+$ and assume that

$$S^{j-1} := X_j \bigcap \{u \in X : ||u|| = 1\}.$$

Then by Proposition 2.7, for any $u \in S^{j-1}$ there exists unique $t_u > 0$ such that $t_u u \in N_\lambda^-$. Thus $\psi : S^{j-1} \to N_\lambda^-$ given by $\psi(u) = t_u u$, is well defined. Since $\beta''_{u,\lambda}(t_u) < 0$, by the implicit function theorem the mapping $u \to t_u$ is continuous. Therefore, $A_j := \psi(S^{j-1})$ is homeomorphic to S^{j-1}. Using the properties of genus we have $\gamma(A_j) = \gamma(S^{j-1}) = j$ (see [20], Section II.5) and this implies that Γ_j is not empty for $j \geq 1$. □

3. The case $p(\alpha+1) < q$

In this section, first we prove some properties of the Nehari manifold and fibring maps. Throughout this section we assume that $g \geq 0$ on Ω.

Lemma 3.1. *Suppose that $\lambda > 0$. Then*

- (*i*) *N_λ is a closed C^1-manifold which is bounded away from zero. Moreover $J_\lambda(u) > 0$ for all $u \in N_\lambda$.*
- (*ii*) *$u \neq 0$ is a critical point of J_λ if and only if it is a critical point of $J_\lambda|_{N_\lambda}$, and $\{u_n\} \subset N_\lambda$ is a $(PS)_c$ sequence for J_λ if and only if it is a $(PS)_c$ sequence for $J_\lambda|_{N_\lambda}$*
- (*iii*) *$J_\lambda|_{N_\lambda}$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.*

Proof. (i) Let $u \in N_\lambda$, then then from the Sobolev inequality we have

$$||u||^{p(\alpha+1)} \leq \lambda||f||_\infty S_q^{\frac{-q}{p}}||u||^q + \left(\frac{||g||_\infty S_\gamma^{\frac{-\gamma}{p}}}{\gamma}\right)^{2r+1}||u||^{\gamma(2r+1)}.$$

Since $p(\alpha+1) < q < \gamma(2r+1)$, the above inequality implies that N_λ is bounded away from 0. Now we show that it is a closed C^1-manifold. Similar to the proof of Lemma 2.6, we consider $\varphi_\lambda : X \to \mathbb{R}$ defined by (2.12). Then $N_\lambda = \varphi_\lambda^{-1}(0) \setminus \{0\}$. Since N_λ is bounded away from 0, N_λ is closed. Now we prove that every point of N_λ is regular for φ_λ. Arguing by contradiction, assume that $u \in N_\lambda$ with $\langle \varphi'_\lambda(u), u\rangle = 0$. Then

$$||u||^{p(\alpha+1)} = \lambda \int_\Omega f(x)|u|^q dx + \left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1},$$

and

$$p(\alpha+1)||u||^{p(\alpha+1)} = q\lambda \int_\Omega f(x)|u|^q dx + \gamma(2r+1)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}.$$

Consequently

$$(p(\alpha+1)-q)||u||^{p(\alpha+1)} = (\gamma(2r+1)-q)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}. \tag{3.1}$$

Since $\int_\Omega g(x)|u|^\gamma dx \geq 0$, from (3.1) we get $u = 0$ and this contradicts $u \in N_\lambda$. Thus every point of N_λ is regular for φ_λ. Also for each $u \in N_\lambda$ we get

$$\begin{aligned} J_\lambda(u) =& \left(\frac{1}{p(\alpha+1)} - \frac{1}{q}\right)||u||^{p(\alpha+1)} \\ &+ \left(\frac{1}{q} - \frac{1}{\gamma(2r+1)}\right)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1} \\ \geq& \left(\frac{1}{p(\alpha+1)} - \frac{1}{q}\right)||u||^{p(\alpha+1)}. \end{aligned} \tag{3.2}$$

Hence, $J_\lambda(u) \geq 0$ and it completes the proof.
(ii) The proof is similar to proof of Lemma 2.6 (ii) and we omit it.
(iii) Let $\{u_n\} \subset N_\lambda$ be a $(PS)_c$ sequence for J_λ. Then similar to (3.2), we have

$$J_\lambda(u_n) \geq \left(\frac{1}{p(\alpha+1)} - \frac{1}{q}\right)||u_n||^{p(\alpha+1)}.$$

Thus $\{u_n\}$ is bounded and there exists $u \in X$ such that $u_n \rightharpoonup u$ and up to a subsequence. Similar to the proof of Lemma 3.2 in [13], using the compact embedding $X \hookrightarrow L^s(\Omega)$ for $1 \leq s < p^*$ we can prove that $u_n \to u$ strongly in X. □

In the next lemma, we study the behavior of fibering maps.

Lemma 3.2. *Let $\lambda > 0$ and $u \in X$. If $\int_\Omega g(x)|u|^\gamma dx > 0$, then the equation $\beta'_{u,\lambda}(t) = 0$ has a unique positive solution t_u such that $t_u u \in N_\lambda$.*

Proof. Consider $u \in X$ with $\int_\Omega g(x)|u|^\gamma dx > 0$. Since $p(\alpha+1) < q < \gamma(2r+1)$, we get $\lim_{t\to\infty} \beta'_{u,\lambda}(t) = -\infty$ and $\beta'_{u,\lambda}(t) > 0$ for $t > 0$ small enough. Then there exists $t_u > 0$ such that $\beta'_{u,\lambda}(t_u) = 0$ and $t_u u \in N_\lambda$. If $\beta'_{u,\lambda}(t_1) = \beta'_{u,\lambda}(t_2) = 0$, then (2.8) holds and consequently we have

$$(t_1t_2)^{p(\alpha+1)}\big(t_2^{q-p(\alpha+1)} - t_1^{q-p(\alpha+1)}\big)||u||^{p(\alpha+1)} =$$
$$(t_1t_2)^q\big(t_1^{\gamma(2r+1)-q} - t_2^{\gamma(2r+1)-q}\big)\left[\frac{1}{\gamma}\int_\Omega g(x)|u|^\gamma dx\right]^{2r+1}.$$

Thus from the condition $\int_\Omega g(x)|u|^\gamma dx > 0$, we obtain $t_1 = t_2$. Then $\beta'_{u,\lambda}(t) = 0$ has a unique positive solution. □

Proof of Theorem 1.2. . Let $\lambda > 0$. Using Lemma 3.1, N_λ is a closed symmetric C^1-submanifold, $0 \notin N_\lambda$ and J_λ is bounded from below on N_λ. Then by Theorem 2.8 and Lemma 3.1 (ii), the proof will be complete if we show that Γ_j defined by

$$\Gamma_j := \{A \subset N_\lambda : A = -A,\ A \text{ is compact},\ \gamma(A) \geq j\},$$

is nonempty for $j \geq 1$. Similar to the proof of Theorem 1.1 (ii), let X_j be a subspace of X spanned by j linearly independent functions $v_k \in C_0^\infty(\Omega)$ such that supp $v_k \subset \Omega_g^+$ and assume that

$$S^{j-1} := X_j \bigcap \{u \in X : ||u|| = 1\}.$$

For any $u \in S^{j-1}$, we have $\int_\Omega g(x)|u|^\gamma dx > 0$. Then by Lemma 3.2, there exists a unique $t_u > 0$ such that $t_u u \in N_\lambda$. Thus Similar to the proof of Theorem 1.1 (ii), we have $\Gamma_j \neq \emptyset$ for $j \geq 1$. □

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