ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 3, pp. 627–642

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 3, pp. 627–642
Online ISSN: 1735-8515

T-DUAL RICKART MODULES

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(Communicated by Omid Ali S. Karamzadeh)

Abstract. We introduce the notions of T-dual Rickart and strongly T-dual Rickart modules. We provide several characterizations and investigate properties of each of these concepts. It is shown that every free (respectively, finitely generated free) R-module is T-dual Rickart if and only if $\overline{Z}^2(R)$ is a direct summand of R and $\mathrm{End}(\overline{Z}^2(R))$ is a semisimple (resp. regular) ring. It is shown that, while a direct summand of a (strongly) T-dual Rickart module inherits the property, direct sums of T-dual Rickart modules do not. Moreover, when a direct sum of T-dual Rickart modules is T-dual Rickart, is included. Examples illustrating the results are presented.
Keywords: Dual Rickart modules; t-lifting modules; t-dual Baer modules; T-dual Rickart modules; strongly T-dual Rickart modules.
MSC(2010): Primary: 16D10; Secondary: 16D70, 16D40, 16S50.

1. **Introduction**

The notion of Baer rings was introduced by Kaplansky in 1955 [11]. A ring R is called *right Baer* (resp. *left Baer*) if the right (resp. left) annihilator of any nonempty subset of R is generated by an idempotent. Baer property is left-right symmetric for every ring. This notion was extended to Rickart rings by Hattori [10]. A ring R is said to be *right Rickart* (also known as p.p ring) if the right annihilator of any single element of R is generated by an idempotent, as a right ideal of R. Rickart rings play an important role in the structure theory of rings. The study of Rickart and Baer rings has its roots in functional analysis with close links to C^*-algebras and Von Neumann algebras.

Recently, the notions of Baer and Rickart rings were extended and studied in general module-theoretic setting by Rizvi, Roman and Lee ([13] and [17]). The notion of t-Baer modules was introduced by Asgari and Haghany [2]. They

Article electronically published on June 29, 2016.
Received: 24 September 2013, Accepted: 15 March 2015.
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used second singular submodule in their definition. The notion of T-Rickart modules was introduced in [7], as a generalization of t-Baer modules.

In [21], Keskin-Tütüncü and Tribak introduced the concept of dual Baer modules. A module M is called a *dual Baer* module if for every right ideal I of $S = \mathrm{End}(M)$, $\sum_{f \in I} \mathrm{Im}(f)$ is a direct summand of M, equivalently, $D_S(N) = \{f \in S : f(M) \subseteq N\}$ is a direct summand of S, for every submodule N of M. The notion of dual Rickart modules was introduced by Lee, Rizvi and Roman [14].

In [1], the authors introduced the notions of t-lifting modules and t-dual Baer modules, which are generalizations of lifting modules. Motivated from these notions and T-Rickart modules we present the new class of modules named T-dual Rickart modules and investigate new results.

In Section 3, we give an equivalent condition for a module to be T-dual Rickart and show that a direct summand of a T-dual Rickart module is T-dual Rickart. It is shown that every free (resp. finitely generated free) R-module is T-dual Rickart if and only if $\overline{Z}^2(R)$ is a direct summand of R and $\mathrm{End}_R(\overline{Z}^2(R))$ is a semisimple (resp. regular) ring. We introduce the notion of relative T-dual Rickart modules to show that over a right perfect ring, every R-module is t-lifting if and only if every R-module is T-dual Rickart. Also, it is shown that every T-dual Rickart module M has summand sum property (SSP) for direct summands which are contained in $\overline{Z}^2(M)$.

In Section 4, we show that a direct sum of T-dual Rickart modules is not a T-dual Rickart module, in general. Moreover, we investigate the question: When are the direct sums of T-dual Rickart modules, also T-dual Rickart?

In Section 5, the notion of strongly T-dual Rickart module is defined and several characterizations of such modules are given. We show that each direct summand of a strongly T-dual Rickart module is strongly T-dual Rickart, and direct sum of arbitrary strongly T-dual Rickart modules is strongly T-dual Rickart under necessary and sufficient conditions.

2. **Preliminaries**

Throughout all rings (not necessarily commutative rings) have identity and all modules are unital right modules. For the sake of completeness, we state some definitions and notations used throughout this paper. Let M be a module over a ring R. For submodules N and K of M, $N \leq K$ denotes N is a submodule of K and $\mathrm{End}(M)$ denotes the ring of right R-module endomorphisms of M. In what follows, by $\leq^{\oplus}$ and $E(M)$ we denote, respectively, a module direct summand and the injective hull of M. The symbols $\mathbb{Z}$, $\mathbb{Z}_{\mathrm{n}}$ and $\mathbb{Q}$ stand for the ring of integers, the ring of residues modulo n and the ring of rational numbers, respectively.

In [19], Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = \cap\{Ker(\varphi) : \varphi \in \mathrm{Hom}(M, N), \text{ where } N \text{ is small in its injective hull}\}.$$

Also, $\overline{Z}^2(M)$ is defined as $\overline{Z}(\overline{Z}(M))$.

Definition 2.1. (1) Let M be a module. Let N and L be submodules of M. N is called a *supplement* of L if it is minimal with the property $M = N + L$, or equivalently, $M = N + L$ and $N \cap L \ll N$. A module M is called *amply supplemented* if, for any submodules $A,\ B$ of M with $M = A + B$ there exists a supplement P of A such that $P \subseteq B$ (see [6]).

(2) A submodule N of M is called *t-small* in M, denoted by $N \ll_t M$, if for every submodule K of M, $\overline{Z}^2(M) \subseteq N + K$ implies that $\overline{Z}^2(M) \leq K$ (see [1]).

(3) A submodule N of M is called *t-coclosed* in M if $N/K \ll_t M/K$ implies that $N = K$ (see [1]).

(4) A module M is called *t-lifting* if every submodule N of M contains a direct summand K of M such that $N/K \ll_t M/K$ (see [1]).

(5) A module M is called a *cosingular* (resp. *noncosingular*) module, if $\overline{Z}(M) = 0$ (resp. $\overline{Z}(M) = M$) (see [19]).

(6) A module M is called *dual Baer* if for every $N \leq M$, there exists an idempotent e in $S = \mathrm{End}(M)$ such that $D(N) = \{f \in \mathrm{End}(M) : f(M) \subseteq N\} = eS$ (see [21]).

(7) A module M is called *dual Rickart* if for each $\varphi \in \mathrm{End}(M)$, $\varphi(M)$ is a direct summand of M (see [14]).

(8) A module M is said to be t-dual Baer if $I(\overline{Z}^2(M))$ is a direct summand of M for every right ideal I of $\mathrm{End}(M)$ (see [1]).

(9) An idempotent $e \in R$ is called left *semicentral* if $re = ere$, for each $r \in R$, or equivalently, eR is an ideal of R. The set of all left semicentral idempotents of R will be denoted by $S_l(R)$. If $e^2 = e \in \mathrm{End}(M)$, then $e \in S_l(End(M))$ if and only if eM is a fully invariant direct summand (see [4], [5]).

(10) An R-module M is said to have the *strong summand sum property* (resp. *summand sum property*), denoted briefly by SSSP (resp. SSP), if the sum of any family of (resp. two) direct summands of M is a direct summand of M (see [21]).

The following proposition is used in the sequel.

Proposition 2.2. *(1) [19, Proposition 2.1] Let M and N be two R-modules and $\{M_i\}_{i\in I}$ a class of R-modules. Then we have the following:*

(i) If $N \leq M$, then $\overline{Z}(N) \subseteq \overline{Z}(M)$ and $\overline{Z}(M/N) \supseteq (\overline{Z}(M) + N)/N$.

(ii) If $f : M \to N$ is a homomorphism, then $f(\overline{Z}(M)) \subseteq \overline{Z}(N)$.

(iii) $\overline{Z}(M/\overline{Z}(M)) = 0$.

(iv) $\overline{Z}(\oplus_{i\in I} M_i) = \oplus_{i\in I}\overline{Z}(M_i)$.

(2) [22, Corollary 3.2] Let M be a module. Then $\mathrm{End}(M)$ is a regular ring if and only if $\mathrm{Im}(\varphi)$ and $\mathrm{Ker}(\varphi)$ are direct summands of M for each $\varphi \in \mathrm{End}(M)$.

(3) [22, Theorem 3.5] Let R be a ring. Then R is semisimple Artinian if and only if R possesses an infinitely generated free module F such that $\mathrm{End}(F)$ is a regular ring.

3. **T-dual Rickart modules**

Motivated by the definitions of dual Rickart modules, t-dual Baer modules and T-Rickart modules, we define the T-dual Rickart notion.

Definition 3.1. An R-module M is called *T-dual Rickart*, if $\varphi(\overline{Z}^2(M)) \leq^{\oplus} M$, for any $\varphi \in \mathrm{End}(M)$.

Clearly, every cosingular module is T-dual Rickart. One can easily show that the notions of dual Rickart module and T-dual Rickart module coincide for every noncosingular module.

In the following, we give an equivalent condition for T-dual Rickart modules.

Theorem 3.2. *The following are equivalent for an R-module M with $S = \mathrm{End}(M)$.*

(1) M is a T-dual Rickart module;

(2) $\overline{Z}^2(M)$ is a direct summand of M and $\overline{Z}^2(M)$ is a dual Rickart module.

Proof. (1) $\Rightarrow$ (2) Since M is T-dual Rickart, $id_M(\overline{Z}^2(M)) = \overline{Z}^2(M) \leq^{\oplus} M$, where id_M is the identity element of S. We show that $\overline{Z}^2(M)$ is dual Rickart. Let $\overline{Z}^2(M) = eM$ for some $e^2 = e \in S$ and $\varphi \in \mathrm{End}(\overline{Z}^2(M))$. As $\mathrm{End}(\overline{Z}^2(M)) = eSe$, $\varphi = e\psi e$ for some $\psi \in S$. Since M is T-dual Rickart, $\psi(\overline{Z}^2(M)) \leq^{\oplus} M$. An inspection shows that $\varphi(\overline{Z}^2(M)) = e\psi e(\overline{Z}^2(M)) = \psi(\overline{Z}^2(M)) \leq^{\oplus} M$, because $\psi(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$. By modularity, $\varphi(\overline{Z}^2(M)) \leq^{\oplus} \overline{Z}^2(M)$. Hence $\overline{Z}^2(M)$ is dual Rickart.

(2) $\Rightarrow$ (1) Suppose that $\overline{Z}^2(M)$ is a direct summand of M and $\overline{Z}^2(M)$ is a dual Rickart module. Then there exists $e^2 = e \in S$ such that $\overline{Z}^2(M) = eM$. Let $\varphi \in S$. As $e\varphi e \in \mathrm{End}(\overline{Z}^2(M))$ and $\overline{Z}^2(M)$ is dual Rickart, $e\varphi e(\overline{Z}^2(M)) \leq^{\oplus} \overline{Z}^2(M)$. We show that $\varphi(\overline{Z}^2(M)) = e\varphi e(\overline{Z}^2(M))$. Since $\overline{Z}^2(M) = eM$, $e(\overline{Z}^2(M)) = \overline{Z}^2(M)$. So $\varphi e(\overline{Z}^2(M)) = \varphi(\overline{Z}^2(M))$. As $\varphi(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$, $e\varphi e(\overline{Z}^2(M)) = e\varphi(\overline{Z}^2(M)) = \varphi(\overline{Z}^2(M))$. Hence $\varphi(\overline{Z}^2(M)) = e\varphi e(\overline{Z}^2(M)) \leq^{\oplus} \overline{Z}^2(M)$. Since $\overline{Z}^2(M) \leq^{\oplus} M$, $\varphi(\overline{Z}^2(M)) \leq^{\oplus} M$. Therefore M is T-dual Rickart. □

In the following, we present a characterization of amply supplemented T-dual Rickart modules.

Proposition 3.3. *Let M be an amply supplemented module. Then M is T-dual Rickart if and only if there exists $N \leq^{\oplus} M$ such that $\varphi(\overline{Z}^2(M))/N \ll_t M/N$, for each $\varphi \in \mathrm{End}(M)$.*

Proof. Suppose that M is a T-dual Rickart module. Then it is clear that for each $\varphi \in \mathrm{End}(M)$, there exists $N \leq^{\oplus} M$ such that $\varphi(\overline{Z}^2(M))/N \ll_t M/N$.

Conversely, let $\varphi \in \mathrm{End}(M)$. By [1, Corollary 2.6], $\varphi(\overline{Z}^2(M))$ is a t-closed submodule of M. Since $\varphi(\overline{Z}^2(M))/N \ll_t M/N$ for some $N \leq^{\oplus} M$, $\varphi(\overline{Z}^2(M)) = N \leq^{\oplus} M$. Hence M is T-dual Rickart. □

The notions of dual Rickart modules and T-dual Rickart modules are the same over a right V-ring.

Proposition 3.4. *Let R be a right V-ring and M an R-module. Then M is T-dual Rickart if and only if it is a dual Rickart module.*

Proof. Since every module over a right V-ring R is noncosingular by [19, Proposition 2.5], M is T-dual Rickart if and only if it is dual Rickart.

□

Example 3.5. (1) The $\mathbb{Z}$-module $\mathbb{Z}$ is a cosingular module; hence it is T-dual Rickart. However it is not a dual Rickart module.

(2) Consider $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_p$ as $\mathbb{Z}$-module (p is a prime integer). Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$. Then M is not a dual Rickart $\mathbb{Z}$-module by [14, Example 2.10]. As $\overline{Z}^2(M) = \mathbb{Z}_{p^\infty}$ is a dual Rickart module, M is a T-dual Rickart module by Theorem 3.2.

Following result shows that direct summands of a T-dual Rickart module inherit the property.

Theorem 3.6. *Let M be a T-dual Rickart module and $N \leq^{\oplus} M$. Then N is a T-dual Rickart module.*

Proof. Assume that $N \leq^{\oplus} M$ say $M = N \oplus N'$. Hence $\overline{Z}^2(M) = \overline{Z}^2(N) \oplus \overline{Z}^2(N')$. Let $\varphi \in \mathrm{End}(N)$ and $e : M \to N$ be natural projection. Then $\varphi e \in \mathrm{End}(M)$. As M is a T-dual Rickart module, $\varphi e(\overline{Z}^2(M)) \leq^{\oplus} M$. As $\varphi(\overline{Z}^2(N)) = \varphi e(\overline{Z}^2(M))$, $\varphi(\overline{Z}^2(N)) \leq^{\oplus} N$. Therefore N is T-dual Rickart. □

The following reformulated proposition characterizes t-dual Baer modules in terms of the SSSP and the T-dual Rickart property for modules.

Proposition 3.7. *Let M be a module. Then M is t-dual Baer if and only if M is T-dual Rickart and M has SSSP for direct summands which are contained in $\overline{Z}^2(M)$.*

Proof. See [1, Theorem 3.2]. □

It is clear that every t-dual Baer is T-dual Rickart. Following examples show that the converse is not true.

Example 3.8. The ring $R = \prod_{n=1}^{\infty} \mathbb{Z}_2$ is a regular ring. Hence R is a dual Rickart R-module. As R is commutative, R is a V-ring. Hence, by Proposition 3.4, R is a T-dual Rickart R-module. Also it is not t-dual Baer (dual Baer) by [14, Example 2.8].

Example 3.9. Let R be a free algebra generated over a field by $\{x_i : i \in I\}$ with $|I| \geq 2$. Then by [12, Example 2.23(i), (j)], R is a right hereditary ring which is not right Noetherian. As R is not right Noetherian, by [3, Theorem 1.3], there exists a family $\{S_n : n \in \mathbb{N}\}$ of simple right R-modules such that $\oplus_{n=1}^{\infty} E(S_n)$ is not injective. Let $M = E(\oplus_{n=1}^{\infty} E(S_n))$. By [20, Example 2.4], M is not dual Baer. As M is noncosingular, M is not t-dual Baer. However, M is T-dual Rickart (see Theorem 3.17).

The following result shows that the notions of t-dual Baer and T-dual Rickart coincide for modules whose endomorphism rings have no infinite set of nonzero orthogonal idempotents.

Proposition 3.10. *Let M be a module and* $\mathrm{End}(M)$ *has no infinite set of nonzero orthogonal idempotents. Then the following are equivalent:*
(1) M is t-dual Baer;
(2) M is T-dual Rickart.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let M be a T-dual Rickart module. Then $M = \overline{Z}^2(M) \oplus N$ where $\overline{Z}^2(M)$ is dual Rickart by Theorem 3.2. As $\overline{Z}^2(M)$ is fully invariant in M by Proposition 2.2(1), $\mathrm{Hom}(\overline{Z}^2(M), N) = 0$, hence

$$\mathrm{End}(M) = \begin{pmatrix} \mathrm{End}(\overline{Z}^2(M)) & \mathrm{Hom}(N, \overline{Z}^2(M)) \\ 0 & \mathrm{End}(N) \end{pmatrix},$$

because $\overline{Z}^2(M) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S_l(\mathrm{End}(M))$.

Since $\mathrm{End}(M)$ has no infinite set of nonzero orthogonal idempotents, $\mathrm{End}(\overline{Z}^2(M))$ has no infinite set of nonzero orthogonal idempotents. Hence by [14, Theorem 4.2], $\overline{Z}^2(M)$ is dual Baer. Therefore [1, Theorem 3.2] implies that M is t-dual Baer. $\square$

Definition 3.11. An R-module M is called *T-dual Rickart relative to N* (or *N-T-dual Rickart*) if $\varphi(\overline{Z}^2(M)) \leq^{\oplus} N$ for each homomorphism $\varphi : M \to N$.

In view of the above definition, a right R-module M is T-dual Rickart if and only if M is T-dual Rickart relative to M. Clearly, if M or N are cosingular, then M is T-dual Rickart relative to N.

Theorem 3.12. *Let M and N be two R-modules. Then M is T-dual Rickart relative to N if and only if for every direct summand L of M and any submodule K of N, L is T-dual Rickart relative to N.*

Proof. Let $L = eM$ for some $e^2 = e \in \mathrm{End}(M)$ and $K \leq N$. Let M be T-dual Rickart relative to N, we show that L is T-dual Rickart relative to K. Let $\varphi : L \to K$ be a homomorphism. Then $\varphi e \in \mathrm{Hom}(M, N)$. Thus

$\varphi e(\overline{Z}^2(M)) \leq^{\oplus} N$. Since $\varphi e(\overline{Z}^2(M)) \subseteq K$, $\varphi e(\overline{Z}^2(M)) \leq^{\oplus} K$. We show that $\overline{Z}^2(L) = e\overline{Z}^2(M)$. Let $M = L \oplus L'$. Then $\overline{Z}^2(M) = \overline{Z}^2(L) \oplus \overline{Z}^2(L')$. Hence $e\overline{Z}^2(M) = e\overline{Z}^2(L) = \overline{Z}^2(L)$. Therefore $\varphi e(\overline{Z}^2(M)) = \varphi(\overline{Z}^2(L)) \leq^{\oplus} K$. Thus L is T-dual Rickart relative to K. The converse is clear. □

Corollary 3.13. *Let M be an R-module. Then the following are equivalent.*

(1) M is T-dual Rickart;

(2) For any $N \leq M$, each $L \leq^{\oplus} M$ is T-dual Rickart relative to N;

(3) If L and N are direct summands of M, then for any $\varphi \in \mathrm{Hom}(L, N)$, $\varphi|_L(\overline{Z}^2(L)) \leq^{\oplus} N$.

Proof. It is clear from Theorem 3.12. □

In the following proposition, we prove that a T-dual Rickart module M has SSP for direct summands that are contained in $\overline{Z}^2(M)$.

Proposition 3.14. *Let M be a T-dual Rickart module. Then the following statements hold true.*

(1) If L and N are direct summands of M with $L \subseteq \overline{Z}^2(M)$, then $L + N$ is a direct summand of M.

(2) M has SSP for direct summands that are contained in $\overline{Z}^2(M)$.

Proof. (1) Let $L \subseteq \overline{Z}^2(M)$ be a direct summand of M say $M = L \oplus L'$ and $N = eM$ for some $e^2 = e \in \mathrm{End}(M)$. We show that $\overline{Z}^2(L) = L$. As $M = L \oplus L'$, $\overline{Z}^2(M) = \overline{Z}^2(L) \oplus \overline{Z}^2(L')$. Since $\overline{Z}^2(L) \subseteq L \subseteq \overline{Z}^2(M)$, $L = \overline{Z}^2(L) \oplus L \cap \overline{Z}^2(L') = \overline{Z}^2(L)$. Consider the projection $1 - e : M \to (1 - e)M$. By Corollary 3.13, $(1-e)|_L(\overline{Z}^2(L)) = (1-e)|_L(L)$ is a direct summand of M. Since $(1-e)|_L(L) = (L+N)\cap(1-e)M$, $M = ((L+N)\cap(1-e)M)\oplus T$ for some $T \leq M$. Hence by modular law, $(1-e)M = ((L+N)\cap(1-e)M) \oplus (T \cap (1-e)M)$. So $M = N\oplus(1-e)M = N+((L+N)\cap(1-e)M)\oplus(T\cap(1-e)M) = (L+N)+(T\cap(1-e)M)$. Since $(L+N)\cap(T\cap(1-e)M) = 0$, $M = (N+L)\oplus(T\cap(1-e)M)$. Hence $N + L \leq^{\oplus} M$.

(2) Apply (1). □

The converse of Proposition 3.14 is not true, in general, as shown below.

Example 3.15. Let R be a nonregular right V-domain (it is known that, there is a field F with derivation δ such that the differential polynomial ring $F[x, \delta]$ is a nonregular right V-domain [15, C29]). As R is not regular, R is not a dual Rickart R-module. Since R is a right V-domain, R is noncosingular, by [19, Proposition 2.5]. Hence R is not T-dual Rickart. However, R has SSP for direct summands that are contained in $\overline{Z}^2(R) = R$, because R is a domain.

The following theorem gives a condition equivalent to being T-dual Rickart for an R-module M in terms of $\sum_{\varphi\in I}\varphi(\overline{Z}^2(M))$, where I is a finitely generated right ideal of $\mathrm{End}(M)$.

Theorem 3.16. *An R-module M is T-dual Rickart if and only if $\sum_{\varphi\in I}\varphi(\overline{Z}^2(M))$ is a direct summand of M for every finitely generated right ideal I of $\mathrm{End}(M)$.*

Proof. Let I be a finitely generated right ideal of $\mathrm{End}(M)$ generated by $\varphi_1,...,\varphi_n$. Since M is T-dual Rickart, $\varphi_i(\overline{Z}^2(M)) \leq^{\oplus} M$ for each $1 \leq i \leq n$. By Proposition 3.14, M has SSP for direct summands which are contained in $\overline{Z}^2(M)$. Since $\varphi_i(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$, $\sum_{\varphi\in I}\varphi(\overline{Z}^2(M)) = \varphi_1(\overline{Z}^2(M))+...+\varphi_n(\overline{Z}^2(M)) \leq^{\oplus} M$. The converse is clear. □

As a consequence of [14, Theorem 2.29] and [19, Proposition 2.7], we characterize rings R for which every injective R-module is noncosingular T-dual Rickart.

Proposition 3.17. *Let R be a ring. Then the following are equivalent:*
(1) R is right hereditary;
(2) Every injective R-module is noncosingular and T-dual Rickart.

Proof. Clear from [14, Theorem 2.9] and [19, Proposition 2.7]. □

Next, we characterize the class of rings R for which every finitely generated free R-module is T-dual Rickart.

Theorem 3.18. *The following conditions are equivalent for a ring R:*
(1) Every finitely generated free R-module is T-dual Rickart;
(2) Every finitely generated projective R-module is T-dual Rickart;
(3) R is a T-dual Rickart R-module;
(4) $\overline{Z}^2(R) \leq^{\oplus} R$ and $S = \mathrm{End}_R(\overline{Z}^2(R))$ is a regular ring.

Proof. (1) ⇒ (2) Let M be a finitely generated projective R-module. Thus $M \leq^{\oplus} F$ for some finitely generated free R-module F. By (1), F is T-dual Rickart. Hence M is T-dual Rickart, by Theorem 3.6.

(2) ⇒ (3) Is clear.

(3) ⇒ (4) By Theorem 3.2, $\overline{Z}^2(R) \leq^{\oplus} R$ and $\overline{Z}^2(R)$ is a dual Rickart module. Thus for each $\varphi \in \mathrm{End}(\overline{Z}^2(R))$, $\mathrm{Im}(\varphi) \leq^{\oplus} \overline{Z}^2(R)$. We show that $\mathrm{Ker}(\varphi) \leq^{\oplus} \overline{Z}^2(R)$. As $\overline{Z}^2(R)/\mathrm{Ker}(\varphi) \cong \mathrm{Im}(\varphi) \leq^{\oplus} \overline{Z}^2(R)$ and $\overline{Z}^2(R)$ is a projective module, the exact sequence $0 \to \mathrm{Ker}(\varphi) \to \overline{Z}^2(R) \to \overline{Z}^2(R)/\mathrm{Ker}(\varphi) \to 0$ splits. Hence $\mathrm{Ker}(\varphi) \leq^{\oplus} \overline{Z}^2(R)$. Thus by Proposition 2.2(2), $\mathrm{End}(\overline{Z}^2(R))$ is a regular ring.

(4) $\Rightarrow$ (1) Let $F = R^{(n)}$ be a free R-module where $n \in \mathbb{N}$. Then $\overline{Z}^2(F) = \overline{Z}^2(R)^{(n)} \leq^{\oplus} F$. We show that $\overline{Z}^2(F)$ is dual Rickart. Let $S = \mathrm{End}(\overline{Z}^2(F))$. As $\mathrm{End}(\overline{Z}^2(R))$ is regular, $S = Mat_n(\mathrm{End}(\overline{Z}^2(R)))$ (the $n \times n$ matrix ring over $\mathrm{End}(\overline{Z}^2(R))$ is regular. Hence by Proposition 2.2(2), $\overline{Z}^2(F)$ is a dual Rickart module. Therefore by Theorem 3.2, F is T-dual Rickart. □

Next, we characterize several important classes of rings in term of every free module is T-dual Rickart.

Theorem 3.19. *The following conditions are equivalent for a ring R:*

(1) Every free R-module is T-dual Rickart;

(2) Every projective R-module is T-dual Rickart;

(3) $\overline{Z}^2(R) \leq^{\oplus} R$ *and* $S = \mathrm{End}_R(\overline{Z}^2(R))$ *is a semisimple ring.*

Proof. (1) $\Rightarrow$ (2) Is clear from Theorem 3.6.

(2) $\Rightarrow$ (3) As R is T-dual Rickart, $\overline{Z}^2(R) \leq^{\oplus} R$ say $R = \overline{Z}^2(R) \oplus R'$. Let $F = R^{(\Lambda)}$ be an infinitely generated free R-module. By (2), F is T-dual Rickart and so $\overline{Z}^2(F) = \overline{Z}^2(R)^{(\Lambda)}$ is dual Rickart by Theorem 3.2. Similar to the proof of Theorem 3.18, $\mathrm{End}(\overline{Z}^2(F))$ is regular. As $\mathrm{End}_R(\overline{Z}^2(F)) = \mathrm{End}_S(S^{(\Lambda)})$ is regular, S is a semisimple ring by Proposition 2.2(3).

(3) $\Rightarrow$ (1) Let $F = R^{(\Lambda)}$ be a free R-module. As $\overline{Z}^2(R) \leq^{\oplus} R$, $\overline{Z}^2(F) = \overline{Z}^2(R)^{(\Lambda)} \leq^{\oplus} F$. By (3), $\mathrm{End}(\overline{Z}^2(R))$ is a semisimple ring. Hence $\mathrm{End}_R(\overline{Z}^2(F)) = \mathrm{End}_R(\overline{Z}^2(R)^{(\Lambda)}) = \mathrm{End}_S(S^{(\Lambda)})$ is a regular ring, because $S^{(\Lambda)}$ is a semisimple S-module (it is known that endomorphism ring of every semisimple module is a regular ring [23, 20.6]). Therefore by Proposition 2.2(2), $\overline{Z}^2(F)$ is a dual Rickart. Hence by Theorem 3.2, F is T-dual Rickart. □

In the following theorem we extend the [1, Theorem 3.12].

Theorem 3.20. *Let R be a right perfect ring. Then the following are equivalent:*

(1) Every R-module is t-lifting;

(2) Every R-module is t-dual Baer;

(3) Every R-module is T-dual Rickart.

Proof. (1) $\Leftrightarrow$ (2) It is clear from [1, Theorem 3.12].

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Since over a right perfect ring every R-module is amply supplemented by [23, 43.9], for an R-module M, it suffices to show that every t-closed submodule is a direct summand of M by [1, Theorem 2.9]. Let N be a t-closed submodule of M. Hence N is noncosingular and $\overline{Z}^2(N) = N$ by [1, Proposition 2.5]. Consider R-module $M \oplus N$. By (3), $M \oplus N$ is T-dual Rickart. By Theorem 3.12, N is T-dual Rickart relative to M. Let $\mathrm{i} : N \to M$ be the

inclusion homomorphism from N to M. As N is T-dual Rickart relative to M, $\mathrm{i}(\overline{Z}^2(N)) = \mathrm{i}(N) = N \leq^{\oplus} M$. Hence every t-closed submodule is a direct summand of M and M is a t-lifting module. □

4. **Direct sum of T-dual Rickart modules**

This section is devoted to investigate when direct sums of T-dual Rickart modules are T-dual Rickart. In the following examples, it is shown that a direct sum of T-dual Rickart modules is not T-dual Rickart, in general.

Example 4.1. Let F be a field and V be an infinite dimensional vector space over F. Set $J = \{x \in \mathrm{End}_F(V) : \dim_F(xV) < \infty\}$ and $R = F + J$. By [9, Example 6.19], R is regular. Also $J = Soc(R_R)$ and J is essential in R_R (see [9, Page 180]). As R is regular and J is semisimple, R and J are dual Rickart R-modules. By [9, Example 6.19], R is a right V-ring. Hence by Proposition 3.4, R and J are T-dual Rickart. We claim that $R \oplus J$ is not T-dual Rickart. Otherwise, if $R \oplus J$ is T-dual Rickart, then by Corollary 3.13, J is T-dual Rickart relative to R. Let $i : J \to R$ be the inclusion homomorphism from J to R. Then $i(\overline{Z}^2(J)) = i(J) = J \leq^{\oplus} R$, a contradiction. Thus $R \oplus J$ is not T-dual Rickart.

Example 4.2. Let $R = \prod_{i=1}^{\infty} F_i$ with $F_i = F$ is a field for all $i \geq 1$. Let $M_1 = R$ and $M_2 = \oplus_{i=1}^{\infty} F_i$. By [14, Example 5.1], M_1 and M_2 are dual Rickart and $M_1 \oplus M_2$ is not dual Rickart. As R is a V-ring, by [19, Proposition 2.5], every R-module is noncosingular. Hence by Proposition 3.4, M_1 and M_2 are T-dual Rickart, but $M_1 \oplus M_2$ is not T-dual Rickart.

When a direct sum of two or more T-dual Rickart modules is also T-dual Rickart is considered in the following.

Proposition 4.3. *Let $\{M_i\}_{i=1}^n$ and N be modules. If N has SSP for direct summands which are contained in $\overline{Z}^2(N)$, then $\oplus_{i=1}^n M_i$ is T-dual Rickart relative to N if and only if M_i is T-dual Rickart relative to N for all $1 \leq i \leq n$.*

Proof. The sufficiency is clear from Theorem 3.12. For the necessity, let $\varphi : \oplus_{i=1}^n M_i \to N$. Then $\varphi = (\varphi_i)_{i=1}^n$ where φ_i is a homomorphism from M_i to N for each $1 \leq i \leq n$. By assumption $\varphi_i(\overline{Z}^2(M_i)) \leq^{\oplus} N$ for each $1 \leq i \leq n$. Since $\overline{Z}^2(\oplus_{i=1}^n M_i) = \oplus_{i=1}^n \overline{Z}^2(M_i)$ and N has SSP for direct summands which are contained in $\overline{Z}^2(N)$, $\varphi(\overline{Z}^2(\oplus_{i=1}^n M_i)) = \sum_{i=1}^n \varphi_i(\overline{Z}^2(M_i)) \leq^{\oplus} N$ (because $\varphi_i(\overline{Z}^2(M_i)) \subseteq \overline{Z}^2(N)$). Thus $\oplus_{i=1}^n M_i$ is T-dual Rickart relative to N. □

Corollary 4.4. *Let $\{M_i\}_{i=1}^n$ be modules. Then $\oplus_{i=1}^n M_i$ is T-dual Rickart relative to M_j $(1 \leq j \leq n)$ if and only if M_i is T-dual Rickart relative to M_j for each $1 \leq i \leq n$.*

In the following theorem, we present conditions under which M_i is T-dual Rickart relative to $\oplus_{j=1}^{n} M_j$.

Theorem 4.5. *Let $\{M_i\}_{i=1}^{n}$ and N be modules and for each $i \geq j$ with $1 \leq i, j \leq n$, M_i is M_j-projective. Then N is T-dual Rickart relative to $\oplus_{j=1}^{n} M_j$ if and only if N is T-dual Rickart relative to M_j for all $1 \leq j \leq n$.*

Proof. $\Rightarrow$ Is clear from Theorem 3.12.

$\Leftarrow$ Assume that N is T-dual Rickart relative to M_j for each $1 \leq j \leq n$. We use induction on n. Let $n = 2$ and N be T-dual Rickart relative to M_1 and M_2. Let φ be a homomorphism from N to $M_1 \oplus M_2$. Then $\varphi = \pi_1\varphi + \pi_2\varphi$, where π_i is the natural projection from $M_1 \oplus M_2$ to M_i $(i = 1, 2)$. As N is T-dual Rickart relative to M_2, $\pi_2\varphi(\overline{Z}^2(N)) \leq^{\oplus} M_2$. Let $M_2 = \pi_2\varphi(\overline{Z}^2(N)) \oplus M_2'$ for some $M_2' \leq M_2$. Hence $M_1 \oplus M_2 = M_1 \oplus \pi_2\varphi(\overline{Z}^2(N)) \oplus M_2'$. Since M_2 is M_1-projective, $\pi_2\varphi(\overline{Z}^2(N))$ is M_1-projective. As $M_1 + \varphi(\overline{Z}^2(N)) = M_1 \oplus \pi_2\varphi(\overline{Z}^2(N)) \leq^{\oplus} M_1 \oplus M_2$, there exists $T \subseteq \varphi(\overline{Z}^2(N))$ such that $M_1 + \varphi(\overline{Z}^2(N)) = M_1 \oplus T$, by [16, Lemma 4.47]. Thus $\varphi(\overline{Z}^2(N)) = \varphi(\overline{Z}^2(N)) \cap M_1 \oplus T$. Since N is T-dual Rickart relative to M_1, $\pi_1\varphi(\overline{Z}^2(N)) = M_1 \cap (M_2 + \varphi(\overline{Z}^2(N))) = M_1 \cap \varphi(\overline{Z}^2(N)) \leq^{\oplus} M_1$. Therefore $\varphi(\overline{Z}^2(N)) \leq^{\oplus} M_1 \oplus T$. Since $M_1 \oplus T = M_1 \oplus \varphi(\overline{Z}^2(N)) \leq^{\oplus} M_1 \oplus M_2$, $\varphi(\overline{Z}^2(N)) \leq^{\oplus} M_1 \oplus M_2$. Thus N is T-dual Rickart relative to $M_1 \oplus M_2$. Now, assume that N is T-dual Rickart relative to $\oplus_{j=1}^{n} M_j$. We show that N is T-dual Rickart relative to $M_{n+1} \oplus (\oplus_{j=1}^{n} M_j)$. Since M_{n+1} is M_j-projective for each $1 \leq j \leq n$, M_{n+1} is $\oplus_{j=1}^{n} M_j$-projective. As N is T-dual Rickart relative to M_{n+1}, N is T-dual Rickart relative to $\oplus_{j=1}^{n+1} M_j$ by a similar argument for the case $n = 2$. $\square$

We remark that we use ideas of the proof of [14, Theorem 5.5] to prove Theorem 4.5 and extend it to T-dual Rickart modules.

Corollary 4.6. *Let $\{M_i\}_{i=1}^{n}$ be modules and M_i is M_j-projective for all $i \geq j$ with $1 \leq i, j \leq n$. Then $\oplus_{i=1}^{n} M_i$ is T-dual Rickart if and only if M_i is T-dual Rickart relative to M_j for all $1 \leq i, j \leq n$.*

Proof. $\Rightarrow$ is clear from Theorem 3.12.

$\Leftarrow$ Assume that M_i is T-dual Rickart relative to M_j for all $1 \leq j \leq n$. Now $\oplus_{i=1}^{n} M_i$ is T-dual Rickart relative to M_j for all $1 \leq j \leq n$ by Corollary 4.4. Hence by Theorem 4.5, $\oplus_{i=1}^{n} M_i$ is T-dual Rickart. $\square$

5. Strongly T-dual Rickart modules

In this section, we introduce the notions of strongly T-dual Rickart modules and strongly dual Rickart modules. A number of characterizations of such modules and basic results are provided.

Definition 5.1. (a) An R-module M is called strongly dual Rickart if $\mathrm{Im}(\varphi)$ is a fully invariant direct summand of M for each $\varphi \in \mathrm{End}(M)$.

(b) An R-module M is called strongly T-dual Rickart if $\varphi(\overline{Z}^2(M))$ is a fully invariant direct summand of M for each $\varphi \in \mathrm{End}(M)$.

It is clear that every cosingular module is strongly T-dual Rickart. Also a noncosingular module is strongly T-dual Rickart if and only if M is strongly dual Rickart.

Theorem 5.2. *Let M be an R-module. Then the following are equivalent:*

(1) M is strongly dual Rickart;

(2) M is dual Rickart and each direct summand of M is fully invariant;

(3) M is dual Rickart and $\mathrm{End}(M)$ *is an abelian ring.*

Proof. (1) $\Rightarrow$ (2) Let M be a strongly dual Rickart module. Then it is clear that M is dual Rickart. Let $N \leq^{\oplus} M$. Then $N = eM$ for some $e^2 = e \in \mathrm{End}(M)$. Since $eM = \mathrm{Im}(e)$, N is fully invariant. Hence every direct summand of M is fully invariant.

(2) $\Rightarrow$ (3) Since every direct summand of M is fully invariant, every idempotent of $\mathrm{End}(M)$ is left semicentral. Let $e^2 = e \in \mathrm{End}(M)$. Then e and $1-e$ are left semicentral. Hence e is central. Thus every idempotent of $\mathrm{End}(M)$ is central.

(3) $\Rightarrow$ (1) Is clear. □

We give some characterizations of strongly T-dual Rickart modules.

Theorem 5.3. *The following statements are equivalent for an R-module M:*

(1) M is strongly T-dual Rickart;

(2) M is T-dual Rickart and each direct summand of M which is contained in $\overline{Z}^2(M)$ is fully invariant;

(3) $M = \overline{Z}^2(M) \oplus M'$, where $\overline{Z}^2(M)$ is strongly dual Rickart;

(4) $M = \overline{Z}^2(M) \oplus M'$ and for each $\varphi \in \mathrm{End}(M)$, $\varphi(\overline{Z}^2(M))$ is a fully invariant direct summand of $\overline{Z}^2(M)$;

(5) $M = \overline{Z}^2(M) \oplus M'$, where $\overline{Z}^2(M)$ is dual Rickart and every direct summand of $\overline{Z}^2(M)$ is fully invariant in $\overline{Z}^2(M)$;

(6) $M = \overline{Z}^2(M) \oplus M'$, where $\overline{Z}^2(M)$ is dual Rickart and $\mathrm{End}(\overline{Z}^2(M))$ *is abelian.*

Proof. (1) $\Rightarrow$ (2) Let M be a strongly T-dual Rickart module. It is clear that M is T-dual Rickart. Let K be a direct summand of M which is contained in $\overline{Z}^2(M)$. Hence $K = eM$ for some $e^2 = e \in \mathrm{End}(M)$. We show that $K = e\overline{Z}^2(M)$. Since $M = eM \oplus (1-e)M$, we have $\overline{Z}^2(M) = \overline{Z}^2(eM) \oplus \overline{Z}^2((1-e)M)$ and $e\overline{Z}^2(M) = e\overline{Z}^2(eM) = \overline{Z}^2(eM) = \overline{Z}^2(K) \subseteq K$. As $K \subseteq \overline{Z}^2(M)$, $K \subseteq$

$e\overline{Z}^2(M)$. Therefore $K = e\overline{Z}^2(M)$. By (1), $e\overline{Z}^2(M)$ is a fully invariant direct summand of M, hence K is fully invariant in M.

(2) $\Rightarrow$ (3) By Theorem 3.2, $M = \overline{Z}^2(M) \oplus M'$ and $\overline{Z}^2(M)$ is dual Rickart. Let $N \leq^{\oplus} \overline{Z}^2(M)$. Then $N \leq^{\oplus} M$, because $\overline{Z}^2(M) \leq^{\oplus} M$. By (2), N is fully invariant in M. We show that N is fully invariant in $\overline{Z}^2(M)$. Let $\overline{Z}^2(M) = eM$ for some $e^2 = e \in \mathrm{End}(M)$. Hence $\mathrm{End}(\overline{Z}^2(M)) = e\mathrm{End}(M)e$. Let $f \in \mathrm{End}(\overline{Z}^2(M))$. Then $f = ege$ for some $g \in \mathrm{End}(M)$. Thus $f(N) = ege(N) \subseteq N$, because N is fully invariant in M. Therefore every direct summand of $\overline{Z}^2(M)$ is fully invariant. This implies that $\overline{Z}^2(M)$ is strongly dual Rickart by Theorem 5.2.

(3) $\Rightarrow$ (4) Let $\varphi \in \mathrm{End}(M)$. We can take $\varphi \in \mathrm{End}(\overline{Z}^2(M))$, because $\overline{Z}^2(M)$ is fully invariant in M. By (3), $\overline{Z}^2(M)$ is strongly dual Rickart, so $\varphi(\overline{Z}^2(M))$ is a fully invariant direct summand of $\overline{Z}^2(M)$.

(4) $\Rightarrow$ (5) Let $\varphi \in \mathrm{End}(\overline{Z}^2(M))$ and $\overline{Z}^2(M) = eM$ for some $e^2 = e \in \mathrm{End}(M)$. Then $\varphi e \in \mathrm{End}(M)$. By (4), $\varphi e(\overline{Z}^2(M)) = \varphi(\overline{Z}^2(M))$ is a fully invariant direct summand of $\overline{Z}^2(M)$. Thus $\overline{Z}^2(M)$ is strongly dual Rickart. Theorem 5.2 gives that every direct summand of $\overline{Z}^2(M)$ is fully invariant.

(5) $\Rightarrow$ (6) It is clear from proof of Theorem 5.2.

(6) $\Rightarrow$ (1) Let $\varphi \in \mathrm{End}(M)$ and $\overline{Z}^2(M) = eM$ for some $e^2 = e \in \mathrm{End}(M)$. Then $e\varphi e \in \mathrm{End}(\overline{Z}^2(M))$. Thus $e\varphi e(\overline{Z}^2(M))$ is a fully invariant direct summand of $\overline{Z}^2(M)$. Since $\overline{Z}^2(M) \leq^{\oplus} M$, $e\varphi e(\overline{Z}^2(M)) \leq^{\oplus} M$. As $\varphi(\overline{Z}^2(M)) \subseteq \overline{Z}^2(M)$, $e\varphi e(\overline{Z}^2(M)) = \varphi(\overline{Z}^2(M))$. Since $\overline{Z}^2(M)$ is fully invariant in M and $\varphi(\overline{Z}^2(M))$ is fully invariant in $\overline{Z}^2(M)$, $\varphi(\overline{Z}^2(M))$ is fully invariant in M. Hence M is strongly T-dual Rickart.

$\square$

It is clear that strongly T-dual Rickart modules are T-dual Rickart, but the converse is not true as the following example shows.

Example 5.4. Consider $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-module. It is clear that M is T-dual Rickart, by Proposition 3.17, however M is not strongly T-dual Rickart, by Theorem 5.3, because $\mathrm{End}(\overline{Z}^2(M)) = \mathrm{End}(M)$ is not abelian.

Next, we see that a direct summand of a strongly T-dual Rickart module inherits the property.

Proposition 5.5. *(1) Every direct summand of a strongly dual Rickart module M is strongly dual Rickart.*

(2) Every direct summand of a strongly T-dual Rickart module M is strongly T-dual Rickart.

Proof. (1) Let $N = eM$ for some $e^2 = e \in \mathrm{End}(M)$ and $\varphi \in \mathrm{End}(N)$. Then $\varphi e \in \mathrm{End}(M)$. Therefore $\varphi e(M) = \varphi(N)$ is a fully invariant direct summand of M. By modularity $\varphi(N) \leq^{\oplus} N$. As $\varphi(N)$ is fully invariant in M and $N \leq^{\oplus} M$, $\varphi(N)$ is fully invariant in N. Therefore N is strongly dual Rickart.

(2) Let M be a strongly T-dual Rickart module and $N \leq^{\oplus} M$ say $M = N \oplus N'$. By Theorem 5.3, $\overline{Z}^2(M)$ is strongly dual Rickart. Since $\overline{Z}^2(M) = \overline{Z}^2(N) \oplus \overline{Z}^2(N')$, $\overline{Z}^2(N)$ is strongly dual Rickart by (1). As $\overline{Z}^2(M) \leq^{\oplus} M$, $\overline{Z}^2(N) \leq^{\oplus} N$. Hence N is strongly T-dual Rickart by Theorem 5.3. □

Proposition 5.6. *Let $M = \oplus_{i\in I} M_i$. Then M is strongly dual Rickart if and only if*

(i) M_i is strongly dual Rickart for each $i \in I$.

(ii) For each distinct $i, j \in I$, $\mathrm{Hom}(M_i, M_j) = 0$.

Proof. By Proposition 5.5, for each $i \in I$, M_i is strongly dual Rickart. By Theorem 5.2, each direct summand of M is fully invariant, so for each $i \neq j \in I$, $\mathrm{Hom}(M_i, M_j) = 0$.

Conversely, since for each $i \neq j \in I$, $\mathrm{Hom}(M_i, M_j) = 0$, $\mathrm{End}(M) = \oplus_{i\in I}\mathrm{End}(M_i)$. Let $f \in \mathrm{End}(M)$. Then $f = \oplus_{i\in I} f_i$ where $f_i \in \mathrm{End}(M_i)$ for each $i \in I$. Since M_i is strongly dual Rickart, $f_i(M_i) = e_i M_i$ for some $e_i \in S_l(\mathrm{End}(M_i))$. This implies that $f(M) = \oplus_{i\in I} f_i(M_i) = \oplus_{i\in I} e_i(M_i)$. As $e_i(M_i)$ is a direct summand of M_i for each $i \in I$, $f(M)$ is a direct summand of M. As $\mathrm{End}(M) = \oplus_{i\in I}\mathrm{End}(M_i)$ and for each $i \in I$, $\mathrm{End}(M_i)$ is an abelian ring, $\mathrm{End}(M)$ is an abelian ring. Hence M is strongly dual Rickart by Theorem 5.2. □

Theorem 5.7. *Let $M = \oplus_{i\in I} M_i$. Then the following are equivalent for module M.*

(1) M is strongly dual Rickart;

(2) For each $i \in I$, M_i is strongly T-dual Rickart and for each $i \neq j \in I$, $\mathrm{Hom}(\overline{Z}^2(M_i), \overline{Z}^2(M_j)) = 0$.

Proof. (1) ⇒ (2) Proposition 5.5 implies that for each $i \in I$, M_i is strongly T-dual Rickart. Also Theorem 5.3 gives $\overline{Z}^2(M) = \oplus_{i\in I}\overline{Z}^2(M_i)$ is strongly dual Rickart. Hence for each $i \neq j \in I$, $\mathrm{Hom}(\overline{Z}^2(M_i), \overline{Z}^2(M_j)) = 0$, by Proposition 5.6.

(2) ⇒ (1) By (2) and Theorem 5.3, for each $i \in I$, $M_i = \overline{Z}^2(M_i) \oplus M_i'$ and $\overline{Z}^2(M_i)$ is strongly dual Rickart. Hence $\overline{Z}^2(M) = \oplus_{i\in I}\overline{Z}^2(M_i) \leq^{\oplus} M$. As for each $i \neq j \in I$, $\mathrm{Hom}(\overline{Z}^2(M_i), \overline{Z}^2(M_j)) = 0$, $\overline{Z}^2(M) = \oplus_{i\in I}\overline{Z}^2(M_i)$ is strongly dual Rickart by Proposition 5.6. Hence M is strongly T-dual Rickart by Theorem 5.3. □

Theorem 5.8. *Let R be a ring and F be a free R-module. The following are equivalent:*

(1) F is strongly T-dual Rickart;
(2)(i) R_R is strongly T-dual Rickart and $rank(F) = 1$, or
(ii) $\overline{Z}^2(R) = 0$.

Proof. (1) $\Rightarrow$ (2) Let F be a free R-module and strongly T-dual Rickart with $rank(F) \geq 2$. By Theorem 5.2, R_R is strongly T-dual Rickart. Hence by Theorem 5.3, $R = \overline{Z}^2(R) \oplus R'$. Let $F = R^{(I)}$, then $F = \overline{Z}^2(R)^{(I)} \oplus R'^{(I)}$, where $\overline{Z}^2(R)^{(I)}$ is strongly dual Rickart. Hence its direct summands must be fully invariant. Therefore $Hom(\overline{Z}^2(R), \overline{Z}^2(R)) = 0$ by Proposition 5.6 and so $\overline{Z}^2(R) = 0$.

(2) $\Rightarrow$ (1) Is clear. □

Acknowledgments

The authors are grateful to the referee for suggesting ways to make this paper more readable.

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