ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 3, pp. 643–658

Title:

The existence of global attractor for a Cahn-Hilliard/Allen-Cahn equation

Author(s):

H. Tang, C. Liu and Z. Zhao

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 3, pp. 643–658
Online ISSN: 1735-8515

THE EXISTENCE OF GLOBAL ATTRACTOR FOR A CAHN-HILLIARD/ALLEN-CAHN EQUATION

H. TANG*, C. LIU AND Z. ZHAO

(Communicated by Asadollah Aghajani)

Abstract. In this paper, we consider a Cahn-Hilliard/Allen-Cahn equation. By using the semigroups and the classical existence theorem of global attractors, we give the existence of the global attractor in H^k $(0 \leq k < 5)$ space of this equation, and it attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.
Keywords: Cahn-Hilliard/Allen-Cahn equation, existence, global attractor.
MSC(2010): Primary: 35B41; Secondary: 35K35, 35K55.

1. Introduction

In this article, we consider a scalar Cahn-Hilliard/Allen-Cahn equation

$$u_t = -\Delta[\gamma\Delta u - f(u)] + [\Delta u - f(u)],\ \gamma > 0,\ x \in \Omega, \tag{1.1}$$

with the boundary condition

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \tag{1.2}$$

and the initial value condition

$$u(x, 0) = u_0(x),\ \text{in}\, \Omega, \tag{1.3}$$

where Ω is a smooth bounded domain in $R^n(n \leq 2)$, $\gamma > 0$ is a diffusion constant and $\int_0^u f(s)ds$ is a quartic bistable potential which has zeros at ± 1. In this paper, for simplicity, we set $f(u) = u^3 - u$.

In recent years, the Cahn-Hilliard/Allen-Cahn equation has been studied in different aspects, such as geometric motion [11], triple-junction motion [12], the global attractor for the related dynamical system [4], discrete approximation with logarithmic entropy [5], the generalized solution [3].

Article electronically published on June 29, 2016.
Received: 19 September 2014, Accepted: 17 March 2015.
*Corresponding author.

Equation (1.1) is introduced by Karali and Katsoulakis [6] as a simplification of a mesoscopic model for multiple microscopic mechanisms model in surface processes. G. Karali and Y. Nagase [7] considered a Cahn-Hilliard/Allen-Cahn equation. They only provided existence of the solution for the deterministic case. Then Antonopoulou, Karali and Millet [2] studied the stochastic case for such an equation in the paper. They proved the existence of a global solution, under a specific sub-linear growth condition for the diffusion coefficient. Path regularity in time and in space was also studied. In addition, Karali and Ricciardi [8] constructed special sequences of solutions to a fourth order nonlinear parabolic equation of the CH/AC equation, converging to the second order Allen-Cahn equation. They studied the equivalence of the fourth order equation with a system of two second order elliptic equations.

The dynamic properties of the equation (1.1), such as the global asymptotical behaviours of solutions and existence of global attractors, are important for the study of fourth-order parabolic system. During the past years, many authors have paid much attention to the attractors of Cahn-Hilliard equation or thin-film equation [9, 14, 17, 18].

In this paper, we discuss the existence of global attractors for problem (1.1)-(1.3). By using the estimates of semigroups and the classical existence theorem of global attractors, we give the two main theorems about the existence of global attractor, then we prove the problem possesses global attractor and the existence of global attractors in the H^k space.

2. Preliminary and the main results

In this article, we assume that $m(u)$ is the average of u, such that

$$m(u) = \frac{1}{|\Omega|}\int_\Omega u(x)dx.$$

We let

$$\mathscr{U}_k = \{u | u \in H^2(\Omega),\ |m(u)| \leq k\}.$$

The following results on global existence of solution to the problem (1.1)-(1.3) have been proved in [7].

Lemma 2.1. *Suppose that Ω is a bounded domain in R^2 with smooth boundary $\partial\Omega$, and $u_0 \in \mathscr{U}_k$, then problem (1.1)-(1.3) admits a unique solution u such that*

$$u \in C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap L^4(\Omega \times (0,T)) \ \textit{for all}\ T > 0.$$

By Lemma 2.1, we can define the operator semigroup $S\{(t)\}_{t\geq 0}$ as

$$S(t)u_0 = u(t), \forall u_0 \in \mathscr{U}_k,\ t \geq 0,$$

where $u(t)$ is the solution of (1.1)-(1.3) corresponding to initial value u_0. It is obviously that the operator semigroup $\{S(t)\}_{t\geq 0}$ is continuous.

The following lemma is the classical existence theorem of global attractor by Temam [16].

Lemma 2.2. *Assume that $S(t)$ is the semigroup generated by Equation (1.1), and the following conditions hold:*

(1) *for any bounded set $A \subset L^2(\Omega)$, there exists a time $t_A \geq 0$ such that $S(t)u_0 \in B$, $\forall u_0 \in A$ and $t > t_A$;*
(2) *for any bounded set $U \subset L^2(\Omega)$ and some $T > 0$ sufficiently large, the set $\overline{\cup_{t\geq T} S(t)u}$ is compact in X.*

Then the ω-limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of Equation (1.1), and $\mathcal{A}$ is connected providing B is connected.

Theorem 2.3. *Assume that Ω denotes an open bounded domain in R^2, then the semi-flow associated with the solution u of the problem (1.1)-(1.3) possesses a global attractor $\mathscr{A}$ in space $\mathscr{U}_k$ which attracts all the bounded set in $\mathscr{U}_k$.*

In order to consider the global attractor for Equation (1.1) in H^k space, we introduce the definition as follows:

$$H = \{u \in L^2(\Omega),\ |m(u)| \leq k\},$$
$$H_{\frac{1}{2}} = \{u \in H^2(\Omega) \cap H,\ u|_{\partial\Omega} = 0\},$$
$$H_1 = \{u \in H^4(\Omega) \cap H,\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}.$$

The following lemmas can be found in [13–15].

Lemma 2.4. *Let $u(t, u_0) = S(t)u_0$ ($u_0 \in H$, $t \geq 0$) be a solution of (1.1), and $S(t)$ be the semigroup generated by (1.1). Let H_α be the fractional order space generated by L and assume*

(1) *for some $\alpha \geq 0$, there is a bounded set $B \subset H_\alpha$, which means that for any $u_0 \in H_\alpha$, there exists $t_{u_0} > 0$ such that $u(t, u_0) \in B$, $\forall t > t_{u_0}$;*
(2) *there is a $\beta > \alpha$, for any bounded set $U \subset H_\beta$, and there are $T > 0$ and $C > 0$, such that $\|u(t, u_0)\|_{H_\beta} \leq C$, $\forall t > T$, $u_0 \in U$.*

Then (1.1) has a global attractor $\mathscr{A} \subset H_\alpha$ which attracts any bounded set of H_α in the H_α-norm.

Lemma 2.5. *Let L be a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $\mathrm{Re}\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathscr{L}^\alpha (\mathscr{L} = -L)$, we have*

(1) *$T(t) : H \to H_\alpha$ is a bounded for all $\alpha \in R$, and $t > 0$;*
(2) *$T(t)\mathscr{L}^\alpha x = \mathscr{L}T(t)(x)$, $\forall x \in H_\alpha$;*
(3) *for each $t > 0$, $\mathscr{L}^\alpha T(t)$ is bounded, and $\|\mathscr{L}^\alpha T(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t}$, where some $\delta > 0$, $C_\alpha > 0$ is a constant only depending on α;*
(4) *The H_α - norm can be defined by $\|x\|_{H_\alpha} = \|\mathscr{L}^\alpha x\|_H$.*

The main results is the following.

Theorem 2.6. *Assume that Ω denotes an open bounded domain in R^2, then for all k satisfying $0 \leq k < 5$, the semi-flow associated with the solution u of the problem (1.1)-(1.3) possesses a global attractor $\mathscr{A}$ in space H^k which attracts all the bounded set H^k of in the H^k-norm.*

3. Proofs of main results

In this section, we prove Theorem 2.3 and 2.6. Firstly, we establish some priori estimates for the solution u of problem (1.1)-(1.3). We always assume that $\{S(t)\}_{t\geq 0}$ is the semigroup generated by the weak solution of Equation (1.1) with initial data $u_0 \in \mathscr{U}_k$. Then, the following lemma can be obtained.

Lemma 3.1. *There exists a bounded set $\mathscr{B}$ whose size depends only on k and Ω in $\mathscr{U}_k$, such that for all the orbits starting from any bonded set B in $\mathscr{U}_k$, $\exists\, t_0 = t_0(B) \geq 0$ s.t. $\forall\, t \geq t_0$, all the orbits will stay in $\mathscr{B}$.*

Proof. It suffices to prove that there is a positive constant C such that for large t, the following holds

$$\|u(t)\|_{H^2} \leq C.$$

We prove the lemma in the following steps.

Step 1. Multiplying (1.1) with u, and integrating it over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|\Delta u\|^2 + \|\nabla u\|^2 + \int_\Omega f(u)udx = -\int_\Omega f'(u)|\nabla u|^2 dx.$$

A simple calculation shows that

$$f'(u) = 3u^2 - 1 \geq -C_0,\ C_0 > 0,$$

hence

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|\Delta u\|^2 + \|\nabla u\|^2 + \|u\|_4^4 \\ \leq\ & \frac{C_0}{2}\|u\|^2 + \|u\|^2 + \frac{C_0}{2}\|\Delta u\|^2, \end{aligned}$$

that is

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + (\gamma - \frac{C_0}{2})\|\Delta u\|^2 \leq (\frac{C_0}{2} + 1)\|u\|^2.$$

In addition, using the Poincaré inequality, we get

$$\begin{aligned} &\|u\|^2 \leq C_1\|\nabla u\|^2 + C_2, \\ &C_1\|\nabla u\|^2 \leq \frac{1}{2}\|u\|^2 + m\|\Delta u\|^2, \end{aligned}$$

where C_1, C_2 only depend on n and m. $m(m > 0)$ is a constant, then

$$\|u\|^2 \leq 2m\|\Delta u\|^2 + 2C_2.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+(\frac{\gamma}{2m}-\frac{C_0}{4m}-\frac{C_0}{2}-1)\|u\|^2\leq(\gamma-\frac{C_0}{2})\frac{C_2}{m}.$$

Let γ satisfy $\frac{\gamma}{2m}-\frac{C_0}{4m}-\frac{C_0}{2}-1\geq 0$, owning to the above inequality, we finally arrive at

$$\|u\|^2\leq e^{-(\frac{\gamma}{2m}-\frac{C_0}{4m}-\frac{C_0}{2}-1)t}\|u_0\|^2+\frac{2C_2(2\gamma-C_0)}{2\gamma-C_0-2C_0m-4m}.$$

Thus, for initial data in any bounded set $B\subset\mathscr{U}_k$, there is a uniform time $t_1(B)$ depending on B such that for $t\geq t_1(B)$,

$$\|u\|^2\leq\frac{4C_2(2\gamma-C_0)}{2\gamma-C_0-2C_0m-4m}. \tag{3.1}$$

Step 2. Multiplying (1.1) with Δu, and integrating it over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\gamma\|\nabla\Delta u\|^2+\int_\Omega|\Delta u|^2dx\\ =\ &-6\int_\Omega u|\nabla u|^2\Delta udx-\int_\Omega(3u^2-1)|\Delta u|^2dx+\int_\Omega(u^3-u)\Delta udx,\end{aligned}$$

that is

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\gamma\|\nabla\Delta u\|^2+\int_\Omega|\Delta u|^2dx\\ \leq\ &3\int_\Omega u^2|\Delta u|^2dx+3\int_\Omega|\nabla u|^4dx-\int_\Omega 3u^2|\Delta u|^2dx\\ &+\int_\Omega|\Delta u|^2dx+\kappa\int_\Omega(u^3-u)^2dx+\kappa\int_\Omega|\Delta u|^2dx\\ \leq\ &C_3\|\nabla u\|_4^4+C_3\|\Delta u\|^2+C_3\int_\Omega(u^3-u)^2dx.\end{aligned}$$

On the other hand, using Nirenberg's inequality, we know

$$\begin{aligned}&\|\nabla u\|_4\leq C_4\|\nabla\Delta u\|^{\frac{1}{3}+\frac{n}{12}}\|u\|^{\frac{2}{3}-\frac{n}{12}}+C'_4\|u\|,\ \ (n\leq 2),\\ &\|u\|_6\leq C_4\|\nabla\Delta u\|^{\frac{n}{9}}\|u\|^{1-\frac{n}{9}}+C'_4\|u\|,\ \ (n\leq 2).\end{aligned}$$

That is

$$\begin{aligned}C_3\|\nabla u\|_4^4&\leq\frac{\gamma}{2}\|\nabla\Delta u\|^2+C_5,\\ C_3\|u\|_6^6&\leq C_6\|\nabla\Delta u\|^2+C'_6.\end{aligned}$$

We also have

$$C_3\|\Delta u\|^2\leq\frac{\gamma}{2}\|\nabla\Delta u\|^2+C_7,$$

and

$$C_3\int_\Omega(u^3-u)^2dx\leq\frac{\gamma}{2}\|\nabla\Delta u\|^2+C_8.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{\gamma}{2}\|\nabla\Delta u\|^2 \le C_5 + C_7 + C_8.$$

On the other hand, we know

$$\|\nabla u\|^2 \le C_9\|\nabla\Delta u\|^{\frac{2}{3}} \le C_9'\|\nabla\Delta u\|^2 + C_9''.$$

So

$$\frac{d}{dt}\|\nabla u\|^2 + \frac{\gamma}{C'_9}\|\nabla u\|^2 \le 2(C_5 + C_7 + C_8) + \frac{\gamma}{C'_9}C_9'',$$

which gives

$$\|\nabla u\|^2 \le e^{-\frac{\gamma}{C'_9}t} + \frac{2(C_5 + C_7 + C_8)C'_9 + \gamma C''_9}{\gamma}.$$

Thus, for initial data in any bounded set $B \subset \mathscr{U}_k$, there is a uniform time $t_2(B)$ depending on B such that for $t \ge t_2(B)$,

$$\|\nabla u(x,t)\|^2 \le 2\frac{2(C_5 + C_7 + C_8)C'_9 + \gamma C''_9}{\gamma}. \tag{3.2}$$

Step 3. Multiplying (1.1) with $\Delta^2 u$ and integrating it over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \gamma\|\Delta^2 u\|^2 + \|\nabla\Delta u\|^2 = \int_\Omega \Delta f(u)\Delta^2 u dx - \int_\Omega f(u)\Delta^2 u dx.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \gamma\|\Delta^2 u\|^2 + \|\nabla\Delta u\|^2 \le C_{10}\|\Delta f(u)\|^2 + C_{11}\|f(u)\|^2 + \frac{1}{2}\|\Delta^2 u\|^2.$$

On the other hand, we know

$$\begin{aligned} C_{10}\|\Delta f(u)\|^2 &\le 2C_{10}\Big(\int_\Omega |f'(u)|^2|\Delta u|^2 dx + \int_\Omega |f''(u)|^2|\nabla u|^4 dx\Big) \\ &\le C_{12}\Big[\Big(\int_\Omega |\Delta u|^6 dx\Big)^{\frac{1}{3}} + \Big(\int_\Omega |\nabla u|^6 dx\Big)^{\frac{2}{3}}\Big], \end{aligned}$$

and

$$\|f(u)\|^2 = \int_\Omega (u^3 - u)^2 dx \le C_{13}.$$

Using Nirenberg's inequality, we have

$$\|\nabla u\|_6 \le C_{14}\|\Delta^2 u\|^{\frac{n}{9}}\|\nabla u\|^{1-\frac{n}{9}} + C'_{14}\|\nabla u\|, (n \le 2),$$
$$\|\Delta u\|_6 \le C_{15}\|\Delta^2 u\|^{\frac{3+n}{9}}\|\nabla u\|^{1-\frac{3+n}{9}} + C'_{15}\|\nabla u\|, (n \le 2).$$

Again by Young's inequality,

$$\|\Delta u\|_6^2 \le C_{16}\|\Delta^2 u\|^{\frac{10}{9}} + C'_{16} \le \varepsilon\|\Delta^2 u\|^2 + C_\varepsilon,$$
$$\|\nabla u\|_6^4 \le C_{17}\|\Delta^2 u\|^{\frac{8}{9}} + C'_{17} \le \varepsilon\|\Delta^2 u\|^2 + C_\varepsilon.$$

Then,

$$\begin{aligned} & \frac{d}{dt}\|\Delta u\|^2 + (2\gamma - 1)\|\Delta^2 u\|^2 + 2\|\nabla\Delta u\|^2 \\ \leq\ & 2C_{12}[(\int_\Omega |\Delta u|^6)^{\frac{1}{3}} + (\int_\Omega |\nabla u|^6)^{\frac{2}{3}}] + 2C_{11}C_{13} \\ \leq\ & 4C_{12}(\varepsilon\|\Delta^2 u\|^2 + C_\varepsilon) + 2C_{11}C_{13}. \end{aligned}$$

Thus,

$$\frac{d}{dt}\|\Delta u\|^2 + (2\gamma - 4C_12\varepsilon - 1)\|\Delta^2 u\|^2 + 2\|\nabla\Delta u\|^2 \leq 4C_{12}C_\varepsilon + 2C_11C_13.$$

For ε small enough, we have

$$2\gamma - 4C_{12}\varepsilon - 1 > 0.$$

And applying regularity theorem of elliptic operator, we get

$$\begin{aligned} & \frac{d}{dt}\|\Delta u\|^2 + C_{18}(2\gamma - 4C_{12}\varepsilon - 1)(\|\Delta u\|^2 + \|\nabla\Delta u\|^2) \\ & \leq 4C_{12}C_\varepsilon + 2C_{11}C_{13}. \end{aligned} \tag{3.3}$$

By Grounwall's inequality, when $\|\Delta u(0)\| \leq R$, we have

$$\begin{aligned} \|\Delta u\|^2 \quad & \leq \quad e^{-C_{18}(2\gamma - 4C_{12}\varepsilon - 1)t}\|\Delta u_0\|^2 + \frac{4C_{12}C_\varepsilon + 2C_{11}C_{13}}{C_{18}(2\gamma - 4C_{12}\varepsilon - 1)} \\ & \leq \quad 2\frac{4C_{12}C_\varepsilon + 2C_{11}C_{13}}{C_{18}(2\gamma - 4C_{12}\varepsilon - 1)}, \end{aligned} \tag{3.4}$$

for $t \geq t_3(B) = \frac{1}{C_18(2\gamma - 4C_12\varepsilon - 1)} \ln \frac{C_18(2\gamma - 4C_{12}\varepsilon - 1)R^2}{4C_{12}C_\varepsilon + 2C_11C_13}$.

Adding (3.1), (3.2), (3.4) together, we obtain

$$\|u(x,t)\|_{H^2} \leq C. \tag{3.5}$$

Let $t_0(B) = \max\{t_1(B), t_2(B), t_3(B)\}$, then the lemma is proved. □

The above lemma implies that $\{S(t)\}_{t>0}$ has a bounded absorbing set in $\mathscr{U}_k$. In what follows, we prove the precompactness of the orbit in $\mathscr{U}_k$.

Lemma 3.2. *For any initial data u_0 in any bounded set $B \subset \mathscr{U}_k$, there is a $T(B) > 0$, such that $\|u(x,t)\|_{H^3} \leq C$, $\forall t \geq T > 0$, which turns out that $\cup_{t\geq T} u(t)$ is relatively compact in $\mathscr{U}_k$.*

Proof. The uniform bound of $H^2(\Omega)$-norm of $u(t)$ has been obtained in Lemma 3.1. In the following, we derive the estimate on $H^3(\Omega) - norm$.

Acting Δ on (1.1), we obtain

$$\Delta u_t = -\Delta^2(\gamma\Delta u - f(u)) + \Delta(\Delta u - f(u)). \tag{3.6}$$

Multiplying (3.6) by $\Delta^2 u$ and integrating on Ω, using the boundary conditions, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2+\gamma\|\nabla\Delta^2 u\|^2+\|\Delta^2 u\|^2\\ =&\int_\Omega\nabla\Delta f(u)\nabla\Delta^2 udx-\int_\Omega\nabla f(u)\nabla\Delta^2 udx,\end{aligned}$$

in which

$$\begin{aligned}&\left|\int_\Omega\nabla\Delta f(u)\nabla\Delta^2 udx\right|\\ \leq&\int_\Omega|f'(u)\nabla\Delta u\nabla\Delta^2 u|dx\\ &+3\int_\Omega|f''(u)\nabla u\Delta u\nabla\Delta^2 u|dx+\int_\Omega|f'''(u)|\nabla u|^2\nabla u\nabla\Delta^2 u|dx\\ \leq&\varepsilon\left(\int_\Omega|\nabla\Delta u\nabla\Delta^2 u|dx+3\int_\Omega|\nabla u\Delta u\nabla\Delta^2 u|dx\right.\\ &\left.+\int_\Omega|\nabla u|^2|\nabla u\nabla\Delta^2 u|dx\right)\\ \leq&\varepsilon'(\|\nabla\Delta u\|^2+\|\nabla u\Delta u\|^2+\|\nabla u\|_6^6)+\varepsilon''\|\nabla\Delta^2 u\|^2,\end{aligned}$$

and

$$\begin{aligned}&\left|\int_\Omega\nabla f(u)\nabla\Delta^2 udx\right|\\ =&\left|\int_\Omega f'(u)\nabla u\nabla\Delta^2 udx\right|\leq\varepsilon\left|\int_\Omega\nabla u\nabla\Delta^2 udx\right|\\ \leq&\varepsilon''(\|\nabla u\|^2+\|\nabla\Delta^2 u\|^2).\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2+\gamma\|\nabla\Delta^2 u\|^2+\|\Delta^2 u\|^2\\ \leq&\varepsilon'(\|\nabla\Delta u\|^2+\|\nabla u\Delta u\|^2+\|\nabla u\|_6^6)\\ &+\varepsilon''\|\nabla\Delta^2 u\|^2+\varepsilon''(\|\nabla u\|^2+\|\nabla\Delta^2 u\|^2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\varepsilon'\|\nabla u\Delta u\|^2\leq C\|\nabla u\|_{L^\infty}^2\|\Delta u\|^2\leq C_{19}\|\nabla u\|_{L^\infty}^2,\\ &\|\nabla u\|_{L^\infty}^2\leq C(\|\nabla\Delta^2 u\|^{\frac{n}{4}}\|\nabla u\|^{2-\frac{n}{4}}+\|\nabla u\|^2)\leq\mu\|\nabla\Delta^2 u\|^2+C_\mu,\\ &\|\nabla u\|_6^6\leq C(\|\nabla\Delta^2 u\|^{\frac{n}{2}}\|\nabla u\|^{6-\frac{n}{2}}+\|\nabla u\|^6)\leq\mu\|\nabla\Delta^2 u\|^2+C_\mu.\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2+\gamma\|\nabla\Delta^2 u\|^2+\|\Delta^2 u\|^2$$

$$
\begin{aligned}
&\le \varepsilon'\|\nabla\Delta u\|^2 + C_{19}\|\nabla u\|^2_{L^\infty} \\
&\quad +\varepsilon'\mu\|\nabla\Delta u\|^2 + \varepsilon' C_\mu + \varepsilon''\|\nabla\Delta^2 u\|^2 + C_{20} + \varepsilon''\|\nabla\Delta^2 u\|^2 \\
&\le (\varepsilon'\mu + 2\varepsilon'' + C_{19}\mu)\|\nabla\Delta^2 u\|^2 + \varepsilon'\|\nabla\Delta u\|^2 \\
&\quad + C_{19}C_\mu + \varepsilon' C_\mu + C_{20}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 + (\gamma - \varepsilon'\mu - 2\varepsilon'' - C_{19}\mu)\|\nabla\Delta^2 u\|^2 \\
&\le \varepsilon'\|\nabla\Delta u\|^2 + C_{19}C_\mu + \varepsilon' C_\mu + C_{20}.
\end{aligned}
\tag{3.7}
$$

Taking μ enough small, we have $\gamma - \varepsilon'\mu - 2\varepsilon'' - C_{19}\mu > 0$,

$$
\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 \le \varepsilon'\|\nabla\Delta u\|^2 + C_{19}C_\mu + \varepsilon' C_\mu + C_{20}. \tag{3.8}
$$

On the other hand, integrating (3.3) between t and $t+1$, using (3.4), we have

$$
\begin{aligned}
&C_{18}(2\gamma - 4C_{12}\varepsilon - 1)\int_t^{t+1}\|\nabla\Delta u\|^2 d\tau \\
&\le \|\Delta u(t)\|^2 + 2\frac{4C_{12}C_\varepsilon + 2C_{11}C_{13}}{C_{18}(2\gamma - 4C_{12}\varepsilon - 1)} \le C.
\end{aligned}
\tag{3.9}
$$

Owning to (3.8), (3.9) and the uniform Gronwall inequality in [16], we get

$$
\|\nabla\Delta u\|^2 \le C, t \ge 1.
$$

The lemma is proved. □

Proof of Theorem 2.3. From above we conclude that $\mathscr{A}_k = \omega(\mathscr{B})$, the ω-limit set of absorbing set $\mathscr{B}$ is a global attractor in $\mathscr{U}_k$. By Lemma 3.2, this global attractor is a bounded set in $H^3(\Omega)$. Thus the proof of Theorem 2.3 is complete. □

Corollary 3.3. *Assume that Ω denotes an open bounded domain in R^2, then we have*

$$
\|u(t)\|_\infty \le C, \tag{3.10}
$$

$$
\|\nabla u(t)\|_\infty \le C. \tag{3.11}
$$

Proof. Based on Lemma 3.1, we conclude (3.10). By (1.2), (1.3), (3.7), a simple calculation shows that $\|\nabla\Delta u\|^2 \le C$.

Therefore, combination with Sobolev's imbedding theorem [1], we have

$$
\|\nabla u\|_\infty \le C.
$$

Then, the corollary is proved. □

Based on [10], it is well-known that the solution $u(t,u_0)$ of problem (1.1)-(1.3) can be written as

$$u(t,u_0)=e^{tL}u_0+\int_0^t e^{(t-\tau)L}G(u)\mathrm{d}x,$$

where let $L=-\Delta^2,\ G(u)=\Delta(f(u)+u)-f(u)$.

Then

$$\begin{aligned}u(t,u_0) &= e^{tL}u_0+\int_0^t e^{(t-\tau)L}\Delta(f(u)+u)\mathrm{d}\tau-\int_0^t e^{(t-\tau)L}f(u)\mathrm{d}\tau\\ &= e^{tL}u_0+\int_0^t e^{(t-\tau)L}\Delta g_1(u)d\tau-\int_0^t e^{(t-\tau)L}g_2(u)d\tau,\end{aligned} \tag{3.12}$$

where $g_1(u)=f(u)+u,\ g_2(u)=f(u)$.

Lemma 3.4. *Assume that Ω denotes an open bounded domain in R^2, then for any bounded set $U\subset H_\alpha$, there exists $C>0$ such that*

$$\|u(t,u_0)\|_{H_\alpha}\le C,\ \forall t\ge 0,\ u_0\in U\subset H_\alpha,\ 0\le\alpha<\frac{5}{4}. \tag{3.13}$$

Proof. For $\alpha=\frac{1}{2}$, this follows Theorem 2.3, i.e., for any bounded set $U\subset H_{\frac{1}{2}}$, there is a constant $C>0$ such that

$$\|u(t,u_0)\|_{H_{\frac{1}{2}}}\le C,\ \forall t\ge 0,\ u_0\in U\subset H_{\frac{1}{2}}.$$

Then, we prove (3.13) for any $\alpha>\frac{1}{2}$. There are following steps.

Step 1. We prove that for any bounded set $U\subset H_\alpha(\frac{1}{2}\le\alpha<\frac{3}{4})$, there exists a constant $C>0$ such that

$$\|u(t,u_0)\|_{H_\alpha}\le C,\ \forall t\ge 0,\ u_0\in U,\ \frac{1}{2}\le\alpha<1. \tag{3.14}$$

In fact, by the compact embedding theorems of Rellich-Koapawes [1]:

$$H_\alpha\hookrightarrow W^{1.2},\ H_\alpha\hookrightarrow C_B^0(\Omega),$$

we have,

$$\begin{aligned}\|g_1(u)\|^2_{H_{\frac{1}{4}}} &= \int_\Omega|\nabla g_1(u)|^2dx\\ &\le \int_\Omega|3u^2\nabla u|^2dx\\ &\le C\int_\Omega(|\ u\ |^2|\ \nabla u\ |^2)\mathrm{d}x\\ &\le C(\|u\|^2_{C_B^0}\|u\|^2_{w^{1.2}})\\ &\le C\|u\|^4_{H_\alpha},\end{aligned}$$

and

$$\begin{aligned}
\|g_2(u)\|^2_{H_{\frac{1}{4}}} &= \int_\Omega |\nabla g_2(u)|^2 dx \\
&\le \int_\Omega |\nabla u + 3u^2 \nabla u|^2 dx \\
&\le C \int_\Omega (|\nabla u|^2 + |u|^2 |\nabla u|^2) \mathrm{d}x \\
&\le C(\|u\|^2_{H_\alpha} + \|u\|^4_{H_\alpha}),
\end{aligned}$$

which implies both $g_1 : H_\alpha \to H_{\frac{1}{4}}$ and $g_2 : H_\alpha \to H_{\frac{1}{4}}$ are bounded, then

$$\begin{aligned}
&\|g_1(u(t, u_0))\|_{H_{\frac{1}{4}}} < C, \ \forall t \ge 0, u_0 \in U, \ \tfrac{1}{2} \le \alpha < \tfrac{3}{4}, \\
&\|g_2(u(t, u_0))\|_{H_{\frac{1}{4}}} < C, \ \forall t \ge 0, u_0 \in U, \ \tfrac{1}{2} \le \alpha < \tfrac{3}{4}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\|u(t, u_0)\|_{H_\alpha} \\
\le\ & C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{2}+\alpha} e^{(t-\tau)L} g_1(u)\|_H \mathrm{d}\tau \\
&+ \int_0^t \|(-L)^{\alpha} e^{(t-\tau)L} g_2(u)\|_H \mathrm{d}\tau \\
\le\ & C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{4}+\alpha} e^{(t-\tau)L}\| \cdot \|g_1(u)\|_{H_{\frac{1}{4}}} \mathrm{d}\tau \\
&\quad + \int_0^t \|(-L)^{\alpha-\frac{1}{4}} e^{t-\tau}\| \cdot \|g_2(u))\|_{H_{\frac{1}{4}}} \mathrm{d}\tau \\
&\le C\|u_0\|_{H_\alpha} + C\int_0^\tau \tau^{-\beta} e^{-\delta\tau} d\tau, \\
&\le C, \forall\, t \ge 0, u_0 \in U \subset H_\alpha,
\end{aligned}$$

where $\beta = \alpha + \frac{1}{4}$ $(0 < \beta < 1)$. Hence (3.14) is proved.

Step 2. We prove that for any bounded set $U \subset H_\alpha(\frac{3}{4} \le \alpha < 1)$, there exists a constant $C > 0$ such that

$$\|u(t, u_0)\|_{H_\alpha} \le C, \ \forall t \ge 0, \ u_0 \in U, \frac{3}{4} \le \alpha < 1. \tag{3.15}$$

In fact, by the compact embedding theorems of Rellich-Koapawes:

$$H_\alpha \hookrightarrow W^{2,2}, \ H_\alpha \hookrightarrow W^{1,4}, \ H_\alpha \hookrightarrow C^0_B(\Omega),$$

we deduce that

$$\begin{aligned}
\|g_1(u)\|^2_{H_{\frac{1}{2}}} &= \int_\Omega |\Delta g_1(u)|^2 dx \\
&\le \int_\Omega |6u(\nabla u)^2 + 3u^2 \Delta u|^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_\Omega(|u|^2|\nabla u|^4+|u|^4|\Delta u|^2)dx \\
&\leq C\int_\Omega((\sup_{x\in\Omega}|u|^2)|\nabla u|^4+(\sup_{x\in\Omega}|u|^4)|\Delta u|^2)dx \\
&\leq C(\|u\|_{H_\alpha}^2\|u\|_{H_\alpha}^4+\|u\|_{H_\alpha}^4\|u\|_{H_\alpha}^2) \\
&\leq C\|u\|_{H_\alpha}^6.
\end{aligned}
$$

$$
\begin{aligned}
\|g_2(u)\|_{H_{\frac{1}{2}}}^2 &= \int_\Omega|\Delta g_2(u)|^2dx \\
&\leq \int_\Omega|\Delta u+6u(\nabla u)^2+3u^2\Delta u|^2dx \\
&\leq C\int_\Omega(|u|^2|\nabla u|^4+|u|^4|\Delta u|^2+|\Delta u|^2)dx \\
&\leq C(\|u\|_{H_\alpha}^6+\|u\|_{H_\alpha}^2).
\end{aligned}
$$

It implies both $g_1 : H_\alpha \to H_{\frac{1}{2}}$ and $g_2 : H_\alpha \to H_{\frac{1}{2}}$ are bounded, then

$$
\|g_1(u(t,u_0))\|_{H_{\frac{1}{2}}} < C,\ \forall t\geq 0, u_0\in U,\ \frac{3}{4}\leq\alpha<1,
$$

$$
\|g_2(u(t,u_0))\|_{H_{\frac{1}{2}}} < C,\ \forall t\geq 0, u_0\in U,\ \frac{3}{4}\leq\alpha<1.
$$

Then, we obtain (3.15). In fact by Lemma 3.2 and (3.12), we obtain

$$
\begin{aligned}
&\|u(t,u_0)\|_{H_\alpha} \\
= &\|e^{tL}u_0-\int_0^t(-L)^{\frac{1}{2}}e^{(t-\tau)L}g_1(u)d\tau-\int_0^t e^{(t-\tau)L}g_2(u)d\tau\|_{H_\alpha} \\
\leq &C\|u_0\|_{H_\alpha}+\int_0^t\|(-L)^{\frac{1}{2}+\alpha}e^{(t-\tau)L}\ g_1(u)d\tau\|_H d\tau \\
&+\int_0^t\|(-L)^\alpha e^{(t-\tau)L}g_2(u)d\tau\|_H d\tau \\
\leq &C\|u_0\|_{H_\alpha}+\int_0^t\|(-L)^\alpha e^{(t-\tau)L}d\tau\|_H\int_0^t\|g_1(u)d\tau\|_{H_{\frac{1}{2}}}d\tau \\
&+\int_0^t\|(-L)^{\alpha-\frac{1}{2}}e^{(t-\tau)L}f(u)d\tau\|_H\int_0^t\|g_2(u)d\tau\|_{H_{\frac{1}{2}}}d\tau \\
\leq &C\|u_0\|_{H_\alpha}+C\int_0^t\tau^{-\beta}e^{-\delta t}d\tau\leq C,
\end{aligned}
$$

where $\beta=\alpha$ $(0<\beta<1)$. Then (3.15) is proved.

Step 3. We prove that for any bounded set $U\subset H_\alpha$ $(1\leq\alpha<\frac{5}{4})$, there exists a constant $C>0$ such that

$$
\|u(t,u_0)\|_{H_\alpha}\leq C,\ \forall t\geq 0,\ u_0\in U\in H_\alpha,\ 1\leq\alpha<\frac{5}{4}. \tag{3.16}
$$

In fact, by the compact embedding theorems of Rellich-koapawes:

$$H_\alpha \hookrightarrow W^{1,6},\ H_\alpha \hookrightarrow W^{1,4},\ H_\alpha \hookrightarrow W^{2,4},\ H_\alpha \hookrightarrow W^{3,2},\ H_\alpha \hookrightarrow C_B^0(\Omega),$$

we have,

$$\begin{aligned}
&\|g_1(u)\|^2_{H_{\frac{3}{4}}} = \int_\Omega |\nabla(\Delta g_1(u))|^2 dx \\
\le& C\int_\Omega (|\nabla u|^6 + |u|^4|\nabla u|^4 + |\Delta u|^4 + |u|^4|\nabla\Delta u|^2)dx \\
\le& C\int_\Omega (|\nabla u|^6 + (\sup_{x\in\Omega}|u|^4)|\nabla u|^4 + |\Delta u|^4 + (\sup_{x\in\Omega}|u|^4)|\nabla\Delta u|^2)dx \\
\le& C(\|u\|^6_{w^{1,6}} + \|u\|^4_{C_B^0}\|u\|^4_{w^{1,4}} + \|u\|^4_{w^{2,4}} + \|u\|^4_{C_B^0}\|u\|^2_{w^{3,2}}) \\
\le& C(\|u\|^2_{H_\alpha} + \|u\|^4_{H_\alpha} + \|u\|^6_{H_\alpha} + \|u\|^8_{H_\alpha}), \\
&\|g_2(u)\|^2_{H_{\frac{3}{4}}} = \int_\Omega |\nabla(\Delta g_2(u))|^2 dx \\
\le& C\int_\Omega (|\nabla\Delta u|^2 + |\nabla u|^6 + |u|^4|\nabla u|^4 + |\Delta u|^4 + |u|^4|\nabla\Delta u|^2)dx \\
\le& C(\|u\|^2_{H_\alpha} + \|u\|^4_{H_\alpha} + \|u\|^6_{H_\alpha} + \|u\|^8_{H_\alpha}),
\end{aligned}$$

which implies both $g_1 : H_\alpha \to H_{\frac{3}{4}}$ and $g_2 : H_\alpha \to H_{\frac{3}{4}}$ are bounded,

$$\|g_1(u(t,u_0))\|_{H_{\frac{3}{4}}} < C,\ \forall t \ge 0, u_0 \in U,\ 1 \le \alpha < \frac{5}{4},$$

$$\|g_2(u(t,u_0))\|_{H_{\frac{3}{4}}} < C,\ \forall t \ge 0, u_0 \in U,\ 1 \le \alpha < \frac{5}{4}.$$

Then, we obtain

$$\begin{aligned}
&\|u(t,u_0)\|_{H_\alpha} \\
=\ & \|e^{tL}u_0 - \int_0^t (-L)^{\frac{1}{2}} e^{(t-\tau)L} g_1(u)d\tau - \int_0^t e^{(t-\tau)L} g_2(u)d\tau\|_{H_\alpha} \\
\le\ & C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\frac{1}{2}+\alpha} e^{(t-\tau)L} g_1(u)d\tau\|_H d\tau \\
&+ \int_0^t \|(-L)^\alpha e^{(t-\tau)L} g_2(u)d\tau\|_H d\tau \\
\le\ & C\|u_0\|_{H_\alpha} + \int_0^t \|(-L)^{\alpha-\frac{1}{4}} e^{(t-\tau)L} d\tau\|_H \int_0^t \|g_1(u)d\tau\|_{H_{\frac{3}{4}}} d\tau \\
&+ \int_0^t \|(-L)^{\alpha-\frac{3}{4}} e^{(t-\tau)L} f(u)d\tau\|_H \int_0^t \|g_2(u)d\tau\|_{H_{\frac{3}{4}}} d\tau \\
\le\ & C\|u_0\|_{H_\alpha} + C\int_0^t \tau^{-\beta} e^{-\delta t} d\tau \\
\le\ & C,\ \forall t \ge 0,\ u_0 \in U \subset H_\alpha,
\end{aligned}$$

where $\beta = \alpha - \frac{1}{4}$ $(0 < \beta < 1)$. Hence (3.16) is proved.

Using the same method as the above, we can prove that for any bounded set $U \subset H_\alpha(\alpha \geq 0)$, there is a constant $C > 0$ such that i.e., for all $\alpha \geq 0$ the semigroup $S(t)$ generated by the problem (1.1)-(1.3) is uniformly compact in H_α. □

Lemma 3.5. *Assume that* Ω *denotes an open bounded in* R^2*, then for any bounded set* $U \subset H_\alpha$ $(0 \leq \alpha < \frac{5}{4})$*, there exists* $T > 0$*, and a constant* $C > 0$*, independent of* u_0*, such that*

$$\|u(t, u_0)\|_{H_\alpha} \leq C, \ \forall t \geq 0, \ u_0 \in U \subset H_\alpha. \tag{3.17}$$

Proof. For $\alpha = \frac{1}{2}$, this follows from Theorem 2.3. Then, we prove (3.17) for any $\alpha > \frac{1}{2}$. The steps is the following:

Step 1. we have

$$u(t, u_0) = e^{(t-T)L}u(T, u_0) - \int_T^t (-L)^{\frac{1}{2}} e^{(t-T)L} g_1(u) d\tau - \int_T^t e^{(t-T)L} g_2(u) d\tau.$$

Let $B \subset H_{\frac{1}{2}}$ be the bounded absorbing set of the problem (1.1)-(1.3), the time such that

$$u(t, u_0) \in B, \ \forall t > t_0 > 0, \ u_0 \in U \subset H_\alpha, \ \alpha \geq \frac{1}{2}.$$

On the other hand, it is known that

$$\|e^{tL}\| \leq Ce^{-d\lambda_1 t},$$

where $\lambda_1 > 0$ is the first eigenvalue of the equation,

$$\begin{cases} -\Delta u = \lambda u, \\ u|_{\partial\Omega} = 0. \end{cases}$$

For any given $T > 0$ and $u_0 \in U \subset H_\alpha(\alpha \geq \frac{1}{2})$. We can obtain

$$\lim_{t \to \infty} \|e^{(t-T)L}u(T, u_0)\|_{H_\alpha} = 0.$$

Then,

$$\begin{aligned}
&\|u(t, u_0)\|_{H_\alpha} \\
\leq\ & \|e^{(t-t_0)L}u(t_0, u_0)\|_{H_\alpha} + \int_{t_0}^t \|(-L)^{\alpha+\frac{1}{4}} e^{(t-T)L}\| \|g_1(u)\|_{H_{\frac{1}{4}}} d\tau \\
& + \int_{t_0}^t \|(-L)^{\alpha-\frac{1}{4}} e^{(t-T)L}\| \|g_2(u)\|_{H_{\frac{1}{4}}} d\tau \\
\leq\ & \|e^{(t-t_0)L}u(t_0, u_0)\|_{H_\alpha} + C\int_{t_0}^t \|(-L)^{\frac{1}{4}+\alpha} e^{(t-T)L}\| d\tau \\
& + C\int_{t_0}^t \|(-L)^{\alpha-\frac{1}{4}} e^{(t-T)L}\| d\tau
\end{aligned}$$

$$\leq \quad \|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C\int_0^{T-t_0} \tau^{-\alpha-\frac{1}{4}}e^{-\delta t}d\tau$$
$$\leq \quad \|e^{(t-t_0)L}u(t_0,u_0)\|_{H_\alpha} + C.$$

We have that (3.17) holds for all $\frac{1}{2} \leq \alpha < \frac{3}{4}$.

Step 2. By the same method as the above steps, we can prove that for any $\frac{3}{4} \leq \alpha < 1$, $1 \leq \alpha < \frac{5}{4}$, the problem (1.1)-(1.3) has a bounded absorbing set in H_α. By the same method, we can obtain that (3.17) holds for all $0 \leq \alpha < \frac{1}{2}$. □

Proof of Theorem 2.6. By Lemma 3.4 and 3.5, the proof of Theorem 2.6 is completed. □

Remark 3.6. For problem(1.1)-(1.3), by the same method as in [13–15], we can prove the existence of global attractor in H^kspace, where $k \in R^+$.

Acknowledgments

The authors would like to express their deep thanks to the referee's valuable suggestions for the revision and improvement of the manuscript. This work is partially supported by the Natural Science Foundation of Changchun Normal University, [2013], NO. 04.

References

[1] R. A. Adams, Sobolev Spaces, Pure and Applied Mathematics, 65, Academic Press, A subsidiary of Harcourt Brace Jovanovich, Publishers, New York-London, 1975.

[2] D. C. Antonopoulou, G. Karali and A. Millet, Existence and regularity of solution for a stochastic Cahn-Hilliard/Allen-Cahn equation with unbounded noise diffusion, *J. Differential Equations* **260** (2016), no. 3, 2383–2417.

[3] J. L. Boldrini and P. N. Da Silva, A generalized solution to a Cahn-Hilliard/Allen-Cahn System, *Electron. J. Differential Equations* **2004** (2004), no. 126, 24 pages.

[4] M. Gokieli and A. Ito, Global attractor for the Cahn-Hilliard/Allen-Cahn system, *Nonlinear Analysis* **52** (2003), no. 7, 1821–1841.

[5] M. Gokieli and L. Marcinkowski, Discrete Approximation of the Cahn-Hilliard/Allen-Cahn System with Logarithmic Entropy, *Japan J. Indust. Appl. Math.* **20** (2003), no. 3, 321–351.

[6] G. Karali and M. A. Katsoulakis, The role of multiple microscopic mechanisms in cluster interface evolution, *J. Differential Equations* **235** (2007), no. 2, 418–438.

[7] G. Karali and Y. Nagase, On the existence of solution for a Cahn-Hilliard/Allen-Cahn equation, *Discrete Contin. Dyn. Syst. Ser. S* **7** (2014), no. 1, 127–137.

[8] G. Karali and T. Ricciardi, On the convergence of a fourth order evolution equation to the Allen-Cahn equation, *Nonlinear Anal.* **72** (2010), no. 11, 4271–4281.

[9] D. Li and C. Zhong, Global attractor for the Cahn-Hilliard system with fast growing nonlinearity, *J. Differential Equations* **149** (1998), no. 2, 191–210.

[10] T. Ma and S. Wang, Stability and Bifurcation of Nonlinear Evolution Equations, Science Press, Beijing, 2006.

[11] A. Novick-Cohen and L. P. Hari, Geometric motion for a degenerate Allen-Cahn/Cahn-Hilliard system: The partial wetting case, *Phys. D* **209** (2005), no. 1-4, 205–235.

[12] A. Novick-Cohen, Triple-junction motion for an Allen-Cahn/Cahn-Hilliard system, *Phys. D* **137** (2000), no. 1-2, 1–24.
[13] L. Song, Y. Zhang and T. Ma, Global attractor of a modified Swift-Hohenberg equation in H^k spaces, *Nonlinear Anal.* **72** (2010), no. 1, 183–191.
[14] L. Song, Y. Zhang and T. Ma, Global attractor of the Cahn-Hilliard equation in H^k spaces, *J. Math. Anal. Appl.* **355** (2009), no. 1, 53–62.
[15] L. Song, Y. He and Y. Zhang, The existence of global attractors for semilinear parabolic equation in H^k spaces, *Nonlinear Anal.* **68** (2008), no. 11, 3541–3549.
[16] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1988
[17] H. Wu and S. M. Zheng, Global attractor for the 1-D thin film equation, *Asymptot. Anal.* **51** (2007), no. 2, 101–111.
[18] X. Zhao and C. Liu, The existence of global attractor for a fourth-order parabolic equation, *Appl. Anal.* **92** (2013), no. 1, 44–59.

(Hui Tang) Department of Mathematics, Jilin University, Changchun 130012, P.R. China;
and
School of Science, Changchun University, Changchun 130022, P.R. China.
E-mail address: tangtangth83@163.com

(Changchun Liu) Department of Mathematics, Jilin University, Changchun 130012, P.R. China.
E-mail address: liucc@jlu.edu.cn

(Zhixin Zhao) Department of Mathematics, Changchun Normal University, Changchun 130032, P.R. China
and
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P.R. China.
E-mail address: jczzx10@163.com