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## Bounding cochordal cover number of graphs via vertex stretching

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# BOUNDING COCHORDAL COVER NUMBER OF GRAPHS VIA VERTEX STRETCHING

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Abstract. In this paper, it is shown that when a special vertex stretching is applied to a graph, the cochordal cover number of the graph increases at most two. As a consequence, it is shown that the induced matching number and cochordal cover number of a special vertex stretching of a graph $G$ are equal provided $G$ is well-covered bipartite or weakly chordal graph.

**Keywords:** Castelnuovo-Mumford regularity, induced matching number, cochordal cover number.
**MSC(2010):** Primary: 13F55; Secondary: 05C70, 05E45.

## 1. **Introduction**

Let $G$ be a simple graph with vertex set $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. A *matching* in a graph $G$ is a set of edges $M$ such that no two edges share a common end. If $M$ is an induced subgraph, the matching is called an *induced matching*. The maximum size of an induced matching in $G$ is called the *induced matching number* of $G$ and denoted by indmatch($G$). The problem of finding a maximum induced matching is known to be NP-hard.

Let $G$ be a graph and $x$ be a vertex in $G$. A *vertex stretching* with respect to $x$ is defined to be the transformation consisting of the following steps:

(1) partition the open neighborhood $N(x)$ of vertex $x$ into two disjoint subsets $Y$ and $Z$ in an arbitrary way;
(2) delete vertex $x$ from the graph together with incident edges;
(3) add a chordless path on $k$ vertices $(a, x_1, \cdots, x_{k-2}, b)$ to the remaining graph;
(4) connect $a$ to each vertex in $Y$ , and connect $b$ to each vertex in $Z$.

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Notice that $k$ is a parameter associated with the transformation. We shall denote the vertex stretching with parameter $k$ by $\mathcal{P}^k$. The graph produced by this operation will be denoted $\mathcal{P}^k(G) = \mathcal{P}^k(G, x)$.

Vertex stretching have been applied to different problems for several goals (see e.g. [1, 4, 8, 12] and [2]).

Although the problem of finding a maximum induced matching is known to be NP-hard, in [11] Lozin showed that $\mathcal{P}^4$ increases the size of a maximum induced matching by one. As a consequence, Lozin showed that the induced matching problem remains NP-hard in some special classes of bipartite graphs such as bipartite graphs with maximum degree 3 or $C_4$-free bipartite graphs.

Let $x$ and $Y$ be non adjacent vertices of a graph $G$. *Identify* $x$ and $y$ means to replace these vertices by a single vertex incident to all the edges which were incident in $G$ to either $x$ or $y$. We denote the resulting graph by $G/\{x, y\}$.

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. The *edge ideal* of $G$ in $R$ is defined by $\mathcal{I}(G) = (x_i x_j \mid \{i, j\} \in E(G))$. In the recent years, several authors have studied the Castelnuovo–Mumford regularity of $R/\mathcal{I}(G)$ denoted by $\mathrm{reg}(R/\mathcal{I}(G))$. It is known that $\mathrm{indmatch}(G) \leq \mathrm{reg}(R/\mathcal{I}(G))$, cf. [7, Lemma 2.2]. Recently, Biyikoglu and Civan [3] get motivated to show that $\mathcal{P}^4$ increases the size of the Castelnuovo-Mumford regularity by one.

In [13] Woodroofe introduced the notion of cochordal cover number of a graph to study the Castelnuovo–Mumford regularity of a graph. A graph $G$ is called *chordal* if every induced cycle in $G$ has length 3, and is *cochordal* if the complement graph $G^c$ is chordal. The *cochordal cover number* of $G$ is the minimum number of cochordal subgraphs required to cover the edges of $G$ and it is denoted by $\mathrm{cochord}(G)$.

Combining [7, Lemma 2.2] and [13, Lemma 1] lead to the following inequalities:

$$\mathrm{indmatch}(G) \leq \mathrm{reg}(R/\mathcal{I}(G)) \leq \mathrm{cochord}(G).$$

Now it is natural to ask: what can we say about the cochordal cover number of $\mathcal{P}^4(G)$?. In this note, we show that $\mathcal{P}^4$ increases the cochordal cover number by at most two. Using [11, Corollary 1] this result implies that the computational complexity of the cochordal cover number of arbitrary graphs is equivalent to that of bipartite graphs having sufficiently large girth with maximum degree three.

## 2. **The results**

To prove the main result we need the following lemmas.

**Lemma 2.1.** *Let $G$ be a graph and $x, y$ be two non-adjacent vertices of $G$. If $N(x) \bigcap N(y) = \varnothing$ and $G$ is cochordal graph, then $H = G/\{x, y\}$ is cochordal graph.*

*Proof.* At first note that every vertex of $N_G(x)$ is adjacent to every vertex of $N_G(y)$, since otherwise if $x' \in N_G(x)$ is not adjacent to $y' \in N_G(y)$ in $G$, then $x - y - x' - y' - x$ is an induced cycle of length four in $G^c$, which is a contradiction.
Let $x = y = a$ in graph $H$. Suppose that $C$ is an induced cycle of length $k$ in $(G/\{x, y\})^c$. If $a$ is not a vertex of $C$, then $C$ is a cycle of $G^c$ and hence $k = 3$. Suppose that $a$ is a vertex of $C$ and

$$C : a = v_1 - v_2 - \cdots - v_k - v_1 = a.$$

Then $v_2, v_k \notin N_H(a)$ and $v_3, v_4, \cdots, v_{k-1} \in N_H(a)$. Since $N_H(a) = N_G(x) \bigcup N_G(y)$, we conclude that $\{v_2, v_k\} \subseteq N_{G^c}(x) \bigcap N_{G^c}(y)$ and $v_i \in N_G(x) \bigcup N_G(y)$. Suppose that $v_3 \in N_G(x)$. Since every vertex of $N_G(x)$ is adjacent to every vertex of $N_G(y)$, we conclude that $v_4 \in N_G(x)$. By the same argument we conclude that $\{v_3, v_4, \cdots, v_{k-1}\} \subseteq N_G(x)$. Hence the cycle $x - v_2 - v_3 - \cdots - v_{k-1} - v_k - x$ is an induced cycle of $G^c$. Thus $k = 3$ and $H$ is a chordal graph. □

**Lemma 2.2.** *Let $G$ be a cochordal graph. Then*

$$2 \leq \text{cochord}(\mathcal{P}^4(G)) \leq 3.$$

*Proof.* Let $N_G(x) = X \bigcup Y$ be a partition of neighborhoods of $x$ and $\mathcal{P}^4(G) = \mathcal{P}^4(G, x)$ is obtained from $G$, by deleting the vertex $x$, adding the path $x_1 - a - b - x_2$, joining the vertex $x_1$ to vertices in $X$ and joining the vertex $x_2$ to vertices in $Y$. Define three subgraph $G_1, G_2, G_3$ of $\mathcal{P}^4(G)$ as follows:

$$V(G_1) = X \bigcup \{x_1, a, b\}, E(G_1) = \{x_1a, ab\} \bigcup \{x_1z : z \in X\},$$

$$V(G_2) = Y \bigcup \{x_2, b\}, E(G_2) = \{x_2b\} \bigcup \{x_2z : z \in Y\},$$

$$V(G_3) = V(\mathcal{P}^4(G)) \setminus \{x_1, x_2, a, b\}, E(G_3) = E(\mathcal{P}^4(G)) \setminus (E(G_1) \bigcup E(G_2)).$$

It is not difficult to see that $G_i$'s are cochordal edge-disjoint subgraphs $\mathcal{P}^4(G)$ and cover the edges of $\mathcal{P}^4(G)$. Therefore $\text{cochord}(\mathcal{P}^4(G)) \leq 3$. On the other hand if $X \neq \varnothing$ (or $Y \neq \varnothing$), we can consider $z \in X (z \in Y)$ and by considering four vertices $z, x_1, x_2b$ (or $a$) we have a induced cycle $C_4$ in $\mathcal{P}^4(G)^c$. Hence $2 \leq \text{cochord}(\mathcal{P}^4(G))$. □

**Remark 2.3.** Note that for any graph $G$, the cochord number of $\mathcal{P}^k(G) = \mathcal{P}^k(G, x)$ dependents on $x$ and partition of $N(x)$; for example $\mathcal{P}^4(C_4)$ is $C_7$ or $C_4.P_4$ (this is a graph union of $C_4$ and $P_4$ where their intersection has just one vertex which is the end vertex of $P_4$) and we have $\text{cochord}(C_7) = 3$ and $\text{cochord}(C_4.P_4) = 2$ and we have $\text{cochord}(C_7) = 3$ and $\text{cochord}(C_4.P_4) = 2$.

**Lemma 2.4.** *Let $G$ be a cochordal graph. Then* $\text{cochord}(\mathcal{P}^4(G, x)) = 2$*, if one of the following holds:*

*i) One of the partitions of $N(x)$ is empty;*
*ii) Induced subgraph $\langle N_G(x)\rangle$ is a clique.*

*Proof.* i) By the same notation as in the proof of Lemma 2.2, Suppose that $Y = \varnothing$. Hence two graphs

$$G_1 : x_1 - a - b - x_2, G_2 : \mathcal{P}^4(G) \setminus \{a, b, x_2\}$$

are edge-disjoint cochordal subgraphs of $\mathcal{P}^4(G)$ which cover the edge set of $\mathcal{P}^4(G)$. Hence $\text{cochord}(\mathcal{P}^4(G)) \leq 2$. Therefore $\text{cochord}(\mathcal{P}^4(G)) = 2$
ii) In this case consider the following edge-disjoint subgraphs of $\mathcal{P}^4(G)$:

$$G_1 : x_1 - a - b - x_2, G_2 : \mathcal{P}^4(G) \setminus \{a, b\}.$$

The subgraph $G_1$ is a cochordal graph, since $G_1$ is a path of length three. We prove that the subgraph $G_2$ is a cochordal graph. Suppose that $C$ is an induced cycle of length $k$ in $G_2^c$. If $x_1, x_2 \notin V(C)$, then $C$ is an induced subgraph of $G^c$, and hence $k = 3$. Suppose that $x_1, x_2 \in C$ and hence we can show the cycle $C$ as follows:

$$C : x_1 = v_1 - v_2 - \cdots - v_{k-1} - (v_k = x_2) - x_1 = v_1.$$

Therefore

$$\{v_3, v_4, \cdots, v_{k-1}\} \subseteq N_G(x_1) = X,$$

$$\{v_2, v_3, \cdots, v_{k-2}\} \subseteq N_G(x_2) = Y.$$

But $X \bigcap Y = \varnothing$, implies that $k \leq 4$. Suppose that $k = 4$. Hence $v_2$ is adjacent to $v_3$ in $G_2^c$, which is a contradiction, since $N_G(x)$ is a clique. Therefore $k = 3$. Now consider the case $|V(C) \bigcap \{x_1, x_2\}| = 1$. Without lose of generality assume that $x_1 \in V(C)$ and $x_2 \notin V(C)$ (the proof of the case $x_1 \notin V(C)$ and $x_2 \in V(C)$ is similar). Again suppose that the cycle $C$ is as the following form:

$$C : x_1 = v_1 - v_2 - \cdots - v_k - v_1 = x_1.$$

Hence

$$\{v_3, v_4, \cdots, v_{k-1}\} \subseteq N_G(x_1) = X,$$

Since $N_G(x)$ is a clique and $X \subseteq N_G(x)$, we conclude that $k \leq 4$. If $k = 4$, then $v_3 \in X$ and $v_2, v_4 \notin N_G(x)$. Hence the cycle $x - v_2 - v_3 - v_4 - x$ is an induced cycle of length four in $G^c$, which is a contradiction. Hence $k = 3$ and therefore $G_2$ is a cochordal graph. □

**Corollary 2.5.** *For $n \geq 2, \text{cochord}(\mathcal{P}^4(K_n)) = 2$.*

**Lemma 2.6.** *Let $G$ be a cochordal graph. If $G$ has an induced cycle $C_4$, then there exists a vertex $x$, such that $\text{cochord}(\mathcal{P}^4(G)) = 3$, for some partitions of $N_G(x)$.*

*Proof.* Let $x$ be a vertex of induced cycle $C_4$ of $G$ and $y$ be a neighborhood of $x$ among the vertices of cycle $C_4$. Let $X = N_G(x) \setminus \{y\}$ and $Y = \{y\}$. By this partition of $N_G(x)$, the graph $\mathcal{P}^4(G)$ has an induced cycle $C_7$. If $\text{cochord}(\mathcal{P}^4(G)) = 2$ and $G_1$ and $G_2$ are cochrdal, edge-disjoint subgraphs, which cover the edges of $\mathcal{P}^4(G)$, then $G_1$ or $G_2$ has an induced path $P_5$. Hence one of the subgraphs $G_1$ or $G_2$ are not cochordal graph, which is a contradiction. Hence $\text{cochord}(\mathcal{P}^4(G)) = 3$. □

**Theorem 2.7.** *Let $G$ be a simple graph. Then*

$$\text{cochord}(G) + 1 \leq \text{cochord}(\mathcal{P}^4(G)).$$

*Proof.* Let $x \in V(G), Y, Z \subseteq N(x)$ such that $Y \cup Z = N(x)$ and $Y \cap Z = \varnothing$.
Consider four new vertices $x_1, x_2, a, b$. Suppose that

$$\mathcal{P}^4(G) = (G - x) + \{ay : y \in Y\} + \{bz : z \in Z\} + \{x_1a, x_1x_2, x_2b\}.$$

Let $\text{cochord}(\mathcal{P}^4(G)) = \ell$ and $G_1, \cdots G_\ell$ be edge disjoint subgraphs of $\mathcal{P}^4(G)$ such that $E(\mathcal{P}^4(G)) = \bigcup_1^\ell E(G_i)$ and ${G_i}^c$ is chordal graph for any $1 \leq i \leq \ell$. Since $E(G_i)$s cover the edges of $\mathcal{P}^4(G)$, there exists an $i_0$ such that $x_2b \in E(G_{i_0})$. If $a \notin V(G_{i_0})$, then $N_{G_{i_0}}(a) = \varnothing$ and if $a \in V(G_{i_0})$, since $G_{i_0}^c$ is chordal, then $N_{G_{i_0}}(a) = \{x_1\}$. In any case $N_{G_{i_0}}(a) \subseteq \{x_1\}$. In addition, if the edges $x_1a, x_2b \in E(G_{i_0})$, then $x_1x_2 \in E(G_{i_0})$. Otherwise the edges $x_1x_2, x_1b, ab, x_2a \in E({G_{i_0}}^c)$ and therefore $G_{i_0}^c$ has an induced cycle $C_4$, which is a contradiction. Thus the path $a - x_1 - x_2 - b$ is an induced subgraph of $G_{i_0}$. Since $G_{i_0}^c$ is chordal, we have $G_{i_0} = a - x_1 - x_2 - b$. Now we for any $1 \leq i \leq \ell$ and $i \neq i_0$ set the graph, $H_i := G_i/\{a, b\}$ if $a, b \in V(G_i)$ and $H_i := G_i$, otherwise. Therefore $H_1, H_2, \cdots, H_{i_0-1}, H_{i_0+1}, \cdots, H_\ell$ is a partition of edges of $G$ and each $H_i^c$ is chordal by Lemma 2.1. Hence we conclude that

$$\text{cochord}(G) \leq \text{cochord}(\mathcal{P}^4(G)) - 1.$$

Since $E(G_i)$s cover the edges of $\mathcal{P}^4(G)$, without loss of generality, we can assume $x_2b \in E(G_1)$ and $x_1a \in E(G_2)$. If $x_1x_2 \notin E(G_1) \bigcup E(G_2)$, then there exists $3 \leq i \leq \ell$, such that $x_1x_2 \in E(G_i)$. Since ${G_i}^c$ is a chordal graph, we conclude that $G_i \cong K_2$. Now set

$$H_1 := G_1 - x_2, H_2 := G_2 - x_1.$$

We conclude that the edge sets of subgraphs

$$H_1, H_2, G_3, \cdots, G_{i-1}, G_{i+1}, \cdots, G_\ell$$

is a partition of edges of $G$ and their complements are chordal. Therefore

$$\text{cochord}(G) \leq \text{cochord}(\mathcal{P}^4(G)) - 1.$$

If $x_1x_2 \in E(G_1)$, then $N_{G_1}(b)$ is an independent set, because the complement of $G_2$ is a chordal graph. Suppose that $N_{G_1}(b) = \{x_2, w_1, w_2, \cdots, w_t\}$ and consider the graph

$$H := (\mathcal{P}^4(G) - \{bw_1, bw_2, \cdots, bw_t\}) + \{aw_1, aw_2, \cdots, aw_t\}.$$

Set

$$H_1 := a - x_1 - x_2 - b, H_2 := G_2 + \{aw_1, aw_2, \cdots, aw_t\}$$

and $H_i := G_i$ for $3 \leq i \leq \ell$. It is not difficult to see that the edges of subgraphs $H_1, H_2, \cdots H_\ell$ is a partition of edges of $H$ and $H_i^c$ are chordal for $3 \leq i \leq \ell$. Hence the edges of subgraphs $H_2, H_3, \cdots, H_\ell$ is a partition of edges of $G$ and we conclude that

$$\text{cochord}(G) \leq \text{cochord}(H) \leq \text{cochord}(\mathcal{P}^4(G)) - 1.$$

□

## 3. Open problems

**Problem 3.1.** *Prove or disprove: Let $G$ be a graph with* $\text{cochord}(G) = \ell$. *For any $x \in V(G)$, there exist a cochordal cover $G_1, \cdots G_\ell$, such that there exists $1 \leq i \leq \ell$ which $xv \in E(G_i)$, for any $v \in N(x)$.*

**Problem 3.2.** *For any graph $G$,*

$$\text{cochord}(\mathcal{P}^4(G)) \leq \text{cochord}(G) + 2.$$

If the statement of Problem 3.1 is true, then we can solve the Problem 3.2. Suppose that $\text{cochord}(G) = \ell$ and $G_1, \cdots G_\ell$ be edge disjoint subgraphs of $G$ such that $E(G) = \bigcup_1^l E(G_i)$, and ${G_i}^c$ are chordal graphs. Without loss of generality we can assume that $xv \in E(G_1)$ for any $v \in N(x)$. Hence $\mathcal{P}^4(G_1)$, $G_2, \cdots G_\ell$, partitioned the edges of $\mathcal{P}^4(G)$. Now since $G_1$ is a chordal graph, we conclude that $\text{cochord}(\mathcal{P}^4(G)) \leq \text{cochord}(G) + 2$, by appling Lemma 2.2. Also if the statement of Problem 3.1 is true, then by applying Lemma 2.4, the following lemma is true.

**Lemma 3.3.** *Let $G$ be a graph. Therefore* $\text{cochord}(\mathcal{P}^4(G, x)) = \text{cochord}(G) + 1$, *if one of the following holds:*
*i) One of the partition of $N(x)$ is empty;*
*ii) Induced subgraph $\langle N_G(x) \rangle$ is a clique.*

There are some families of graphs such as, complete graph, paths and cycle of length at least five, with the property $\text{cochord}(\mathcal{P}^4(G, x)) = \text{cochord}(G) + 1$. Hence it is natural to state the following problem.

**Problem 3.4.** *Characterize all graphs with property that for each vertex $x$ and each partitions of $N_G(x)$, we have* $\text{cochord}(\mathcal{P}^4(G, x)) = \text{cochord}(G) + 1$

**Problem 3.5.** *Characterize all graphs $G$ with* $\text{cochord}(\mathcal{P}^4(G, x)) = 2$.

**Problem 3.6.** *Characterize all graphs $G$ for which* $\text{cochord}(\mathcal{P}^4(G)) = 2$.

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