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ON SUBDIFFERENTIAL IN HADAMARD SPACES

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Abstract. In this paper, we deal with the subdifferential concept on Hadamard spaces. Flat Hadamard spaces are characterized, and necessary and sufficient conditions are presented to prove that the subdifferential set in Hadamard spaces is nonempty. Proximal subdifferential in Hadamard spaces is addressed and some basic properties are highlighted. Finally, a density theorem for subdifferential set is established.
Keywords: Subdifferential, Hadamard space, flat space, Hilebert space, convexity.
MSC(2010): Primary: 49J52; Secondary: 49J27, 46N10.

1. Introduction

Nonsmooth analysis, which refers to differential analysis in the absence of differentiability, has grown rapidly in the past decades. Nonsmooth analysis has come to play a vital role in functional analysis, optimization, mechanics, differential equations, etc. The differential analysis in the absence of differentiability is done utilizing the subdifferential notion, which has been introduced in 1960, by Moreau and Rockafellar.

Although there is an abundant literature about nonsmooth analysis and nonsmooth optimization on locally convex topological vector spaces (see e.g. [3,11,14–16] and the references therein), in nonlinear spaces, it is a very young field. In the recent years, this and related topics have become of strong interest in variational analysis, optimization theory, and their applications. It is not only by pure mathematical reasons, but also due to important applications to some classes of problems, which cannot be adequately described in standard Euclidean or Banach frameworks while can be perfectly modeled in the Hadamard space settings. Azagra et al. [4, 5] defined subdifferential concept and some related issues on Riemannian manifolds. Many of the results on Hadamard manifolds remain true for complete CAT(0) space, as a well-known

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and nice nonlinear space. Since such results help in describing some important real world problems which are not linear or yet smooth (e.g. the problem of finding a gradient flow [6]), nowadays there has been a lot of attention to optimization and control theory on these spaces. A crucial problem in optimization and control theory is studying subdifferential notion and its related objects. Due to these, it seems vital to work with subdifferential on CAT(0) spaces to develop the theory. In fact, the concept of subdifferential is very different in nonlinear spaces, because we do not have vector tools no longer. An important example of nonlinear spaces, is metric spaces with nonpositive curvature, developed by Aleksandrov [2]. Hadamard spaces (also named Aleksandrov $\Re_0$ domains or complete CAT(0) spaces) are such spaces that have some nice properties similar to Hilbert spaces, and hence these are widely-used in optimization and nonsmooth analysis.

Two basic tools which are required to develop nonsmooth analysis in Hadamard spaces are dual space and subdifferential. These two important concepts have been defined and investigated in a recently-published paper by Ahmadi-Kakavandi and Amini [1]. An important question about subdifferential set is that under which conditions it is nonempty. In this paper, necessary and sufficient conditions for nonemptyness of the subdifferential set, with respect to convexity, are proved. Using these conditions, a density theorem is given which shows that the set of the points with nonempty subdifferential set is dense in the domain of the function. Moreover, flat Hadamard spaces are characterized and proximal subdifferential in Hadamard spaces is addressed and some of its basic properties are established.

Section 2 contains the preliminaries and section 3 is devoted to the main results.

2. Some guidelines for using standard features

Let $(\mathbf{X}, d)$ be a metric space. A constant speed geodesic is a path γ defined on a closed interval $I \subseteq \mathbb{R}$ into $\mathbf{X}$ such that $d(\gamma(t), \gamma(t')) = v|t - t'|$, for some $v > 0$ and all $t, t' \in I$. v is called the speed of geodesic γ. A geodesic space is a metric space $(\mathbf{X}, d)$ such that there is a geodesic between each pair of its points [10]. If such a geodesic is unique for each two points, we call $\mathbf{X}$ a uniquely geodesic space. An important class of uniquely geodesic spaces is that of Hadamard spaces.

An Hadamard space is a complete simply connected metric space of nonpositive curvature in the sense of Aleksandrov [7, 9, 10]. Equivalently a complete metric space ($\mathbf{X}$,d) is an Hadamard space if and only if it satisfies the CAT(0) inequality, that is: for each pair of points $x, y \in \mathbf{X}$ and $t \in [0, 1]$, there exists $x_t \in \mathbf{X}$ such that

$$d^2(z, x_t) \leq (1 - t)d^2(x, z) + td^2(z, y) - t(1 - t)d^2(x, y),$$

for every $z \in \mathbf{X}$; see [7, 12]. Since each Hadamard space is a uniquely geodesic space, there exists a one to one correspondence between $\mathbf{X} \times \mathbf{X}$ and the set of all geodesic segments in $\mathbf{X}$. We call each ordered pair $(x, y) \in \mathbf{X} \times \mathbf{X}$, a bound vector and denote it by $\overrightarrow{xy}$. We consider $\overrightarrow{0_x} = \overrightarrow{xx}$ as zero bound vector at $x \in \mathbf{X}$ and $-\overrightarrow{xy}$ as the bound vector $\overrightarrow{yx}$. Two bound vectors $\overrightarrow{xy}$ and $\overrightarrow{wz}$ are called admissible if y coincides with w. The summation of two admissible bound vectors $\overrightarrow{xy}$ and $\overrightarrow{yz}$ is defined by $\overrightarrow{xy} + \overrightarrow{yz} = \overrightarrow{xz}$; see [8] for more detail.

Hereafter, assume that $\mathbf{X}$ is an Hadamard space.

The quasi-inner product on Hadamard spaces is defined as follows [8]:

$$\langle \overrightarrow{xy}, \overrightarrow{wz} \rangle = \frac{1}{2} d^2(x, z) + \frac{1}{2} d^2(y, w) - \frac{1}{2} d^2(x, w) - \frac{1}{2} d^2(y, z),$$

for all $x, y, z, w \in \mathbf{X}$.

Suppose that the function f is defined on Hadamard space $\mathbf{X}$ to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The effective domain of f, $domf$, is the set of all $x \in \mathbf{X}$ such that $f(x) < +\infty$ and the epigraph of f is the set $\{(x, r) : f(x) \leq r\} \subseteq X \times R$. Moreover f is called proper, if for all $x \in \mathbf{X}$, $f(x) > -\infty$ and $domf \neq \emptyset$.

Through this paper, we use the notation $[x, y]$ for both closed interval in $\mathbb{R}$ and the image of the geodesic joining x, y in $\mathbf{X}$. For every $x, y \in \mathbf{X}$ and $t \in [0, 1]$, there is a unique $x_t \in [x, y]$ such that $d(x, x_t) = td(x, y)$. We denote such element by $(1-t)x \oplus ty$; see [12]. Clearly $[x, y] = \{x_t : t \in [0, 1]\}$.

A set $C \subseteq \mathbf{X}$ is said to be convex if for every $x, y \in C$, the image of the unique geodesic joining them, is a subset of C; see [10]. Suppose that C is a convex set. The function $f : C \longrightarrow \mathbb{R}$ is a convex function if for each constant speed geodesic γ in C, the function $fo\gamma$ is convex as a function defined on a subset of $\mathbb{R}$ into $\mathbb{R}$; see [10]. Equivalently, f is a convex function if for all $x_0, x_1 \in C$ and $\lambda \in [0, 1]$,

$$f((1-\lambda)x_0 \oplus \lambda x_1) \leq (1-\lambda) f(x_0) + \lambda f(x_1).$$

To define the subdifferential of a convex function in an Hadamard space $\mathbf{X}$, one needs the dual space notion for $\mathbf{X}$. For this purpose, Ahmadi-Kakavandi and Amini [1] introduced an equivalence relation on $\mathbb{R} \times \mathbf{X} \times \mathbf{X}$ defined by $(t, a, b) \sim (s, c, d)$ whenever $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in \mathbf{X}$. They introduced the equivalence class of (t, a, b) as follows:

$$[t\overrightarrow{ab}] := \{(s, c, d) : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle, \ \forall x, y \in \mathbf{X}\}.$$

Using this equivalent class notion, they defined the dual space of $\mathbf{X}$ by

$$\mathbf{X}^* := \{[t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X\}.$$

Considering $x^* = [t\overrightarrow{ab}]$ and $x, y \in \mathbf{X}$, the value of x^* at $\overrightarrow{yx}$ is defined as

$$\langle x^*, \overrightarrow{yx} \rangle := t\langle \overrightarrow{ab}, \overrightarrow{yx} \rangle.$$

The subdifferential of a function $f : \mathbf{X} \longrightarrow \mathbb{R}$ at $x \in \mathbf{X}$, is the set

$$\partial f(x) = \{x^* \in \mathbf{X}^* \ : \ f(y) - f(x) \geq \langle x^*, \overrightarrow{xy} \rangle\}.$$

This set on Hadamard spaces has been investigated in [1] at first.

3. **Main results**

Various important results about subdifferentials, which hold under topological vector spaces, are not valid on Hadamard spaces in general. Due to this, we start this section by introducing the concept of flat Hadamard spaces. After providing a characterization of flat Hadamard spaces, we establish some basic properties of subdifferentials under these spaces, though some of the given results are valid in general Hadamard spaces.

Definition 3.1. An Hadamard space $\mathbf{X}$ is said to be flat if for each $x, y \in \mathbf{X}$ and $\lambda \in [0, 1]$, the following property is satisfied:

$$d^2(z, x_\lambda) = (1 - \lambda)d^2(x, z) + \lambda d^2(y, z) - \lambda(1 - \lambda)d^2(x, y), \tag{3.1}$$

for all $z \in \mathbf{X}$.

The following theorem provides a useful characterization of flat Hadamard spaces.

Theorem 3.2. *An Hadamard space $\boldsymbol{X}$ is flat if and only if for each $x, y \in \boldsymbol{X}$, and $\lambda \in [0, 1]$, we have*

$$\langle \overrightarrow{xx_\lambda}, \overrightarrow{ab} \rangle = \lambda \langle \overrightarrow{xy}, \overrightarrow{ab} \rangle,$$

for all $a, b \in \boldsymbol{X}$.

Proof. Suppose that $\mathbf{X}$ is a flat Hadamard space and $a, b \in \mathbf{X}$. Then we have

$$d^2(a, x_\lambda) - d^2(x, a) = -\lambda d^2(x, a) + \lambda d^2(y, a) - \lambda(1 - \lambda)d^2(x, y),$$

and

$$-d^2(b, x_\lambda) + d^2(x, b) = \lambda d^2(x, b) - \lambda d^2(y, b) + \lambda(1 - \lambda)d^2(x, y).$$

Summing two equations, we get

$$\begin{aligned} d^2(a, x_\lambda) \quad &+ \quad d^2(b, x) - d^2(a, x) - d^2(b, x_\lambda) \\ &= \quad \lambda(d^2(a, y) + d^2(b, x) - d^2(a, x) - d^2(b, y)). \end{aligned}$$

Equivalently $\langle \overrightarrow{xx_\lambda}, \overrightarrow{ab} \rangle = \lambda \langle \overrightarrow{xy}, \overrightarrow{ab} \rangle$.

Conversely, if for all $a, b \in \mathbf{X}$ and $\lambda \in [0, 1]$, $\langle \overrightarrow{xx_\lambda}, \overrightarrow{ab} \rangle = \lambda \langle \overrightarrow{xy}, \overrightarrow{ab} \rangle$, then putting $a = x$, we have

$$d^2(x, b) + d^2(x, x_\lambda) - d^2(x_\lambda, b) = \lambda d^2(x, b) + \lambda d^2(y, x) - \lambda d^2(y, b),$$

for all b. Therefore

$$d^2(x_\lambda, b) = (1 - \lambda)d^2(x, b) + \lambda d^2(y, b) - \lambda d^2(y, x) + d^2(x, x_\lambda).$$

Hence

$$d^2(x_\lambda, b) = (1-\lambda)d^2(x,b) + \lambda d^2(y,b) - \lambda(1-\lambda)d^2(x,y),$$

and the proof is completed. □

It is well-known that, each Hilbert space is a flat Hadamard space. In the following theorem, we introduce some conditions under which a flat Hadamard real vector space is a Hilbert space. Recall that, $\mathbf{X}$ is a flat Hadamard real vector space if it is a real vector space satisfying Definition 3.1. A metric d on $\mathbf{X}$ enjoys the translation invariance property if

$$d(x+z, y+z) = d(x,y)\ \forall x,y,z \in \mathbf{X}.$$

Theorem 3.3. *Suppose that $(\boldsymbol{X}, d)$ is an Hadamard real vector space with translation invariant metric. If $\boldsymbol{X}$ is flat, then it is a Hilbert pace.*

Proof. Suppose that x is an arbitrary point in $\mathbf{X}$ and $l := d(x,0)$. Let γ be the geodesic parameterized by arc length from 0 to x. For each $t \in [0,1]$,

$$d(tx,0) = d(\gamma(tl), \gamma(0)) = tl = td(x,0).$$

If $t > 1$, then

$$d(\frac{1}{t}tx, 0) = \frac{1}{t}d(tx,0).$$

Thus

$$d(tx,0) = td(x,0).$$

For $t < 0$, since d is translation invariant, we get

$$d(tx,0) = d(-tx,0) = -td(x,0) = |t|d(x,0).$$

Therefore $d(tx,0) = |t|d(x,0)$ for all $t \in \mathbb{R}$. Now, we define

$$\|x\| = d(x,0).$$

Then

$$\|x+y\| = d(x+y,0) = d(x,-y) \leq d(x,0) + d(0,-y) = d(x,0) + d(y,0) = \|x\| + \|y\|.$$

Furthermore, $\|x\| = 0$ if and only if $d(x,0) = 0$ if and only if $x = 0$. By the above argument,

$$\|tx\| = |t|\|x\|.$$

Therefore, $\mathbf{X}$ is a normed Hadamard space, and hence it is a Hilbert space. □

Remark 3.4. Although each Hilbert space is a flat Hadamard space, Lurie [13] proved that each flat Hadamrd space is isometric to a (nonempty) closed convex subset of a Hilbert space. It is clear that a flat Hadamrd space may not be a vector space and then it may not be a Hilbert space. For example, consider the closed unit ball in an arbitrary Hilbert space (see p. 14 in [6]).

Some properties of the subdifferentials defined in Topological Vector Spaces (TVSs), can not be generalized to Hadamard spaces. For example, if $\mathbf{X}$ is a locally convex TVS and $\partial f(x) \neq \emptyset$ for all $x \in \mathbf{X}$, then f is convex and conversely, if f is a proper and convex function then for each $x \in int(domf)$ such that f is continuous at x, we have $\partial f(x) \neq \emptyset$ (Theorem 7.12 in [3]). The following theorem gives a sufficient condition for convexity in flat Hadamard spaces based upon the subdifferential notion, though it is not correct in general Hadamard spaces necessarily. For example, considering some special metrics (see Chapter 2 in [9]) the function $f(x) = \langle \overrightarrow{ab}, \overrightarrow{cx} \rangle$ has nonempty subdifferential in each point while it is not necessarily convex.

Theorem 3.5. *Suppose that $\boldsymbol{X}$ is a flat Hadamard space. If $\partial f(x) \neq \emptyset$ for each x, then f is convex.*

Proof. Let $x, y \in \mathbf{X}$, $\lambda \in (0,1)$ be arbitrary and $x^* \in \partial f(x_\lambda)$ in which $x_\lambda = (1-\lambda)x \oplus \lambda y$. Then

$$\begin{aligned} f(y) - f(x_\lambda) &\geq \langle x^*, \overrightarrow{x_\lambda y} \rangle = -\langle x^*, \overrightarrow{y x_\lambda} \rangle \\ &= -(1-\lambda)\langle x^*, \overrightarrow{yx} \rangle = (1-\lambda)\langle x^*, \overrightarrow{xy} \rangle. \end{aligned}$$

Similarly, we get

$$f(x) - f(x_\lambda) \geq -\lambda \langle x^*, \overrightarrow{xy} \rangle.$$

Multiplying the above inequalities by λ and $(1-\lambda)$, respectively, we have

$$f(x_\lambda) \leq (1-\lambda) f(x) + \lambda f(y).$$

Thus f is convex. □

We continue the paper with proximal subgradient notion and some of its basic properties.

Definition 3.6. $x^* \in \mathbf{X}^*$ is a *proximal subgradient* of the function $f : \mathbf{X} \longrightarrow \overline{\mathbb{R}}$ if there exist positive numbers k and r such that

$$f(x) \geq f(x_0) + \langle x^*, \overrightarrow{x_0 x} \rangle - k d^2(x_0, x) \ \forall x \in B(x_0, r).$$

The set of all proximal subgradients of f at x is called *proximal subdifferential* of f at x. We use the notation $\partial_P f(x_0)$ for proximal subdifferential of f at x_0.

The following theorem gives some important properties of the (proximal) subdifferential set. In part (iii) of this theorem, we use the summation on $\mathbf{X}^*$ as defined in [1] (See equation (7) in [1]). The proofs of parts (ii) and (iv) is not difficult and hence omitted.

Theorem 3.7.

(i) *$\partial f(x) \subseteq \partial_P f(x)$ for all $x \in \boldsymbol{X}$. Moreover if f is convex and $\boldsymbol{X}$ is flat, then $\partial f = \partial_P f$.*

(ii) *If f attains its minimum (global) at x_0, then $0_{X^*} \in \partial f(x_0)$.*

(iii) *Let $\lambda, \mu \in \mathbb{R}$. If $\partial_P \lambda f(x) + \partial_P \mu g(x)$ is a subset of $\boldsymbol{X}^*$ (resp. $\partial \lambda f(x) + \partial \mu g(x)$ is a subset of $\boldsymbol{X}^*$), then $\partial_P \lambda f(x) + \partial_P \mu g(x) \subseteq \partial_P(\lambda f + \mu g)(x)$ (resp. $\partial \lambda f(x) + \partial \mu g(x) \subseteq \partial(\lambda f + \mu g)(x)$).*

(iv) $\partial(cf) = c\partial f$ *and* $\partial_P(cf) = c\partial_P f$ *for all* $c > 0$.

Proof. (i) Clearly $\partial f(x) \subseteq \partial_P f(x)$ for all $x \in \mathbf{X}$. Suppose that $x^* \in \partial_P(f)(x_0)$. Then for some positive r, k,

$$f(x) - f(x_0) \geq \langle x^*, \overrightarrow{x_0 x}\rangle - kd^2(x, x_0),$$

for each $x \in B(x_0, r)$. Let z be an arbitrary point of $\mathbf{X}$ and $\lambda \in [0, 1]$. By convexity of f, we have

$$\begin{array}{rcl}(1-\lambda)f(x_0) + \lambda f(z) - f(x_0) & \geq & f(x_\lambda) - f(x_0) \\ & \geq & \langle x^*, \overrightarrow{x_0 x_\lambda}\rangle - kd^2(x_\lambda, x_0) \\ & = & \lambda\langle x^*, \overrightarrow{x_0 z}\rangle - k\lambda^2 d^2(x_0, z),\end{array}$$

in which $x_\lambda = (1-\lambda)x_0 \oplus \lambda z$. Dividing the above inequality by λ and tending λ to 0, we get the desired result.

(ii) If $x_1^* + x_2^* \in \partial_P \lambda f(x) + \partial_P \mu g(x)$, then there exist positive numbers k_1,k_2 and r_1, r_2 such that

$$\lambda f(y) \geq \lambda f(x) + \langle x_1^*, \overrightarrow{xy}\rangle - k_1 d^2(x, y)$$

for all $y \in B(x, r_1)$, and

$$\mu g(y) \geq \mu g(x) + \langle x_2^*, \overrightarrow{xy}\rangle - k_2 d^2(x, y)$$

for all $y \in B(x, r_2)$. Setting $k = k_1 + k_2$ and $r = min\{r_1, r_2\}$ and summing the above two inequalities, the desired result is obtained. The proof for ∂ is similar.

□

One important question arising in this topic is, "when $\partial f(x) \neq \emptyset$?". This question is very important in optimization from both theoretical and computational standpoints. The following theorem addresses this issue. This result holds in general Hadamard spaces.

Theorem 3.8. *Suppose that* f *is lsc and convex; and* $X_0 = (x_0, f(x_0))$ *is the closest point of* $epif$ *to a point* $E = (e, r_e) \notin epi(f)$ *such that* $f(x_0) \neq r_e$. *Then* $\partial f(x_0) \neq \emptyset$.

Proof. Since X_0 is the closest point to E, for each $A = (a, r_a) \in epif$ satisfying $A \neq X_0$, we have $\angle_{X_0}(E, A) \geq \frac{\pi}{2}$ (see Proposition 2.4 in [9]). Consequently

$$\rho^2(A, X_0) + \rho^2(X_0, E) \leq \rho^2(A, E),$$

in which ρ is the metric of the space $\mathbf{X} \times \mathbb{R}$ defined as follows:

$$\rho^2((x_1, r_1), (x_2, r_2)) = d^2(x_1, x_2) + (r_2 - r_1)^2.$$

Thus

$$d^2(a, x_0) + d^2(x_0, e) + (f(x_0) - r_a)^2 + (f(x_0) - r_e)^2 \leq d^2(a, e) + (r_e - r_a)^2.$$

An easy calculation implies

$$\langle \overrightarrow{x_0 e}, \overrightarrow{x_0 a}\rangle \leq (r_a - f(x_0))(f(x_0) - r_e).$$

Since r_a may increase infinitely, $f(x_0) - r_e > 0$. Putting $r_a = f(a)$, we get

$$\langle x^*, \overrightarrow{x_0 a}\rangle \leq f(a) - f(x_0),\ \forall a \in domf,$$

in which

$$x^* = [\frac{1}{f(x_0) - r_e}\overrightarrow{x_0 e}].$$

Clearly the above inequality holds for each $a \notin domf$. Hence $x^* \in \partial f(x_0)$, and the proof is completed. □

The following lemma shows that the assumption of Theorem 3.8 holds for each interior point of $domf$.
In fact, Theorem 3.8 proves that if the point under consideration is the projection of a point outside of $epi(f)$ and these two points do not have the same values on f-axis, then the subdifferential set is nonempty. Lemma 3.9 shows that the assumption considered in Theorem 3.8 can be reduced for interior points. The metric ρ is as used in the previous result.

Lemma 3.9. *Suppose that f is lsc and convex. Assume that $x_0 \in int(domf)$ and $X_0 = (x_0, f(x_0))$ is the nearest point of $epif$ to $Y_0 = (y_0, r_0)$. Then $r_0 \neq f(x_0)$.*

Proof. By contradiction, assume that $r_0 = f(x_0)$. There exists $r > 0$ and $\lambda_0 \in [0, 1]$ such that $B(x_0, r) \subseteq domf$ and

$$(1 - \lambda)x_0 \oplus \lambda y_0 \in B(x_0, r)$$

for each $\lambda \in [0, \lambda_0]$. First suppose that there exists $\lambda_1 \in (0, 1)$ such that

$$x_1 = (1 - \lambda_1)x_0 \oplus \lambda_1 y_0 \in B(x_0, r) \cap [x_0, y_0]$$

and $f(x_0) < f(x_1)$. Put

$$X_1 = (x_1, f(x_1)) \in epif.$$

Then $\angle_{X_0}(Y_0, X_1) \geq \dfrac{\pi}{2}$.
Let

$$\sigma : [0, \infty) \longrightarrow \mathbf{X} \times \mathbb{R}$$

be the unit speed geodesic emanating from X_0 defined by

$$\sigma(t) := (x_0, f(x_0) + t).$$

Setting $\kappa(t) := \rho(\sigma(t), Y_0)$ for $t \geq 0$, let

$$\gamma_t : [0, \kappa(t)] \longrightarrow [\sigma(t), Y_0],$$

be the unit speed geodesic emanating from $\sigma(t)$ to Y_0.
Setting

$$t_1 := \frac{f(x_1) - f(x_0)}{1 - \lambda_1},$$

we have

$$X_1 = (1-\lambda_1)\sigma(t_1) \oplus \lambda_1 Y_0.$$

Moreover

$$\rho^2(X_0, X_1) = d^2(x_0, x_1) + (f(x_1) - f(x_0))^2 = \lambda_1^2 d^2(x_0, y_0) + (f(x_1) - f(x_0))^2,$$

$$\rho^2(Y_0, X_1) = d^2(y_0, x_1) + (f(x_1) - f(x_0))^2 = (1-\lambda_1)^2 d^2(x_0, y_0) + (f(x_1) - f(x_0))^2$$

and

$$\rho^2(X_0, Y_0) = d^2(x_0, y_0).$$

Since $\angle_{X_0}(Y_0, X_1) \geq \frac{\pi}{2}$, we have

$$\rho^2(X_0, X_1) + \rho^2(X_0, Y_0) \leq \rho^2(Y_0, X_1).$$

A simple calculation implies $\lambda_1 \leq 0$, which makes a contradiction and completes the proof in the first case.

Now suppose that $f(x) \leq f(x_0)$ for each $x \in B(x_0, r) \cap [x_0, y_0] \subseteq domf$. Let

$$Y_n = (1 - \frac{1}{n})X_0 \oplus \frac{1}{n} Y_0 = (y_n, r_n).$$

Hence, X_0 is the nearest point of $epif$ to each Y_n (Proposition 2.4 in [9]). Moreover $y_n = f(x_0)$, for all n and the sequence $\{Y_n\}$ is convergent to X_0. If $y_n \in B(x_0, r)$, then $f(y_n) \leq f(x_0)$. Thus $(y_n, r_n) = (y_n, f(x_0)) \in epif$ that is not true (because X_0 is a boundary point of $epif$ and Y_0 does not belong to $epif$). Therefore, $y_n \in (B(x_0, r))^c$. It means that $\{y_n\}$ is a sequence in $(B(x_0, r))^c$ converging to x_0 which is impossible. This completes the proof. □

An important question in variational analysis and nonsmooth optimization is nonemptiness of the subdifferential set. This plays a vital role in sketching numerical algorithms and providing duality results in optimization. We follow the paper by considering this important issue, and it is shown that the set of points with nonempty subdifferential set is dense in the interior of domain of f. In the following theorem,

$$dom(\partial f) = \{x \in domf \ : \ \partial f(x) \neq \emptyset\}.$$

Theorem 3.10. *(Density) Suppose that f is convex and lsc. Then $dom(\partial f)$ is dense in $int(domf)$.*

Proof. For an arbitrary $x_0 \in int(domf)$, $X_0 = (x_0, f(x_0))$ is a boundary point of $epif$. Therefore, there exists a sequence Y_n in $(epif)^c$, convergent to X_0. For each Y_n, there exists a unique closest point $X_n \in epif$. It is not difficult to show that X_n converges to X_0. Therefore, the proof is complete, because of Theorem 3.8. □

A worth studying question is investigating the assumptions under which $dom(\partial f) = int(domf)$, i.e. the subdifferential set be nonempty in each point of $domf$.

We close the paper by some examples. As instances, the Euclidean space $\mathbb{R}^n$ and its convex subsets are $CAT(0)$ spaces. Complete simply connected Riemannian manifolds of nonpositive sectional curvature and $\mathbb{R}$-trees are also well-known examples of $CAT(0)$ spaces. Recall that an $\mathbb{R}$-tree, T, is a metric space such that there is a unique geodesic segment $[a,b]$ joining each pair of points $a,b \in T$ such that $[a,b] \cap [b,c] = \{b\}$ implies $[a,b] \cup [b,c] = [a,c]$. Hilbert spaces, Euclidean Bruhat-Tits buildings and Hyperbolic spaces are also examples of $CAT(0)$ spaces [9].

In the following, some convex functions on Hadamard spaces are addressed. Here, $\mathbf{X}$ is an Hadamard space.

1. Distance functions: Let $x_0 \in \mathbf{X}$ be given. Then two functions $x \mapsto d(x,x_0)$ and $x \mapsto d^2(x,x_0)$ are convex and continuous functions. Also, considering the convex set $C \subseteq \mathbf{X}$, the distance function defined by $d_C(x) = \inf_{c\in C} d(x,c)$ is a convex and 1-Lipschitz function.
2. Displacement functions [7]: For an isometry $\alpha : \mathbf{X} \longrightarrow \mathbf{X}$, the function $D_\alpha : \mathbf{X} \longrightarrow [0,\infty)$ defined by $D_\alpha(x) = d(x,\alpha(x))$ is called the displacement function of α. This function is convex because it maps geodesics to geodesics.
3. Busemann functions [9]: For a geodesic ray $c : [0,\infty) \longrightarrow \mathbf{X}$, the function $b_c : \mathbf{X} \longrightarrow \mathbb{R}$ defined by $b_c(x) = \lim_{t\to\infty} [d(x,c(t)) - t]$, is called the Busemann function associated to the ray c. It is also a well-known convex function.

The following examples deal with the subdifferential set for two functions.

a. Let $x_0 \in \mathbf{X}$ be a fixed and arbitrary point of $\mathbf{X}$. Set $f(x) = d(x_0,x)$. Then, according to Cauchy-Schwarz inequality (see Theorem 1.1 in [1]), we have

$$\left\{[\frac{1}{d(x_0,z)}\overrightarrow{x_0z}] \ : \ z \in \mathbf{X},\ z \neq x_0\right\} \subseteq \partial f(x_0).$$

b. Consider the function $f : \mathbf{X} \longrightarrow \mathbb{R}$ defined by $f(x) = d^2(x,z)$, for some fixed $z \in \mathbf{X}$. Then, $[2\overrightarrow{zx_0}] \in \partial f(x_0)$ because

$$\begin{aligned} f(x) - f(x_0) &= d^2(x,z) - d^2(x_0,z) \\ &\geq d^2(x,z) - d^2(x_0,z) - d^2(x_0,x) = 2\langle \overrightarrow{zx_0}, \overrightarrow{x_0x}\rangle. \end{aligned}$$

As a concluding remark, although we proved some results concerning the nonemptiness and density of the domain of subdifferential mapping under Hadamard spaces, various important issues which are valid for Mordukhovich limiting subdifferential [14] and the Clarke generalized gradient [11] on Banach spaces, are open to study here. One of the most important problems is investigating the chain rule and the mean value theorem for (complete) $CAT(0)$ spaces. The main difficulty in proving these results is due to the properties of the summation operator in dual space.

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