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ITERATIVE SCHEME BASED ON BOUNDARY POINT METHOD FOR COMMON FIXED POINT OF STRONGLY NONEXPANSIVE SEQUENCES

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Abstract. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ and $\{T_n\}$ be sequences of nonexpansive self-mappings of C, where one of them is a strongly nonexpansive sequence. K. Aoyama and Y. Kimura introduced the iteration process $x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n(\alpha_n u + (1 - \alpha_n)T_n x_n)$ for finding the common fixed point of $\{S_n\}$ and $\{T_n\}$, where $u \in C$ is an arbitrarily (but fixed) element in C, $x_0 \in C$ arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. But in the case where $u \notin C$, the iterative scheme above becomes invalid because x_n may not belong to C. To overcome this weakness, a new iterative scheme based on the thought of boundary point method is proposed and the strong convergence theorem is proved. As a special case, we can find the minimum-norm common fixed point of $\{S_n\}$ and $\{T_n\}$ whether $0 \in C$ or $0 \notin C$.
Keywords: minimum-norm common fixed point, strongly nonexpansive mappings, strong convergence, boundary point method, variational inequality.
MSC(2010): Primary: 47H09; Secondary: 47H10, 90C25.

1. Introduction

Let H be a real Hilbert space endowed with an inner product and its induced norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $T : C \to C$ a mapping. $Fix(T)$ denotes the fixed point set of T, that is, $Fix(T) = \{x \in C \,|\, Tx = x\}$. Throughout this paper, $Fix(T)$ is always assumed to be nonempty.

Many iteration processes are often used to approximate a common fixed point of a pair of nonlinear mappings (e.g. $[1-5]$, $[7-9]$). One of them is a classical iterative scheme [1] and is defined as follows: Take an initial guess

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$x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_n(\alpha_n u + (1-\alpha_n) T_n x_n) \tag{1.1}$$

where $u \in C$ is an arbitrarily (but fixed) element in C, $\{S_n\}$ and $\{T_n\}$ are sequences of nonexpansive self-mappings of C, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

The iterative algorithm generated by (1.1) has been widely investigated in extensive literature. For example, S. Takahashi, W. Takahashi and M. Toyoda [3] used the iterative scheme generated by (1.1) to deal with the fixed point problem for a nonexpansive mapping and the zero point problem for a monotone operator. Y. Yao and J. C. Yao [5] used this algorithm in order to solve the fixed point problem for a nonexpansive mapping and the variational inequality (VI) problem for an inverse-strongly monotone mapping. K. Aoyama and Y. Kimura [1] used this iteration process in order to approximate a common fixed point of a pair of sequences of nonexpansive mappings where one of them is a strongly nonexpansive sequence [6].

We next analyze how to get the common fixed point $\{S_n\}$ and $\{T_n\}$. In the case where $u \in C$, it is not hard to obtain $\{x_n\}$ generated by (1.1) converges strongly to the common fixed point of $\{S_n\}$ and $\{T_n\}$ under some assumptions. But, in the case where $u \notin C$, the iteration scheme defined by (1.1) becomes invalid because x_n may not belong to C.

To make up for this weakness, a natural and rational idea is adopting metric projection P_C to modify iterative scheme (1.1) so that the iterative sequence in C. One simple and feasible modified iterative scheme by P_C can be given as follows:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_n P_C(\alpha_n u + (1-\alpha_n) T_n x_n). \tag{1.2}$$

However, in the process of actual calculation, it is difficult to obtain the specific expression of P_C, in general. In view of this shortage, iteration process (1.2) is not a right choice.

Main contributions of this paper are as follows: We extend Aoyama and Kimura's algorithm, extends Aoyama and Kimura's result based on the inspiration of viscosity approximation method (e.g. $[10-22]$), and also avoid the computation of the metric projection P_C. we show that the iterative sequence generated by our proposed algorithm has strongly convergent property under some appropriate conditions. As a special case, the minimum-norm common fixed point of $\{T_n\}$ and $\{S_n\}$ is obtained whether $0 \in C$ or $0 \notin C$.

2. Preliminaries

Throughout this paper, we adopt the following notations.

1) $x_n \to x$: $\{x_n\}$ converges strongly to x, $x_n \rightharpoonup x$: $\{x_n\}$ converges weakly to x.

2) $\mathbb{N}$: the set of positive integers,

3) ∂C: the boundary of C,
4) I: the identity mapping on H,
5) $H \setminus S$: the complementary set of C in H.

We recall some definitions and facts that are needed in our study.

Let C be a nonempty closed convex subset of H and $T : C \to C$ a mapping. A mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$.

Let $\{T_n\}$ be a sequence of self-mappings of C. We use the notation $Fix(\{T_n\})$ to denote the set of common fixed points of $\{T_n\}$, i.e.,$Fix(\{T_n\}) = \bigcap_{n=1}^{\infty} Fix(T_n)$. A sequence $\{z_n\}$ is said to be an *approximate fixed point sequence* of $\{T_n\}$ if $z_n - T_n z_n \to 0$. The set of all *bounded* approximate fixed point sequences of $\{T_n\}$ is denoted by $\widetilde{Fix}(\{T_n\})$ [6]. It is easily see that $\widetilde{Fix}(\{T_n\})$ is nonempty if $Fix(\{T_n\})$ is nonempty. A sequence $\{T_n\}$ is called a *strongly nonexpansive sequence* if each T_n is nonexpansive and

$$x_n - y_n - (T_n x_n - T_n y_n) \to 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$. A sequence $\{T_n\}$ having a common fixed point is said to satisfy the *condition* (Z) if every weak cluster point of $\{x_n\}$ is a common fixed point whenever $\{x_n\} \in \widetilde{Fix}(\{T_n\})$. A sequence $\{T_n\}$ of nonexpansive self-mappings of C is said to satisfy the *condition* (R) if

$$\lim_{n\to\infty} \sup_{y \in D} \|T_{n+1} y - T_n y\| = 0$$

for every nonempty bounded subset D of C [23].

We need some lemmas and facts listed as follows:

Lemma 2.1. *[24] Let K be a closed convex subset of a real Hilbert space H and let P_K be the (metric of nearest point) projection from H onto K (i.e., for $x \in H$, $P_K(x)$ is the only point in K such that $\|x - P_K(x)\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K(x)$ if and only if, for any $y \in K$,*

$$\langle x - z, y - z \rangle \le 0.$$

It is well-known that $Fix(T)$ is closed and convex if T is nonexpansive. So the metric projection $P_{Fix(T)}$ is reasonable and thus there exists a unique element, which is denoted by $x^\dagger$, in $Fix(T)$ such that $\|x^\dagger\| = \inf_{x \in Fix(T)} \|x\|$, that is, $x^\dagger = P_{Fix(T)}(0)$. $x^\dagger$ is called the *minimum-norm fixed point* of T.

Lemma 2.2. *[25] In a real Hilbert space H, the following well-known result holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 2.3. *[26, 27] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space and $\{\beta_n\}$ a sequence in $[0, 1]$. Suppose that $x_{n+1} = (1 - \beta_n) x_n + \beta_n y_n$*

for every $n \in \mathbf{N}$, $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $x_n - y_n \to 0$.

Lemma 2.4. *[1] Let* H *be a real Hilbert space,* C *a nonempty subset of* H*, and* $\{S_n\}$ *and* $\{T_n\}$ *sequences of nonexpansive self-mappings of* C*. Suppose that* $\{S_n\}$ *and* $\{T_n\}$ *satisfy the condition* (R) *and* $\{T_n y : y \in D, n \in \mathbb{N}\}$ *is bounded for any bounded subset* D *of* C*. Then* $\{S_nT_n\}$ *satisfies the condition* (R)*.*

Lemma 2.5. *[6] Let* H *be a real Hilbert space,* C *a nonempty subset of* H*, and* $\{S_n\}$ *and* $\{T_n\}$ *sequences of nonexpansive self-mappings of* C*. Suppose that* $\{S_n\}$ *or* $\{T_n\}$ *is a strongly nonexpansive sequence and* $\widetilde{Fix}(\{S_n\}) \cap \widetilde{Fix}(\{T_n\})$ *is nonempty. Then* $\widetilde{Fix}(\{S_n\}) \cap \widetilde{Fix}(\{T_n\}) = \widetilde{Fix}(\{S_nT_n\})$*.*

The following is a sufficient condition for a real sequence to converge to zero.

Lemma 2.6. *[28, 29] Let* $\{\alpha_n\}$ *be a nonnegative real sequence satisfying:*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n\delta_n + \sigma_n, \quad n = 0, 1, 2\ldots$$

If $\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$,$\{\delta_n\}_{n=1}^{\infty}$ *and* $\{\sigma_n\}_{n=1}^{\infty}$ *satisfy the conditions:*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) *either* $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$;
(3) $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Boundary point function for non-self contraction

Let C be a nonempty closed convex subset of a real Hilbert space H and $V : C \to H$ is a non-self α-contraction, i.e., $\|V(x) - V(y)\| \le \alpha\|x - y\|$ $(0 \le \alpha < 1)$, for $x, y \in C$. To formulate our iterative algorithm, we introduce a function $s : C \to [0,1]$ by the following definition [30]:

$$s(x) = \inf\{\lambda \in [0,1] \mid \lambda x + (1-\lambda)V(x) \in C\}, \quad x \in C. \tag{3.1}$$

Remark 3.1. Now we explain clearly the rationality of the definition of function $s(x)$. Since C is closed and convex, in the case where $V(x) \in C$, it is easy to observe that $s(x) = 0$ for any $x \in C$. But in the case where $V(x) \notin C$, we obtain $s(x)x + (1 - s(x))V(x) \in \partial C$ and $s(x) > 0$ (otherwise, suppose that $s(x)x + (1 - s(x))V(x) \notin \partial C$, for convenience, we set $w = s(x)x + (1 - s(x))V(x)$. Obviously, w is an interior point in C. Furthermore, there exists efficiently small $\delta > 0$ such that the neighborhood of w, $B(w, \delta)$, contained C, i.e., $B(w, \delta) \subset C$, by using the definition of function $s(x)$. Thus $(1 - (s(x) - \delta/2)V(x) + (s(x) - \delta/2)x \in C$ and $s(x) - \delta/2 < s(x)$. This is a contradiction with the definition of function $s(x)$). So the function $s(x)$ above is well-defined.

We need to consider how to calculate the value of function $s(x)$, for any $x \in C$. As a matter of fact, in many practical problems, C is often a level set of a convex function c, that is, $C = \{x \in H \mid c(x) \le a\}$, where a is a constant in $\mathbb{R}^1$. Without loss of generality, suppose that $C = \{x \in H \mid c(x) \le 0\}$ and $V(x) \notin C$. The function $s(x)$ can be expressed by $s(x) = \inf\{\lambda \in (0,1] \mid c(\lambda x + (1-\lambda)V(x)) = 0\}$, for any $x \in C$. To calculate the value of $s(x)$, we only need to solve the algebraic equation $c(\lambda x + (1-\lambda)V(x)) = 0$ with the unknown number λ. Solving the algebraic equation above is generally easier than computing the metric projection P_C in actual calculation process. We have the following an illustrative example to show that this point of view.

Example 3.2. Let $B : H \to H$ be an inverse-strongly monotone bounded linear operator with coefficient $\alpha > 0$, that is, there is a constant $\alpha > 0$ with the property $\langle Bx - By, x - y\rangle \ge \alpha\|Bx - By\|^2$, $\forall x, y \in H$. Define a convex function $c : H \to \mathbb{R}^1$ by

$$c(x) = 1/2\langle Bx - Bu, x - u\rangle - 2\langle x - u, y^*\rangle + \langle Bx^* - Bu, x^* - u\rangle, \quad \forall x \in H,$$

where y^*, u ($y^* \neq 0$) are two given points in H and x^* is the unique solution of the equation $B(x-u) = y^*$. Set $C = \{x \in H \mid c(x) \le 0\}$ and define contraction $V : C \to H$ as $V(x) = u$, $\forall x \in C$. Note that $c(x^*) = -(1/2)\langle Bx^* - Bu, x^* - u\rangle < 0$ and $c(u) = \langle Bx^* - Bu, x^* - u\rangle > 0$, then it is easy to verify that C is a nonempty closed convex subset of H such that $u \notin C$. For a given $x \in C$, we have $c(x) \le 0$. To obtain the value of $s(x)$, let $c(\lambda x + (1-\lambda)u) = 0$, where $\lambda \in (0, 1]$ is a unknown variable. Thus we have the following equation

$$1/2\langle Bx - Bu, x - u\rangle\lambda^2 - 2\langle x - u, y^*\rangle\lambda + \langle Bx^* - Bu, x^* - u\rangle = 0.$$

Consequently, we have

$$\lambda = \frac{2\langle x - u, y^*\rangle \pm \sqrt{4\langle x - u, y^*\rangle^2 - 2\langle Bx - Bu, x - u\rangle\langle Bx^* - Bu, x^* - u\rangle}}{\langle Bx - Bu, x - u\rangle}.$$

Hence,

$$\begin{aligned} s(x) &= \frac{2\langle x - u, y^*\rangle - \sqrt{4\langle x - u, y^*\rangle^2 - 2\langle Bx - Bu, x - u\rangle\langle Bx^* - Bu, x^* - u\rangle}}{\langle Bx - Bu, x - u\rangle} \\ &= \frac{2\langle y^*, x^* - u\rangle}{2\langle x - u, y^*\rangle + \sqrt{4\langle x - u, y^*\rangle^2 - 2\langle Bx - Bu, x - u\rangle\langle y^*, x^* - u\rangle}}. \end{aligned}$$

4. **Iterative scheme based on boundary point method**

Now we give our iteration process based on the thought of boundary point method for finding the common fixed point of $\{S_n\}$ and $\{T_n\}$: Taking $x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$\text{(4.1)} \qquad \begin{cases} x_1 \in C; \\ y_n = \alpha_n((1 - \lambda_n)V(x_n) + \lambda_n x_n) + (1 - \alpha_n)T_n x_n; \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n; \end{cases}$$

where $V : C \to H$ is non-self α-contraction ($0 \le \alpha < 1$), $\lambda_n = s(x_n)$, $n = 1, 2, \ldots$ $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

Since C is closed and convex, algorithm (4.1) induces to algorithm (1.1) on condition of $V(x_n) \in C$. But in the case where $V(x_n) \notin C$, it is easy to verify the sequence $\{x_n\}$ generated by (4.1) belongs to C and $s(x_n)x_n + (1 - s(x_n))Vx_n \in \partial C$ for any positive integer n, so we call our algorithm as *boundary point method.*

Theorem 4.1. *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $\{S_n\}$ and $\{T_n\}$ sequences of nonexpansive self-mappings of C. Let $V : C \to H$ be an α-contraction. Suppose that $F = Fix(\{S_n\}) \cap Fix(\{T_n\})$ is nonempty, both $\{S_n\}$ and $\{T_n\}$ satisfy the conditions* (R) *and* (Z)*, and $\{S_n\}$ or $\{T_n\}$ is a strongly nonexpansive sequence. Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions*

(D1) $\alpha_n \to 0$;
(D2) $\sum_{n=1}^{\infty} \alpha_n(1 - \lambda_n)(1 - \beta_n) = \infty$.

Then $\{x_n\}$ defined by (4.1) *converges strongly to x^* in F verifying*

$$x^* = P_F V(x^*), \tag{4.2}$$

which solves the following variational inequality

$$find\ x^* \in F,\ such\ that\ \langle (I - V)x^*, \bar{x} - x^* \rangle \ge 0\ ,\ \forall\ \bar{x} \in F. \tag{4.3}$$

We divide our detailed proof into several lemmas. In the course of the proof of Lemma 4.2 - Lemma 4.4, we assume that all conditions of Theorem 4.1 are satisfied.

Lemma 4.2. *$\{x_n\}$, $\{T_n x_n\}$ and $\{S_n V_n x_n\}$ are bounded, where $V_n = \alpha_n(1 - \lambda_n)V + \alpha_n\lambda_n I + (1 - \alpha_n)T_n$, $n = 1, 2, \ldots$*

Proof. Take a $z \in F$, we have $S_n z = z$ and $T_n z = z$. Since both S_n and T_n are nonexpansive, we immediately obtain

$$\begin{aligned}
\|S_n V_n x_n - z\| &\le \|V_n x_n - z\| \\
&\le \alpha_n(1 - \lambda_n)(\alpha\|x_n - z\| + \|Vz - z\|) \\
&\quad + \alpha_n\lambda_n\|x_n - z\| + (1 - \alpha_n)\|T_n x_n - z\| \\
&\le \max_n\{\frac{\|Vz - z\|}{1 - \alpha}, \|x_n - z\|\}
\end{aligned}$$

and hence,

$$\begin{aligned}
\|x_{n+1} - z\| &\le \beta_n\|x_n - z\| + (1 - \beta_n)\|S_n V_n x_n - z\| \\
&\le \beta_n\|x_n - z\| + (1 - \beta_n)\max_n\{\frac{\|Vz - z\|}{1 - \alpha}, \|x_n - z\|\} \\
&\le \max\{\frac{\|Vz - z\|}{1 - \alpha}, \|x_n - z\|\}.
\end{aligned}$$

Thus, inductively,

$$\|T_n x_n - z\| \le \|x_n - z\| \le \max\{\frac{\|Vz - z\|}{1-\alpha}, \|x_1 - z\|\}, \quad n = 1, 2, \ldots$$

This shows that $\{x_n\}$ is bounded, so are $\{Vx_n\}$, $\{T_n x_n\}$ and $\{S_n V_n x_n\}$. □

Lemma 4.3. *The following results hold:*

(a) $\widetilde{Fix}(\{S_n V_n\}) = \widetilde{Fix}(\{S_n\}) \cap \widetilde{Fix}(\{T_n\})$;
(b) $\{S_n V_n\}$ *satisfies the condition* (R);
(c) $\{x_n\} \in \widetilde{Fix}(\{S_n\}) \cap \widetilde{Fix}(\{T_n\})$;

where $V_n = \alpha_n(1-\lambda_n)V + \alpha_n\lambda_n I + (1-\alpha_n)T_n$, $n = 1, 2, \ldots$

Proof. We first show (a). Let $\{z_n\}$ be a bounded sequence in C and $z \in F$. If $\{z_n\} \in \widetilde{Fix}(\{T_n\})$, we have

$$\|z_n - V_n z_n\| \le \alpha_n(1-\lambda_n)(\alpha\|z - z_n\| + \|Vz - z_n\|) \to 0, (n \to \infty)$$

and therefore $\{z_n\} \in \widetilde{Fix}(\{V_n\})$. On the other hand, if $\{z_n\} \in \widetilde{Fix}(\{V_n\})$, we obtain

$$\begin{aligned}\|z_n - T_n z_n\| &= \|V_n z_n - T_n z_n\| \\ &\le \alpha_n(1-\lambda_n)\|Vz_n - T_n z_n\| + \alpha_n\lambda_n\|z_n - T_n z_n\| \\ &\le \alpha_n(1-\lambda_n)(\alpha\|z_n - z\| + \|Vz - z\| + \|z - T_n z_n\|) \\ &\quad + \alpha_n\lambda_n(\|z_n - z\| + \|z - T_n z_n\|) \\ &\le \alpha_n \sup_n\{\|Vz - z\| + 2\|z_n - z\|\} \to 0, (n \to \infty)\end{aligned}$$

and hence $\{z_n\} \in \widetilde{Fix}(\{T_n\})$. Then we obviously have $\widetilde{Fix}(\{T_n\}) = \widetilde{Fix}(\{V_n\})$ and $\widetilde{Fix}(\{S_n T_n\}) = \widetilde{Fix}(\{S_n V_n\})$. Since F is nonempty, $\{S_n\}$ or $\{T_n\}$ is a strongly nonexpansive sequence. By Lemma 2.5, it is easy to verify that

$$\widetilde{Fix}(\{S_n V_n\}) = \widetilde{Fix}(\{S_n T_n\}) = \widetilde{Fix}(\{S_n\}) \cap \widetilde{Fix}(\{T_n\}).$$

We next show (b). Since F is nonempty, we have $\{T_n y : y \in D, n = 0, 1, 2, \ldots\}$ is bounded for any nonempty bounded subset D of C, by condition $\{T_n\}$ satisfies the condition (R), there hold

$$\begin{aligned}\|V_n y - z\| &\le (1-\lambda_n)\alpha_n(\alpha\|y - z\| + \|Vz - z\|) \\ &\quad + \lambda_n\alpha_n\|y - z\| + (1-\alpha_n)\|T_n y - z\| \\ &\le \max\{\frac{\|Vz - z\|}{1-\alpha}, \sup_{y\in D}\|y - z\|\}\end{aligned}$$

and

$$\begin{aligned}\sup_{y\in D}\|V_{n+1}y - V_n y\| &\le |(1-\lambda_{n+1})\alpha_{n+1} - (1-\lambda_n)\alpha_n|\|Vy\| \\ &\quad + |\lambda_{n+1}\alpha_{n+1} - \lambda_n\alpha_n|\sup_{y\in D}\|y\| \\ &\quad + \sup_{y\in D}\|(1-\alpha_{n+1})T_{n+1}y - (1-\alpha_n)T_n y\| \\ &\le (|\alpha_{n+1}-\alpha_n| + \alpha_n|\lambda_{n+1}-\lambda_n|)(\sup_{y\in D}\|Vy\| + \sup_{y\in D}\|y\|) \\ &\quad + \sup_{y\in D}\|T_{n+1}y - T_n y\| + |\alpha_{n+1}-\alpha_n|\sup_{y\in D}\|T_n y\| \to 0\end{aligned}$$

for any nonempty bounded subset D of C. Hence, we further obtain that $\{V_n\}$ satisfies the condition (R) and $\{V_n y : y \in D, n = 0, 1, 2, \ldots\}$ is bounded for any nonempty bounded subset D of C. Lemma 2.4 implies that $\{S_n V_n\}$ satisfies the condition (R).

We finally show (c). Using Lemma 4.2, there is a nonempty bounded subset D of C such that $\{x_n\} \in D$, for every $n \in \mathbb{N}$. Set $U_n = S_n V_n$ for every $n \in \mathbb{N}$. Since U_n is nonexpansive and $\{U_n\}$ satisfies the condition (R) by (b), we obtain

$$\begin{aligned}\|U_{n+1}x_{n+1} - U_n x_n\| &\le \|U_{n+1}x_{n+1} - U_n x_{n+1}\| + \|U_n x_{n+1} - U_n x_n\| \\ &\le \sup_{y\in D}\|U_{n+1}y - U_n y\| + \|x_{n+1} - x_n\|\end{aligned}$$

and this implies that

$$\|U_{n+1}x_{n+1} - U_n x_n\| - \|x_{n+1} - x_n\| \le \sup_{y\in D}\|U_{n+1}y - U_n y\| \to 0, \quad (n\to\infty).$$

By Lemma 4.2, Lemma 2.3 and (a), we have

$$\{x_n\} \in \widetilde{Fix}(\{U_n\}) = \widetilde{Fix}(\{S_n\}) \cap \widetilde{Fix}(\{T_n\})$$

and this proof is finished. □

Lemma 4.4. $\limsup_{n\to\infty}\langle (I-V)x^*, T_n x_n - x^*\rangle \ge 0$, *where* x^* *is the unique solution of VI* (4.3).

Proof. To prove this result, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ so that

$$\limsup_{n\to\infty}\langle (I-V)x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle (I-V)x^*, x_{n_k} - x^*\rangle. \tag{4.4}$$

By Lemma 4.3, $\{x_n\}$ is bounded. Without loss of generality, we may further assume that $\{x_{n_k}\}$ converges weakly to a point $\bar{x}$, Since $\{S_n\}$ and $\{T_n\}$ satisfy the condition (Z), Lemma 4.3 implies that $\bar{x} \in F$. Noting that $T_n x_n - x_n \to 0$, (4.4) and applying Lemma 2.1, we arrive at

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-V)x^*, T_nx_n - x^*\rangle &= \limsup_{n\to\infty}(\langle (I-V)x^*, T_nx_n - x_n\rangle \\ &\quad + \limsup_{n\to\infty}\langle (I-V)x^*, x_n - x^*\rangle) \\ &= \lim_{k\to\infty}\langle (I-V)x^*, x_{n_k} - x^*\rangle \\ &= \langle (I-V)x^*, \bar{x} - x^*\rangle \geq 0.\end{aligned}$$

□

By Lemma 4.2 - Lemma 4.4, we finally prove Theorem 4.1.

Proof. Set $V_n = \alpha_n(1-\lambda_n)V + \alpha_n\lambda_n I + (1-\alpha_n)T_n$, for every $n \in \mathbb{N}$. Since $x^* \in F$, and both S_n and T_n are nonexpansive, it is easy to verify that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|S_nV_nx_n - x^*\|^2 \\ &\leq \beta_n\|x_n - x^*\|^2 \\ &\quad + (1-\beta_n)[\|(1-\alpha_n)(T_nx_n - x^*) + \alpha_n\lambda_n(x_n - x^*)\|^2 \\ &\quad + 2\alpha_n(1-\lambda_n)\langle V(x*) - x^*, V_nx_n - x^*\rangle] \\ &\leq [1-\alpha_n(1-\beta_n)(1-\lambda_n)]\|x_n - x^*\|^2 \\ &\quad + 2\alpha_n(1-\beta_n)(1-\lambda_n)\langle V(x^*) - x^*, V_nx_n - x^*\rangle.\end{aligned}$$

By Lemma 4.2 and Lemma 4.4, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle Vx^* - x^*, V_nx_n - x^*\rangle &= \limsup_{n\to\infty}(\langle Vx^* - x^*, T_nx_n - x^*\rangle \\ &\quad + \alpha_n\langle Vx^* - x^*, \lambda_nx_n + (1-\lambda_n)Vx^* - T_nx_n\rangle) \\ &= \limsup_{n\to\infty}\langle Vx^* - x^*, T_nx_n - x^*\rangle \leq 0.\end{aligned} \tag{4.5}$$

Hence,

$$\|x_{n+1} - x^*\|^2 \leq (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\sigma_n,$$

where

$$\gamma_n = \alpha_n(1-\beta_n)(1-\lambda_n), \quad \sigma_n = 2\langle Vx^* - x^*, V_nx_n - x^*\rangle.$$

It is easily check that $\gamma_n \to 0$, $\sum_{n=0}^{\infty}\gamma_n = \infty$ by conditions (D1) and (D2), and $\limsup_{n\to\infty}\sigma_n \leq 0$ by (4.5). By Lemma 2.6, we conclude that $x_n \to x^*$, and the proof is finished. □

Remark 4.5. If $V(x) = u \in C$, a given point, for any $x \in C$, then Theorem 4.1 is induced to Aoyama and Kimura's result [1].

As a direct result of Theorem 4.1, we can find the minimum-norm common fixed point of $\{S_n\}$ and $\{T_n\}$ whether $0 \in C$ or $0 \notin C$ via the following corollary:

Corollary 4.6. *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $\{S_n\}$ and $\{T_n\}$ sequences of nonexpansive self-mappings of C. Suppose that $F = Fix(\{S_n\}) \cap Fix(\{T_n\})$ is nonempty, both $\{S_n\}$ and $\{T_n\}$ satisfy the conditions (R) and (Z), and $\{S_n\}$ or $\{T_n\}$ is a strongly nonexpansive sequence. Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfy the following conditions*

(D1) $\alpha_n \to 0$;

(D2) $\sum_{n=1}^{\infty} \alpha_n(1-\lambda_n)(1-\beta_n) = \infty$;

where $\lambda_n = s(x_n)$, $n = 1, 2, \ldots$. Let $\{x_n\}$ be sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_n(\alpha_n \lambda_n x_n + (1-\alpha_n) T_n x_n) \tag{4.6}$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ defined by (4.6) converges strongly to the minimum-norm common fixed point $P_F(0)$ of $\{S_n\}$ and $\{T_n\}$.

5. Conclusions

This paper presents the definition of boundary point function for non-self contraction and further develops an iterative scheme based on boundary point method for finding the common fixed point of a pair strongly nonexpansive sequences. Our algorithm overcomes the shortcoming of $u \notin C$ in Aoyama and Kimura's iterative scheme and does not involve the computation of the metric projection P_C, so it is easy to realize in the actual computational process. In addition, strong convergence theorem of the iterative scheme (4.1) is obtained under some appropriate conditions. In particular, we do not require any additional conditions on parametric sequence $\{\lambda_n\}$ except available condition $\sum_{n=1}^{\infty} \alpha_n(1-\lambda_n)(1-\beta_n) = \infty$. As a special result, our proposed algorithm can find the minimum-norm common fixed point of a pair of strongly nonexpansive sequences $\{S_n\}$ and $\{T_n\}$ whether $0 \in C$ or $0 \notin C$.

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