ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 3, pp. 761–777

Title:

Applications of subordination theory to starlike functions

Author(s):

K. Sharma and V. Ravichandran

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 3, pp. 761–777
Online ISSN: 1735-8515

APPLICATIONS OF SUBORDINATION THEORY TO STARLIKE FUNCTIONS

K. SHARMA AND V. RAVICHANDRAN*

(Communicated by Ali Abkar)

Abstract. Let p be an analytic function defined on the open unit disc $\mathbb{D}$ with $p(0)=1$. The conditions on α and β are derived for $p(z)$ to be subordinate to $1+4z/3+2z^2/3 =: \varphi_C(z)$ when $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)/p(z)$ is subordinate to e^z. Similar problems were investigated for $p(z)$ to lie in a region bounded by lemniscate of Bernoulli $|w^2-1|=1$ when the functions $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)$, $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)/p(z)$ or $p(z)+\beta zp'(z)/p^2(z)$ are subordinates to $\varphi_C(z)$. Related results for p to be in the parabolic region bounded by the $\operatorname{Re} w=|w-1|$ are investigated.

Keywords: convex and starlike functions, cardioid, parabolic starlike, lemniscate of Bernoulli, subordination.
MSC(2010): Primary: 30C80; Secondary: 30C45.

1. Introduction

Let $\mathcal{A}$ be the class of all functions f analytic in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For an analytic function φ with positive real part in $\mathbb{D}$ with $\varphi(0)=1$ and $\varphi'(0)>0$, let

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1+\frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}.$$

These classes unify various classes of starlike and convex functions. Shanmugam [18] studied the convolution properties of these classes when φ is convex while Ma and Minda [8] investigated the growth, distortion and coefficient estimates under less restrictive assumption that φ is starlike and $\varphi(\mathbb{D})$ is symmetric with

Article electronically published on June 29, 2016.
Received: 13 February 2015, Accepted: 27 April 2015.
*Corresponding author.

respect to the real axis. Notice that, for $-1 \le B < A \le 1$, the class $\mathcal{S}^*[A,B] := \mathcal{S}^*((1+Az)/(1+Bz))$ is the class of Janowski starlike functions [6, 13]. For $0 \le \alpha < 1$, the class $\mathcal{S}^*[1-2\alpha,-1] =: \mathcal{S}^*(\alpha)$ is the familiar class of starlike functions of order α, introduced by Robertson [16]. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the classes of starlike and convex functions respectively. If the function $\varphi_{PAR} : \mathbb{D} \to \mathbb{C}$ is given by

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad \operatorname{Im}\sqrt{z} \ge 0 \tag{1.1}$$

then $\varphi_{PAR}(\mathbb{D}) = \left\{w = u + iv : v^2 < 2u - 1\right\} = \{w : \operatorname{Re} w > |w-1|\} =: \Omega_P$. The class $\mathcal{C}(\varphi_{PAR})$ is the class of uniformly convex functions introduced by Goodman [4]. The corresponding class $\mathcal{S}_P := \mathcal{S}^*(\varphi_{PAR})$ of parabolic starlike functions, introduced by Rønning [17], consists of function $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}.$$

Sokól and Stankiewicz [23] have introduced and studied the class $\mathcal{S}_L^* = \mathcal{S}^*(\sqrt{1+z})$; the class $\mathcal{S}_L^*$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $\Omega_L := \left\{w \in \mathbb{C} : |w^2 - 1| < 1\right\}$. There has been several works [1, 3, 5, 14, 19, 21, 22] related to these classes. Similarly, the class $\mathcal{S}_C^* := \mathcal{S}^*(\varphi_C)$, where $\varphi_C(z) = 1 + 4z/3 + 2z^2/3$ was introduced and studied recently in [15, 20]. Precisely, $f \in \mathcal{S}_C^*$ provided $zf'(z)/f(z)$ lies in the region bounded by the cardioid

$$\Omega_C := \left\{w = u + iv : (9u^2 + 9v^2 - 18u + 5)^2 - 16(9u^2 + 9v^2 - 6u + 1) = 0\right\}.$$

Another class $\mathcal{S}_e^* := \mathcal{S}^*(e^z)$, introduced recently by Mendiratta *et al.* [10], consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$.

A convex function is starlike of order 1/2; analytically,

$$p(z) + zp'(z)/p(z) \prec (1+z)/(1-z) \Longrightarrow p(z) \prec 1/(1-z).$$

Similarly, a sufficient condition for a function p to be a function with positive real part is that $p(z) + zp'(z)/p(z) \prec \mathcal{R}(z)$, where $\mathcal{R}$ is the open door mapping given by

$$\mathcal{R}(z) := \frac{1+z}{1-z} + \frac{2z}{1-z^2}.$$

Several authors have investigated similar results for functions to belong to certain regions in right half plane. For example, Ali *et al.* [2] determined the condition on β for $p(z) \prec \sqrt{1+z}$ when $1 + \beta zp'(z)/p^n(z)$ with $n = 0, 1, 2$ or $(1-\beta)p(z) + \beta p^2(z) + \beta zp'(z)$ is subordinate to $\sqrt{1+z}$. For related results, see [1–3, 7, 12, 22]. We investigate a similar problem for regions that were considered recently by many authors, including parabolic and lemniscate regions associated with the classes $\mathcal{S}_P$ and $\mathcal{S}_L^*$, respectively. Precisely we determine conditions on α and β so that $p(z) \prec \varphi_C(z)$ when $(1-\alpha)p(z) +$

$\alpha p^2(z) + \beta zp'(z)/p(z) \prec e^z$. Conditions on α and β are also determined so that $(1-\alpha)p(z) + \alpha p^2(z) + \beta zp'(z)$ or $(1-\alpha)p(z) + \alpha p^2(z) + \beta zp'(z)/p(z)$ or $p(z) + \beta zp'(z)/p^2(z) \prec \varphi_C(z)$ implies $p(z) \prec \sqrt{1+z}$. We also find condition on β so that $1+\beta zp'(z)$ is subordinate to $\varphi_C(z)$ or $\sqrt{1+z}$ implies $p(z) \prec \varphi_{PAR}(z)$. Our results yield several sufficient conditions for $f \in \mathcal{A}$ to belong to the class $\mathcal{S}_P$, $\mathcal{S}^*_C$ or $\mathcal{S}^*_L$.

We need the following lemmas to prove our results.

Lemma 1.1. *[11, Corollary 3.4h, p.135] Let q be univalent in $\mathbb{D}$, and let φ be analytic in a domain D containing $q(\mathbb{D})$. Let $zq'(z)\varphi(q(z))$ be starlike. If p is analytic in $\mathbb{D}$, $p(0) = q(0)$ and $zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z))$, then $p \prec q$ and q is the best dominant.*

Lemma 1.2. *[11, Theorem 3.4i, p.134] Let q be univalent in $\mathbb{D}$ and let φ and ν be analytic in a domain D containing $q(\mathbb{D})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{D})$. Set $Q(z) := zq'(z)\varphi(q(z))$, $h(z) := \nu(q(z)) + Q(z)$. Suppose that (i) either h is convex or $Q(z)$ is starlike univalent in $\mathbb{D}$ and (ii) $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in \mathbb{D}$. Let p be analytic in $\mathbb{D}$ with $p(0) = q(0)$ and $p(\mathbb{D}) \subset \mathbb{D}$. If p satisfies*

$$\nu(p(z)) + zp'(z)\varphi(p(z)) \prec \nu(q(z)) + zq'(z)\varphi(q(z)), \tag{1.2}$$

then $p \prec q$ and q is the best dominant.

2. Results associated with starlikeness

Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. In the first result, we find the conditions on α and β so that $p(z) \in \Omega_C$, whenever $(1-\alpha)p(z) + \alpha p^2(z) + \beta zp'(z)/p(z) \prec e^z$.

Theorem 2.1. *Let the function p be analytic in $\mathbb{D}$ with $p(0) = 1$. Let $\alpha, \beta \in \mathbb{R}$ such that either (i) $3(e-3)/(2e) < \alpha < (e-3)/6$, $\beta > 9(e-3-6\alpha)/8$, or (ii) $(e-3)/6 \leq \alpha \leq 0$, $\beta > 0$ holds. If the function p satisfies*

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta \frac{zp'(z)}{p(z)} \prec e^z$$

then $p(z) \prec \varphi_C(z)$.

Proof. The function $q : \mathbb{D} \to \mathbb{C}$ defined by $q(z) = \varphi_C(z) = 1 + 4(z + z^2/2)/3$ is univalent in $\mathbb{D}$. Let $h : \mathbb{D} \to \mathbb{C}$ be defined by

$$h(z) := (1-\alpha)q(z) + \alpha q^2(z) + \beta \frac{zq'(z)}{q(z)}$$

$$= (1-\alpha)\left(1 + \frac{4z}{3} + \frac{2z^2}{3}\right) + \alpha\left(1 + \frac{4z}{3} + \frac{2z^2}{3}\right)^2 + \frac{4\beta z(1+z)}{3+4z+2z^2}. \tag{2.1}$$

The proof is by showing that (a)

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta \frac{zp'(z)}{p(z)} \prec (1-\alpha)q(z) + \alpha q^2(z) + \beta \frac{zq'(z)}{q(z)} \tag{2.2}$$

implies that $p(z) \prec q(z)$ and (b) the subordination $\psi(z) := e^z \prec h(z)$ holds.

(a) The subordination (2.2) is the same as (1.2) if we define the functions ν, φ by $\nu(w) = (1-\alpha)w + \alpha w^2$ and $\varphi(w) = \beta/w$. The function ν is analytic in $\mathbb{C}$. Since $\beta > 0$, φ is analytic in $\mathbb{C} \setminus \{0\}$ and $\varphi(w) \neq 0$. Consider the functions Q and h defined as follows:

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta z q'(z)}{q(z)} = \frac{4\beta z(1+z)}{3+4z+2z^2} \tag{2.3}$$

and

$$h(z) = \nu(q(z)) + Q(z) = (1-\alpha)q(z) + \alpha q^2(z) + Q(z). \tag{2.4}$$

The equation (2.3) gives

$$\frac{zQ'(z)}{Q(z)} = \frac{z}{1+z} + \frac{3-2z^2}{3+4z+2z^2} =: K(z).$$

Substituting $x = \cos t$ $(t \in [-\pi, \pi])$, we have

$$\operatorname{Re}(K(e^{it})) = \frac{1}{2} + \frac{5+4\cos t}{29+40\cos t+12\cos 2t} = \frac{1}{2} + \frac{5+4x}{24x^2+40x+17} \geq \frac{11}{18} > 0.$$

This together with the minimum principle for harmonic functions shows that the function Q is starlike univalent in $\mathbb{D}$. Using (2.3) and (2.4), we get

$$\frac{zh'(z)}{Q(z)} = \frac{1-\alpha}{\beta}q(z) + \frac{2\alpha}{\beta}q^2(z) + \frac{zQ'(z)}{Q(z)} = M(z) + K(z),$$

where

$$M(z) = ((1-\alpha)/\beta)q(z) + (2\alpha/\beta)q^2(z).$$

We show that $\operatorname{Re}(zh'(z)/Q(z)) > 0, z \in \mathbb{D}$ when $\alpha, \beta \in \mathbb{R}$ satisfy the conditions in (i) or (ii) in the hypothesis. For $t \in [-\pi, \pi]$, we have

$$\begin{aligned}\operatorname{Re}(M(e^{it})) = (&9 + 9\alpha + 12(1+3\alpha)\cos t \\ &+ (6+50\alpha)\cos 2t + 32\alpha\cos 3t + 8\alpha\cos 4t)/9\beta =: H(\cos t).\end{aligned}$$

We need to prove that $H(x) \geq 0$ in the interval $-1 \leq x \leq 1$ for cases (i) and (ii), where

$$H(x) = (3 - 33\alpha + 12(1-5\alpha)x + 12(1+3\alpha)x^2 + 128\alpha x^3 + 64\alpha x^4)/9\beta.$$

Since, $H(1) = (3+15\alpha)/\beta$ and $H(-1) = (3-\alpha)/9\beta$, $H(1)$ and $H(-1)$ both are non- negative for $-1/5 \leq \alpha \leq 3, \beta > 0$. A calculation shows that $H'(x) = 0$ if

$$\begin{aligned}x = x_0 = -\frac{1}{2} &- \frac{1152\alpha - 5760\alpha^2}{4608(2^{\frac{1}{3}}\alpha(16\alpha^3 + \sqrt{2}\sqrt{\alpha^3 - 15\alpha^4 + 75\alpha^5 + 3\alpha^6})^{\frac{1}{3}})} \\ &+ \frac{(16\alpha^3 + \sqrt{2}\sqrt{\alpha^3 - 15\alpha^4 + 75\alpha^5 + 3\alpha^6})^{\frac{1}{3}}}{4(2^{\frac{2}{3}}\alpha)}\end{aligned}$$

and

$$H''(x) = (768x\alpha + 8(-16+96x^2)\alpha + 4(6+50\alpha))/9\beta.$$

Clearly for both the cases (i) and (ii), $H''(x_0) < 0$, $H(1) \geq 0$ and $H(-1) \geq 0$. Therefore, $H(x) \geq \min(H(1), H(-1)) \geq 0$ for $-1 \leq x \leq 1$. This shows that $\operatorname{Re}(zh'(z)/Q(z)) > 0, z \in \mathbb{D}$ and therefore, $h(z) - 1$ is close-to-convex function and hence univalent in $\mathbb{D}$. If the subordination (2.2) holds, Lemma 1.2 shows that $p(z) \prec q(z)$.

(b) We now show that $\psi(z) := e^z \prec h(z)$ holds. The subordination $\psi(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C}\backslash\overline{\psi(\mathbb{D})} = \{w \in \mathbb{C} : |\log w| > 1\}$. Set $w = u+iv = h(e^{it})$, where $t \in [-\pi, \pi]$. Then, the inequality $|\log w| > 1$ reduces to

$$f(t) := (\log(u^2+v^2))^2 + 4(\arg(u+iv))^2 - 4 > 0. \tag{2.5}$$

By definition of h given in (2.1), we get

$$\begin{aligned} u = \frac{1}{9(29+40\cos t+12\cos 2t)}\Big(&537+372\alpha+216\beta \\ &+4(225+239\alpha+81\beta)\cos t+2(261+611\alpha+54\beta)\cos 2t \\ &+96(2+11\alpha)\cos 3t+4(9+142\alpha)\cos 4t \\ &+176\alpha\cos 5t+24\alpha\cos 6t\Big) \end{aligned}$$

and

$$\begin{aligned} v =\frac{1}{9(29+40\cos t+12\cos 2t)}\Big(&4(147+511\alpha+45\beta \\ &+(225+907\alpha+54\beta)\cos t+8(12+77\alpha)\cos 2t \\ &+2(9+148\alpha)\cos 3t+88\alpha\cos 4t+12\alpha\cos 5t)\sin t\Big). \end{aligned}$$

Since $f(t)$ is an even function of t, it is enough to show that $f(t) > 0$ for $t \in [0, \pi]$. It can be easily verified that for both the cases (i) and (ii), the function $f(t)$ attains its minimum value either at $t = 0$ or $t = \pi$. So, we need to show that both $f(0)$ and $f(\pi)$ are positive in either cases. Note that

$$f(0) = -4 + 4(\arg(27+54\alpha+8\beta))^2 + (\log((27+54\alpha+8\beta)^2/81))^2 \tag{2.6}$$

and

$$f(\pi) = -4 + 4(\arg(3-2\alpha))^2 + (\log((3-2\alpha)^2/81))^2. \tag{2.7}$$

For the case (i), the relation $\beta > 9(e-3-6\alpha)/8$ gives $27+54\alpha+8\beta > 9e$ so that $\arg(27+54\alpha+8\beta) = 0$ and $(\log((27+54\alpha+8\beta)^2/81))^2 > (2\log e)^2 = 4$. Thus, the use of (2.6) yields $f(0) > 0$. The conditions $\alpha < (e-3)/6$ and $3(e-3)/(2e) < \alpha$ lead to $3-2\alpha > 4-e/3 > 0$ and $(3-2\alpha)^2/81 < 1/e^2$ respectively which further implies that $\arg(3-2\alpha) = 0$ and $(\log((3-2\alpha)^2/81))^2 > 4$ respectively. Hence, by using (2.7), we get $f(\pi) > 0$.

For the case (ii), the condition $(e-3)/6 \leq \alpha$ gives $27+54\alpha+8\beta > 8\beta + 9e > 9e$. So, proceeding as in the case (i), we get $f(0) > 0$. Using the fact that $\alpha \leq 0$, we get $3-2\alpha > 0$ and hence $\arg(3-2\alpha) = 0$. Observe that

$\alpha \geq (e-3)/6 > 3(e-3)/(2e)$. Thus, again proceeding as in the case (i), we get $f(\pi) > 0$. This completes the proof. □

By taking $p(z) = zf'(z)/f(z)$, $p(z) = z^2 f'(z)/f^2(z)$ and $p(z) = f'(z)$, the above theorem gives the following:

Example 2.2. Let $\alpha, \beta \in \mathbb{R}$ such that either (i) $3(e-3)/(2e) < \alpha < (e-3)/6$, $\beta > 9(e-3-6\alpha)/8$, or (ii) $(e-3)/6 \leq \alpha \leq 0$, $\beta > 0$ holds.

(1) If the function $f \in \mathcal{A}$ satisfies the subordination

$$(1-\alpha-\beta)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)^2 + \beta\left(1+\frac{zf''(z)}{f'(z)}\right) \prec e^z$$

then $f \in \mathcal{S}^*_C$.

(2) If the function $f \in \mathcal{A}$ satisfies the subordination

$$\left((1-\alpha)+\alpha\frac{z^2 f'(z)}{f^2(z)}\right)\frac{z^2 f'(z)}{f^2(z)} + \beta\left(\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right) \prec e^z$$

then $z^2 f'(z)/f^2(z) \prec \varphi_C(z)$.

(3) If the function $f \in \mathcal{A}$ satisfies the subordination

$$((1-\alpha)+\alpha f'(z))f'(z) + \beta\frac{zf''(z)}{f'(z)} \prec e^z$$

then $f'(z) \prec \varphi_C(z)$.

In the next two theorems, we compute the conditions on β so that $p(z) \in \Omega_L$, whenever

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z) \quad or \quad (1-\alpha)p(z)+\alpha p^2(z)+\beta\frac{zp'(z)}{p(z)} \in \Omega_C,$$

where p is an analytic function defined on $\mathbb{D}$ with $p(0) = 1$.

Theorem 2.3. *Let α, $\beta \in \mathbb{R}$ satisfying $-1/(2\sqrt{2}-1) \leq \alpha \leq 1$ and $\beta > -2(2-3\sqrt{2}-2\alpha+2\sqrt{2}\alpha)$. If the function p is analytic in $\mathbb{D}$ with $p(0) = 1$ and satisfies $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z) \prec \varphi_C(z)$ then $p(z) \prec \sqrt{1+z}$.*

Proof. Let q be the convex univalent function defined by $q(z) = \sqrt{1+z}$. Then it is clear that $\beta zq'(z)$ is starlike. We will prove the result by showing that (a)

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z) \prec (1-\alpha)q(z)+\alpha q^2(z)+\beta zq'(z) \tag{2.8}$$

implies that $p(z) \prec q(z)$ and (b)

$$\begin{aligned}\varphi_C(z) &:= 1+\frac{4z}{3}+\frac{2z^2}{3} \prec (1-\alpha)q(z)+\alpha q^2(z)+\beta zq'(z)\\ &= (1-\alpha)\sqrt{1+z}+\alpha(1+z)+\frac{\beta z}{2\sqrt{1+z}} =: h(z).\end{aligned} \tag{2.9}$$

(a) To prove (2.8), define $\nu(w)=(1-\alpha)w+\alpha w^2$ and $\varphi(w)=\beta$. The function ν is analytic in $\mathbb{C}$. Since $\beta>0$, φ is analytic in $\mathbb{C}\setminus\{0\}$ and $\varphi(w)\neq 0$. The function Q defined by

$$Q(z):=zq'(z)\varphi(q(z))=\beta zq'(z)=\frac{\beta z}{2\sqrt{1+z}} \tag{2.10}$$

is starlike of order $3/4$ and for the function h defined by

$$h(z):=\nu(q(z))+Q(z)=(1-\alpha)q(z)+\alpha q^2(z)+Q(z), \tag{2.11}$$

we have

$$\frac{zh'(z)}{Q(z)}=\frac{1-\alpha}{\beta}+\frac{2\alpha}{\beta}q(z)+\frac{zQ'(z)}{Q(z)}.$$

Using the fact that $0<\operatorname{Re}q(z)<\sqrt{2}$, $z\in\mathbb{D}$, we have the following two cases:

Case 1: $0\leq\alpha\leq 1$. In this case, we have

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)>\frac{1-\alpha}{\beta}+\frac{3}{4}>0.$$

Case 2: $-1/(2\sqrt{2}-1)\leq\alpha<0$. In this case, we have

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)>\frac{1-\alpha}{\beta}+\frac{2\sqrt{2}\alpha}{\beta}+\frac{3}{4}>0.$$

This shows that $\operatorname{Re}(zh'(z)/Q(z))>0, z\in\mathbb{D}$ and therefore, $h(z)-1$ is close-to-convex function and hence univalent in $\mathbb{D}$. If the subordination (2.8) holds, Lemma 1.2 shows that $p(z)\prec q(z)$.

(b) We now show that (2.9) holds. Clearly,

$$\varphi_C(\mathbb{D})=\left\{w\in\mathbb{C}:|-2+\sqrt{6w-2}|<2\right\}.$$

The subordination $\varphi_C(z)\prec h(z)$ holds if $\partial h(\mathbb{D})\subset\mathbb{C}\setminus\overline{\varphi_C(\mathbb{D})}$. Thus, by using the definition of h as given in (2.9), the subordination $\varphi_C(z)\prec h(z)$ holds if for $t\in[-\pi,\pi]$, we have

$$\left|\sqrt{-2+6(1-\alpha)\sqrt{1+e^{it}}+6\alpha(1+e^{it})+\frac{3\beta e^{it}}{\sqrt{1+e^{it}}}}-2\right|>2. \tag{2.12}$$

By writing

$$w=-2+6(1-\alpha)\sqrt{1+e^{it}}+6\alpha(1+e^{it})+3\beta e^{it}(1+e^{it})^{-\frac{1}{2}}, \tag{2.13}$$

we see that the condition (2.12) holds if $|\sqrt{w}-2|>2$ or equivalently if $|w|>4\operatorname{Re}(\sqrt{w})$. On further simplification after substituting $w=u+iv$, (2.12) holds if

$$(u^2+v^2-8u)^2-64(u^2+v^2)>0. \tag{2.14}$$

Using (2.13), we get

$$u = -2 + 6(1-\alpha)\sqrt{2\cos(t/2)}\cos(t/4) + 6\alpha(1+\cos t) + 3\beta\cos(3t/4)(2\cos(t/2))^{-\frac{1}{2}} \tag{2.15}$$

and

$$v = 6(1-\alpha)\sqrt{2\cos(t/2)}\sin(t/4) + 6\alpha\sin t + 3\beta\sin(3t/4)(2\cos(t/2))^{-\frac{1}{2}}. \tag{2.16}$$

Using (2.15) and (2.16) in (2.14), we get

$$\begin{aligned} g(t) := -64\Big(&\Big(-2 + 6\sqrt{2}(1-\alpha)\cos(t/4)\sqrt{\cos(t/2)} + \frac{3\beta\cos(3t/4)}{\sqrt{2}\sqrt{\cos(t/2)}} \\ &+ 6\alpha(1+\cos t)\Big)^2 + \Big(6\sqrt{2}(1-\alpha)\sqrt{\cos(t/2)}\sin(t/4) + \frac{3\beta\cos(3t/4)}{\sqrt{2}\sqrt{\cos(t/2)}} \\ &+ 6\alpha\sin t\Big)^2\Big) + \Big(-8\Big(-2 + 6\sqrt{2}(1-\alpha)\cos(t/4)\sqrt{\cos(t/2)} \\ &+ \frac{3\beta\cos(3t/4)}{\sqrt{2}\sqrt{\cos(t/2)}} + 6\alpha(1+\cos t)\Big) \\ &+ \Big(-2 + 6\sqrt{2}(1-\alpha)\cos(t/4)\sqrt{\cos(t/2)} + \frac{3\beta\cos(3t/4)}{\sqrt{2}\sqrt{\cos(t/2)}} \\ &+ 6\alpha(1+\cos t)\Big)^2 + \Big(6\sqrt{2}(1-\alpha)\sqrt{\cos(t/2)}\sin(t/4) \\ &+ \frac{3\beta\sin(3t/4)}{\sqrt{2}\sqrt{\cos(t/2)}} + 6\alpha\sin t\Big)^2\Big)^2 > 0. \end{aligned}$$

Observe that $g(t) = g(-t)$ for all $t \in [-\pi, \pi]$ and $g(t)$ attains its minimum value at $t = 0$. A calculation shows that

$$\begin{aligned} g(0) = &\frac{3}{16}(4 - 12\sqrt{2} + 12(-2+\sqrt{2})\alpha - 3\sqrt{2}\beta)^3 \\ &\times (12 - 4\sqrt{2} + 4(-2+\sqrt{2})\alpha - \sqrt{2}\beta). \end{aligned} \tag{2.17}$$

Note that the condition $\beta > -2(2 - 3\sqrt{2} - 2\alpha + 2\sqrt{2}\alpha)$ is equivalent to $12 - 4\sqrt{2} + 4(-2+\sqrt{2})\alpha - \sqrt{2}\beta < 0$ and $4 - 12\sqrt{2} + 12(-2+\sqrt{2})\alpha - 3\sqrt{2}\beta < 0$. Thus, the use of (2.17) yields $g(0) > 0$ which implies that $g(t) > 0$ for all $t \in [0, \pi]$. Hence the result follows. □

Theorem 2.4. *Let α, $\beta \in \mathbb{R}$ satisfying $0 \le \alpha \le 1$ and $\beta > 4(3 - \sqrt{2} + (\sqrt{2} - 2)\alpha)$. If p is an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta\frac{zp'(z)}{p(z)} \prec \varphi_C(z)$$

then $p(z) \prec \sqrt{1+z}$.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$. Proceeding as in Theorem 2.3, the result is proved by showing that (a)

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta \frac{zp'(z)}{p(z)} \prec (1-\alpha)q(z) + \alpha q^2(z) + \beta \frac{zq'(z)}{q(z)} \tag{2.18}$$

implies that $p(z) \prec q(z)$ and (b)

$$\begin{aligned} \varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} &\prec (1-\alpha)q(z) + \alpha q^2(z) + \frac{\beta z q'(z)}{q(z)} \\ &= (1-\alpha)\sqrt{1+z} + \alpha(1+z) + \frac{\beta z}{2(1+z)} =: h(z). \end{aligned} \tag{2.19}$$

(a) Let us define $\nu(w) = (1-\alpha)w + \alpha w^2$ and $\varphi(w) = \beta/w$. Clearly $\beta > 0$. The functions ν and φ are analytic in $\mathbb{C}\setminus\{0\}$ which includes $q(\mathbb{D})$ and $\varphi(w) \neq 0$. Next, define the functions Q and h by

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1+z)} \tag{2.20}$$

and

$$h(z) := \nu(q(z)) + Q(z) = (1-\alpha)q(z) + \alpha q^2(z) + Q(z). \tag{2.21}$$

Since Q is a Möbius transformation, the function Q is convex. Further using (2.20) and (2.21), we get

$$\frac{zh'(z)}{Q(z)} = \frac{1-\alpha}{\beta} q(z) + \frac{2\alpha}{\beta} q^2(z) + \frac{zQ'(z)}{Q(z)}.$$

Since $0 < \operatorname{Re} q(z) < \sqrt{2}$ and $0 < \operatorname{Re} q^2(z) < 2$, $z \in \mathbb{D}$, we have

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > \sqrt{2}\left(\frac{1-\alpha}{\beta}\right) + \frac{4\alpha}{\beta} > 0.$$

Therefore, $h(z) - 1$ is close-to-convex function and hence univalent in $\mathbb{D}$. If the subordination (2.18) holds, Lemma 1.2 shows that $p(z) \prec q(z)$.

(b) We now claim that (2.19) holds. Note that

$$\varphi_C(\mathbb{D}) = \left\{ w \in \mathbb{C} : |-2 + \sqrt{6w-2}| < 2 \right\}.$$

The subordination $\varphi_C(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\varphi_C(\mathbb{D})}$. Using the definition of h given in (2.19), the subordination $\varphi_C(z) \prec h(z)$ holds if for $t \in [-\pi, \pi]$, the following condition holds

$$\left| \sqrt{-2 + 6(1-\alpha)\sqrt{1+e^{it}} + 6\alpha(1+e^{it}) + \frac{3\beta e^{it}}{1+e^{it}}} - 2 \right| > 2. \tag{2.22}$$

Let

$$w = u + iv = -2 + 6(1-\alpha)\sqrt{1+e^{it}} + 6\alpha(1+e^{it}) + \frac{3\beta e^{it}}{1+e^{it}}. \tag{2.23}$$

Proceeding as in Theorem 2.3, the condition (2.22) holds if (2.14) holds. From (2.23), we get

$$u = -2 + 6(1-\alpha)\sqrt{2\cos(t/2)}\cos(t/4) + 6\alpha(1+\cos t) + \frac{3\beta}{2}$$

and

$$v = 6(1-\alpha)\sqrt{2\cos(t/2)}\sin(t/4) + 6\alpha\sin t + \frac{3\beta}{2}\tan t/2.$$

Using these above expressions for u and v, the condition (2.14) takes the following form

$$\begin{aligned} k(t) := -64\Big(&\big(-2 + (3\beta)/2 + 6\sqrt{2}(1-\alpha)\cos(t/4)\sqrt{\cos(t/2)} \\ &+ 6\alpha(1+\cos t)\big)^2 + \big(6\sqrt{2}(1-\alpha)\sqrt{\cos(t/2)}\sin(t/4) + 6\alpha\sin t \\ &+ (3/2)\beta\tan(t/2)\big)^2\Big) + \Big(-8\big(-2 + (3\beta)/2 \\ &+ 6\sqrt{2}(1-\alpha)\cos(t/4)\sqrt{\cos(t/2)} + 6\alpha(1+\cos t)\big) + \big(-2 + (3\beta)/2 \\ &+ 6\sqrt{2}(1-\alpha)\cos(t/4)\sqrt{\cos(t/2)} + 6\alpha(1+\cos t)\big)^2 \\ &+ \big(6\sqrt{2}(1-\alpha)\sqrt{\cos(t/2)}\sin(t/4) + 6\alpha\sin t + (3/2)\beta\tan(t/2)\big)^2\Big)^2 \\ &> 0. \end{aligned}$$

Note that $k(t) = k(-t)$, so it is enough to show that $k(t) > 0$ for $t \in [0, \pi]$. Also note that $k(t)$ is an increasing function of t. A calculation shows that

$$\begin{aligned} k(0) = \frac{3}{16}&(4 - 12\sqrt{2} + 12(-2+\sqrt{2})\alpha - 3\beta)^3 \\ &\times (12 - 4\sqrt{2} + 4(-2+\sqrt{2})\alpha - \beta). \end{aligned} \tag{2.24}$$

Consider the given relation $\beta > 4(3 - \sqrt{2} - 2\alpha + \sqrt{2}\alpha)$ which is same as $12 - 4\sqrt{2} + 4(-2+\sqrt{2})\alpha - \beta < 0$ and $4 - 12\sqrt{2} + 12(-2+\sqrt{2})\alpha - 3\beta < 0$. By using (2.24), we get $k(0)$ is positive which implies that $k(t)$ is positive for all $t \in [0, \pi]$. This completes the proof. □

Next result depicts the condition on β so that $p(z) \in \Omega_L$, whenever $p(z) + \beta zp'(z)/p^2(z) \in \Omega_C$.

Theorem 2.5. *Let $\beta \in \mathbb{R}$ satisfying $\beta > 4(-2 + 3\sqrt{2})$. If p is an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying*

$$p(z) + \beta\frac{zp'(z)}{p^2(z)} \prec \varphi_C(z)$$

then $p(z) \prec \sqrt{1+z}$.

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$. Proceeding as in Theorem 2.3, we will prove the result by showing that (a)

$$p(z) + \beta \frac{zp'(z)}{p^2(z)} \prec q(z) + \beta \frac{zq'(z)}{q^2(z)} \tag{2.25}$$

implies that $p(z) \prec q(z)$ and (b)

$$\begin{aligned} \varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} &\prec q(z) + \beta \frac{zq'(z)}{q^2(z)} \\ &= \sqrt{1+z} + \frac{\beta z}{2(1+z)^{\frac{3}{2}}} =: h(z). \end{aligned} \tag{2.26}$$

(a) The subordination (2.25) is same as (1.2) if we define $\nu(w) = w$ and $\varphi(w) = \beta/w^2$. Clearly, the functions ν and φ are analytic in $\mathbb{C} \setminus \{0\}$ which includes $q(\mathbb{D})$ and $\varphi(w) \neq 0$. Consider the functions Q and h defined as follows:

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta zq'(z)}{q^2(z)} = \frac{\beta z}{2(1+z)^{\frac{3}{2}}} \tag{2.27}$$

and

$$h(z) := \nu(q(z)) + Q(z) = q(z) + Q(z). \tag{2.28}$$

Since $z/(1-z)^{2-2\alpha} \in S^*(\alpha)$, the function Q is starlike in $\mathbb{D}$. Using (2.27) and (2.28), we get

$$\frac{zh'(z)}{Q(z)} = \frac{1}{\beta} q^2(z) + \frac{zQ'(z)}{Q(z)}$$

which further gives

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > \frac{1}{4} > 0.$$

Hence, h is univalent in $\mathbb{D}$. If the subordination (2.25) holds, then from Lemma 1.2, it follows that $p(z) \prec q(z)$.

(b) We now show that (2.26) holds. Proceeding as in Theorem 2.3 and by using the definition of h given in (2.26), the subordination $\varphi_C(z) \prec h(z)$ holds if for $t \in [-\pi, \pi]$, the following condition holds

$$\left| \sqrt{-2 + 6\sqrt{1+e^{it}} + \frac{3\beta e^{it}}{(1+e^{it})^{\frac{3}{2}}}} - 2 \right| > 2. \tag{2.29}$$

Set

$$w = u + iv = -2 + 6\sqrt{1+e^{it}} + \frac{3\beta e^{it}}{(1+e^{it})^{\frac{3}{2}}}$$

so that

$$u = -2 + 6\sqrt{2\cos(t/2)}\cos(t/4) + 3\beta\cos(t/4)(2\cos(t/2))^{-\frac{3}{2}} \tag{2.30}$$

and

$$v = 6\sqrt{2\cos(t/2)}\sin(t/4) + 3\beta\sin(t/4)(2\cos(t/2))^{-\frac{3}{2}}. \tag{2.31}$$

Proceeding as in Theorem 2.3, the condition (2.29) holds if (2.14) holds. After using (2.30) and (2.31) in (2.14), we get

$$\begin{aligned} g(t) := -64\Big(\Big(-2+\frac{3\beta\cos(t/4)}{2\sqrt{2}(\cos(t/2))^{\frac{3}{2}}}+6\sqrt{2}\cos(t/4)\sqrt{\cos(t/2)}\Big)^2 \\ +\Big(\frac{3\beta\sin(t/4)}{2\sqrt{2}(\cos(t/2))^{\frac{3}{2}}}+6\sqrt{2}\sqrt{\cos(t/2)}\sin(t/4)\Big)^2\Big) \\ +\Big(-8\big(-2+\frac{3\beta\cos(t/4)}{2\sqrt{2}(\cos(t/2))^{\frac{3}{2}}}+6\sqrt{2}\cos(t/4)\sqrt{\cos(t/2)}\Big) \\ +\Big(-2+\frac{3\beta\cos(t/4)}{2\sqrt{2}(\cos(t/2))^{\frac{3}{2}}}+6\sqrt{2}\cos(t/4)\sqrt{\cos(t/2)}\Big)^2 \\ +\Big(\frac{3\beta\sin(t/4)}{2\sqrt{2}(\cos(t/2))^{\frac{3}{2}}}+6\sqrt{2}\sqrt{\cos(t/2)}\sin(t/4)\Big)^2\Big)^2>0. \end{aligned}$$

Since $g(t)$ is an even function of t, we will consider $g(t)$ for $t\in[0,\pi]$. It can be easily seen that the function $g(t)$ attains its minimum value at $t=0$. A simple calculation shows that

$$g(0)=\frac{3}{256}(8(-3+\sqrt{2})+\sqrt{2}\beta)(-8+24\sqrt{2}+3\sqrt{2}\beta)^3. \tag{2.32}$$

The relation $\beta>4(-2+3\sqrt{2})$ gives $8(-3+\sqrt{2})+\sqrt{2}\beta>0$ and $-8+24\sqrt{2}+3\sqrt{2}\beta>0$ so that (2.32) yields $g(0)>0$. Hence, we conclude that $g(t)>0$ for $t\in[0,\pi]$. □

By taking $p(z)=zf'(z)/f(z)$ in Theorems 2.3, 2.4, and 2.5, we obtain the following example.

Example 2.6. Let $f\in\mathcal{A}$. Then the following are sufficient conditions for $f\in\mathcal{S}^*_L$.

(1) Let $-1/(2\sqrt{2}-1)\le\alpha\le1$ and $\beta>-2(2-3\sqrt{2}-2\alpha+2\sqrt{2}\alpha)$. The function f satisfies the subordination

$$\left((1-\alpha)+(\alpha-\beta)\frac{zf'(z)}{f(z)}\right)\frac{zf'(z)}{f(z)}+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi_C(z).$$

(2) Let $0\le\alpha\le1$ and $\beta>4(3-\sqrt{2}+(\sqrt{2}-2)\alpha)$. The function f satisfies the subordination

$$\left(1-\alpha-\beta+\alpha\frac{zf'(z)}{f(z)}\right)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\varphi_C(z).$$

(3) Let $\beta>4(-2+3\sqrt{2})$. The function f satisfies the subordination

$$\frac{zf'(z)}{f(z)}-\beta+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)/\left(\frac{zf'(z)}{f(z)}\right)\prec\varphi_C(z).$$

By taking $p(z) = f'(z)$ in Theorems 2.3, 2.4, and 2.5 respectively, we obtain the following example.

Example 2.7. Let $f \in \mathcal{A}$. Then the following are sufficient conditions for $f'(z) \prec \sqrt{1+z}$.

(1) Let $-1/(2\sqrt{2}-1) \leq \alpha \leq 1$ and $\beta > -2(2-3\sqrt{2}-2\alpha+2\sqrt{2}\alpha)$. The function f satisfies the subordination
$$(1-\alpha)f'(z) + \alpha(f'(z))^2 + \beta z f''(z) \prec \varphi_C(z).$$

(2) Let $0 \leq \alpha \leq 1$ and $\beta > 4(3-\sqrt{2}+(\sqrt{2}-2)\alpha)$. The function f satisfies the subordination
$$(1-\alpha)f'(z) + \alpha(f'(z))^2 + \beta \frac{zf''(z)}{f'(z)} \prec \varphi_C(z).$$

(3) Let $\beta > 4(-2+3\sqrt{2})$. The function f satisfies the subordination
$$f'(z) + \beta \frac{zf''(z)}{(f'(z))^2} \prec \varphi_C(z).$$

In the following theorem, condition on β is obtained so that $1+\beta z p'(z) \in \Omega_C$ implies $p(z) \in \Omega_P$, where p is an analytic function in $\mathbb{D}$ with $p(0) = 1$.

Theorem 2.8. *Let $\beta \in \mathbb{R}$ satisfying $\beta < -2\pi$. If the function p is analytic in $\mathbb{D}$ with $p(0) = 1$ satisfies*
$$1 + \beta z p'(z) \prec \varphi_C(z)$$
then $p(z) \prec \varphi_{PAR}(z)$, where the function $\varphi_{PAR}(z)$ is defined by (1.1).

Proof. Define the function $q : \mathbb{D} \to \mathbb{C}$ as $q(z) = \varphi_{PAR}(z)$ with $q(0) = 1$. Let us define $\varphi(w) = \beta$ and $Q(z) = zq'(z)\varphi(q(z)) = \beta z q'(z)$. Since q is the convex univalent function, Q is starlike in $\mathbb{D}$. It follows from Lemma 1.1 that the subordination
$$1 + \beta z p'(z) \prec 1 + \beta z q'(z)$$
implies $p(z) \prec q(z)$. The theorem is proved by showing that

$$\begin{aligned} \varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} &\prec 1 + \beta z q'(z) \\ &= 1 - \frac{4\beta}{\pi^2} \frac{\sqrt{z}}{z-1} \log \frac{1+\sqrt{z}}{1-\sqrt{z}} =: h(z). \end{aligned} \tag{2.33}$$

Proceeding as in Theorem 2.3 and by using the definition of h given in (2.33), the subordination $\varphi_C(z) \prec h(z)$ holds if for $t \in [-\pi, \pi]$, the following condition holds

$$\left| \sqrt{4 - \frac{24\beta}{\pi^2} \frac{e^{it/2}}{e^{it}-1} \log \frac{1+e^{it/2}}{1-e^{it/2}}} - 2 \right| > 2. \tag{2.34}$$

Set

$$w = u + iv = 4 - \frac{24\beta}{\pi^2}\frac{e^{it/2}}{e^{it}-1}\log\frac{1+e^{it/2}}{1-e^{it/2}} = 4 + \frac{12\beta i}{\pi^2}\csc\frac{t}{2}\log\left(i\cot\frac{t}{4}\right).$$

Clearly,

$$u = 4 - \frac{1}{\pi^2}\left(12\beta\csc\frac{t}{2}\arg\left(i\cot\frac{t}{4}\right)\right) \tag{2.35}$$

and

$$v = \frac{1}{\pi^2}\left(12\beta\csc\frac{t}{2}\log\left|\cot\frac{t}{4}\right|\right). \tag{2.36}$$

Proceeding as in Theorem 2.3, the condition (2.34) holds if (2.14) holds. Substituting the values of u and v given by (2.35) and (2.36) respectively in (2.14), we get

$$\begin{aligned} f(t) := &-16\pi^4((\pi^2 - 3\beta\arg(i\cot(t/4))\csc(t/2))^2 \\ &+ 9\beta^2\csc^2(t/2)(\log|\cot(t/4)|)^2) + \csc^4(t/2)(\pi^4(-1+\cos t) \\ &+ 18\beta^2((\arg(i\cot(t/4)))^2 + (\log|\cot(t/4)|)^2))^2 > 0. \end{aligned} \tag{2.37}$$

Note that $f(t)$ is an even function of t so we will take $t \in [0,\pi]$. Since for $t \in [0,\pi]$, we have $\arg(i\cot(t/4)) = \pi/2$ and $\log|\cot(t/4)| = \log\cot(t/4)$, the condition (2.37) further reduces to

$$\begin{aligned} f(t) = &-16\pi^4((\pi^2 - 3\beta(\pi/2)\csc(t/2))^2 + 9\beta^2\csc^2(t/2)(\log(\cot(t/4)))^2) \\ &+ \csc^4(t/2)(\pi^4(-1+\cos t) + (9/2)\beta^2(\pi^2 + 4(\log(\cot(t/4)))^2))^2 > 0. \end{aligned}$$

It can be easily verified that f is decreasing function of t. The relation $\beta < -2\pi$ implies $2\pi - 3\beta > 0$ and $2\pi + \beta < 0$ so that $f(\pi) = -3\pi^4(2\pi-3\beta)^3(2\pi+\beta)/4 > 0$. Therefore, we conclude that $f(t) > 0$ for $t \in [0,\pi]$. □

We close this section by obtaining the conditions on β so that $p(z) \in \Omega_P$, whenever $1 + \beta zp'(z) \in \Omega_L$.

Theorem 2.9. *Let p be an analytic function defined on $\mathbb{D}$ and $p(0) = 1$. Let $|\beta - \pi| > \sqrt{2}\pi$. If the function p satisfies the subordination*

$$1 + \beta zp'(z) \prec \sqrt{1+z},$$

then the function p satisfies the subordination

$$p(z) \prec \varphi_{PAR}(z)$$

where the function $\varphi_{PAR}(z)$ is defined by (1.1).

Proof. Let q be the convex univalent function $\varphi_{PAR}(z)$ defined by (1.1). Proceeding as in Theorem 2.8, the result is proved by showing that

$$\sqrt{1+z} \prec 1 + \beta zq'(z) = 1 - \frac{4\beta}{\pi^2}\frac{\sqrt{z}}{z-1}\log\frac{1+\sqrt{z}}{1-\sqrt{z}} =: h(z). \tag{2.38}$$

Set $\psi(z)=\sqrt{1+z}$. The subordination $\psi(z)\prec h(z)$ holds if $\partial h(\mathbb{D})\subset\mathbb{C}\backslash\overline{\psi(\mathbb{D})}=\{w\in\mathbb{C}:|w^2-1|>1\}$. For $t\in[-\pi,\pi]$, let

$$\begin{aligned} w=u+iv=h(e^{it}) &= 1-\frac{4\beta}{\pi^2}\frac{e^{it/2}}{e^{it}-1}\log\frac{1+e^{it/2}}{1-e^{it/2}} \\ &= 1+\frac{2\beta i}{\pi^2}\csc\frac{t}{2}\log\left(i\cot\frac{t}{4}\right). \end{aligned} \tag{2.39}$$

The subordination $\psi(z)\prec h(z)$ holds if $|h^2(e^{it})-1|>1$ which holds if

$$u^2+v^2-2>0. \tag{2.40}$$

From (2.39), we get

$$u=1-\frac{2\beta}{\pi^2}\csc(t/2)\arg(i\cot(t/4)) \quad \text{and} \quad v=\frac{2\beta}{\pi^2}\csc(t/2)\log|\cot(t/4)|.$$

After substituting these values of u and v in (2.40), we get

$$\begin{aligned} f(t) &:= -2+\big(1-\frac{2\beta}{\pi^2}\arg(i\cot(t/4))\csc(t/2)\big)^2 \\ &\quad +\frac{4\beta^2}{\pi^4}\csc^2(t/2)(\log|\cot(t/4)|)^2>0. \end{aligned} \tag{2.41}$$

Since $f(t)=f(-t)$, we will consider $t\in[0,\pi]$. Therefore, the condition (2.41) further reduces to

$$f(t):=-2+(-1+\frac{\beta}{\pi}\csc(t/2))^2+\frac{4\beta^2}{\pi^4}\csc^2(t/2)(\log(\cot(t/4)))^2>0.$$

Note that

$$\begin{aligned} f'(t)=\frac{\beta\csc^3(t/2)}{2\pi^4}\Big(&\pi^3\sin t-\beta\big(8\log(\cot(t/4)) \\ &+2\cos(t/2)(\pi^2+4(\log(\cot(t/4)))^2)\big)\Big). \end{aligned}$$

Since $f'(t)<0$ and $f(\pi)=-2+(-1+\beta/\pi)^2>0$, we conclude that $f(t)>0$ for $t\in[0,\pi]$. □

As applications of Theorems 2.8 and 2.9, we have the following examples.

Example 2.10. Let $f\in\mathcal{A}$. Then the following are sufficient conditions for $f\in\mathcal{S}_P$.

(1) Let $\beta<-2\pi$. The function f satisfies the subordination

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec\varphi_C(z).$$

(2) Let $|\beta-\pi|>\sqrt{2}\pi$. The function f satisfies the subordination

$$1+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec\sqrt{1+z}.$$

Example 2.11. Let $f \in \mathcal{A}$. Then the following are sufficient conditions for $f'(z) \prec \varphi_{PAR}(z)$, where the function $\varphi_{PAR}(z)$ is given by (1.1).

(1) Let $\beta < -2\pi$. The function f satisfies the subordination

$$1 + \beta z f''(z) \prec \varphi_C(z).$$

(2) Let $|\beta - \pi| > \sqrt{2}\pi$. The function f satisfies the subordination

$$1 + \beta z f''(z) \prec \sqrt{1+z}.$$

Acknowledgments

The authors are thankful to Lee See Keong for his comments on an earlier draft of this paper. The work presented here was supported by a grant from University of Delhi.

References

[1] R. M. Ali, N. E. Cho, N. Jain and V. Ravichandran, Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination, *Filomat* **26** (2012), no. 3, 553–561.

[2] R. M. Ali, N. E. Cho, V. Ravichandran and S. Sivaprasad Kumar, Differential subordination for functions associated with the lemniscate of Bernoulli, *Taiwanese J. Math.* **16** (2012), no. 3, 1017–1026.

[3] R. M. Ali, N. K. Jain and V. Ravichandran, Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane, *Appl. Math. Comput.* **218** (2012), no. 11, 6557–6565.

[4] A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.* **56** (1991), no. 1, 87–92.

[5] W. Janowski, Some extremal problems for certain families of analytic functions, I, *Ann. Polon. Math.* **28** (1973) 297–326.

[6] W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.* **23** (1970/1971) 159–177.

[7] S. Sivaprasad Kumar, V. Kumar, V. Ravichandran and N. E. Cho, Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli, *J. Inequal. Appl.* **2013** (2013) 13 pages.

[8] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal., I Int. Press, Cambridge, 1994.

[9] R. Mendiratta, S. Nagpal and V. Ravichandran, A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli, *Internat. J. Math.* **25** (2014), no. 9, 17 pages.

[10] R. Mendiratta, S. Nagpal and V. Ravichandran, On a subclass of strongly starlike functions associated with exponential function, *Bull. Malays. Math. Sci. Soc.* **38** (2015), no. 1, 365–386.

[11] S. S. Miller and P. T. Mocanu, Differential Subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[12] E. Paprocki and J. Sokół, The extremal problems in some subclass of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* **20** (1996) 89–94.

[13] Y. Polatoğlu and M. Bolcal, Some radius problem for certain families of analytic functions, *Turkish J. Math.* **24** (2000), no. 4, 401–412.

[14] V. Ravichandran, F. Rønning and T. N. Shanmugam, Radius of convexity and radius of starlikeness for some classes of analytic functions, *Complex Variables Theory Appl.* **33** (1997), no. 1-4, 265–280.

[15] V. Ravichandran and K. Sharma, Sufficient conditions for starlikeness, *J. Korean Math. Soc.* **52** (2015), no. 4, 727–749.

[16] M. S. Robertson, Certain classes of starlike functions, *Michigan Math. J.* **32** (1985), no. 2, 135–140.

[17] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.* **118** (1993), no. 1, 189–196.

[18] T. N. Shanmugam, Convolution and differential subordination, *Internat. J. Math. Math. Sci.* **12** (1989), no. 2, 333–340.

[19] T. N. Shanmugam and V. Ravichandran, Certain properties of uniformly convex functions, Computational Methods And Function Theory 1994 (Penang), 319–324, Ser. Approx. Decompos., 5 World Sci. Publ., River Edge, 1995.

[20] K. Sharma, N. K. Jain and V. Ravichandran, Starlike functions associated with a cardioid, *Afr. Mat.* (2015) DOI: 10.1007/s13370-015-0387-7.

[21] J. Sokół, Coefficient estimates in a class of strongly starlike functions, *Kyungpook Math. J.* **49** (2009), no. 2, 349–353.

[22] J. Sokół, Radius problems in the class $\mathcal{SL}$, *Appl. Math. Comput.* **214** (2009), no. 2, 569–573.

[23] J. Sokółand J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, *Zeszyty Nauk. Politech. Rzeszowskiej Mat.* **19** (1996) 101–105.

(K. Sharma) Department of Mathematics, Atma Ram Sanatan Dharma College, University of Delhi, Delhi 110021, India.

E-mail address: kanika.divika@gmail.com

(V. Ravichandran) Department of Mathematics, University of Delhi, Delhi–110007, India.

E-mail address: vravi@maths.du.ac.in;vravi68@gmail.com