Title:
Applications of subordination theory to starlike functions

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APPLICATIONS OF SUBORDINATION THEORY TO STARLIKE FUNCTIONS

K. SHARMA AND V. RAVICHANDRAN

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Abstract. Let \( p \) be an analytic function defined on the open unit disc \( D \) with \( p(0) = 1 \). The conditions on \( \alpha \) and \( \beta \) are derived for \( p(z) \) to be subordinate to \( 1 + 4z/3 + 2z^2/3 =: \phi_C(z) \) when \( (1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) \) is subordinate to \( e^z \). Similar problems were investigated for \( p(z) \) to lie in a region bounded by lemniscate of Bernoulli \( |w^2 - 1| = 1 \) when the functions \( (1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) \) or \( p(z) + \beta z p'(z)/p^2(z) \) are subordinates to \( \phi_C(z) \). Related results for \( p \) to be in the parabolic region bounded by the Re \( w = |w - 1| \) are investigated.

Keywords: convex and starlike functions, cardioid, parabolic starlike, lemniscate of Bernoulli, subordination.

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1. Introduction

Let \( \mathcal{A} \) be the class of all functions \( f \) analytic in the unit disc \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). Let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of univalent functions. For an analytic function \( \varphi \) with positive real part in \( D \) with \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \), let

\[
\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \varphi(z) \right\}
\]

and

\[
\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \right\}.
\]

These classes unify various classes of starlike and convex functions. Shanmugam [18] studied the convolution properties of these classes when \( \varphi \) is convex while Ma and Minda [8] investigated the growth, distortion and coefficient estimates under less restrictive assumption that \( \varphi \) is starlike and \( \varphi(D) \) is symmetric with
The class \( C(1.1) \)

is the function \( C \) and \( S \) then functions, introduced by Rnning [17]. The corresponding class \( S : = S^*(0) \) and \( C : = C(0) \) are the classes of starlike and convex functions respectively. If the function \( \varphi_{PAR} : \mathbb{D} \to \mathbb{C} \) is given by

\[
\varphi_{PAR}(z) : = 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{\frac{z}{1 - z}}} \right)^2, \quad \text{Im} \sqrt{z} \geq 0
\]

then \( \varphi_{PAR}(\mathbb{D}) = \{ w = u + iv : v^2 < 2u - 1 \} = \{ w : \text{Re} w > |w - 1| \} = : \Omega_p \).

The class \( C(\varphi_{PAR}) \) is the class of uniformly convex functions introduced by Goodman [4]. The corresponding class \( S_p : = S^*(\varphi_{PAR}) \) of parabolic starlike functions, introduced by Ronning [17], consists of functions \( f \in \mathbb{A} \) satisfying

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{D}.
\]

Sokól and Stankiewicz [23] have introduced and studied the class \( S_L^* = S^*(\sqrt{1 + z}) \);

the class \( S^*_L \) consists of functions \( f \in \mathbb{A} \) such that \( zf'(z)/f(z) \) lies in the region bounded by the right-half of the lemniscate of Bernoulli given by \( \Omega_L : = \{ w \in \mathbb{C} : |w^2 - 1| < 1 \} \). There has been several works [1, 3, 5, 14, 19, 21, 22] related to these classes. Similarly, the class \( S_C^* : = S^*(\varphi_C) \), where \( \varphi_C(z) = 1 + 4z/3 + 2z^2/3 \) was introduced and studied recently in [15, 20]. Precisely, \( f \in S_C^* \) provided \( zf'(z)/f(z) \) lies in the region bounded by the cardioid

\[
\Omega_C : = \{ w = u + iv : (9u^2 + 9v^2 - 18u + 5)^2 - 16(9u^2 + 9v^2 - 6u + 1) = 0 \}.
\]

Another class \( S^*_C : = S^*(e^z) \), introduced recently by Mendiratta et al. [10], consists of functions \( f \in \mathbb{A} \) satisfying the condition \( |\log (zf'(z)/f(z))| < 1 \).

A convex function is starlike of order 1/2; analytically,

\[
p(z) + zp'(z)/p(z) < (1 + z)/(1 - z) \implies p(z) < 1/(1 - z).
\]

Similarly, a sufficient condition for a function \( p \) to be a function with positive real part is that \( p(z) + zp'(z)/p(z) < \mathcal{R}(z) \), where \( \mathcal{R} \) is the open door mapping given by

\[
\mathcal{R}(z) : = \frac{1 + z}{1 - z} + \frac{2z}{1 - z^2}.
\]

Several authors have investigated similar results for functions to belong to certain regions in right half plane. For example, Ali et al. [2] determined the condition on \( \beta \) for \( p(z) < \sqrt{1 + z} \) when \( 1 + \beta zp'(z)/p^n(z) \) with \( n = 0, 1, 2 \) or \( (1 - \beta)p(z) + \beta p^2(z) + \beta zp'(z) \) is subordinate to \( \sqrt{1 + z} \). For related results, see [1–3, 7, 12, 22]. We investigate a similar problem for regions that were considered recently by many authors, including parabolic and lemniscate regions associated with the classes \( S_P \) and \( S^*_L \), respectively. Precisely we determine conditions on \( \alpha \) and \( \beta \) so that \( p(z) < \varphi_C(z) \) when \( (1 - \alpha)p(z) + \beta p^2(z) + \beta zp'(z) \) is subordinate to \( \sqrt{1 + z} \). For related results, see [1–3, 7, 12, 22].
\[ \alpha p^2(z) + \beta z p'(z)/p(z) < e^z. \]

Conditions on \( \alpha \) and \( \beta \) are also determined so that \((1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) \) or \((1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) \) or \( p(z) + \beta z p'(z)/p^2(z) < \varphi_C(z) \) implies \( p(z) < \sqrt{1 + z} \). We also find condition on \( \beta \) so that \( 1 + \beta z p'(z) \) is subordinate to \( \varphi_C(z) \) or \( \sqrt{1 + z} \) implies \( p(z) < \varphi_{PAR}(z) \).

Our results yield several sufficient conditions for \( f \in A \) to belong to the class \( S_P, S^*_C \) or \( S^*_L \).

We need the following lemmas to prove our results.

**Lemma 1.1.** [11, Corollary 3.4h, p.135] Let \( q \) be univalent in \( \mathbb{D} \), and let \( \varphi \) be analytic in a domain \( D \) containing \( q(\mathbb{D}) \). Let \( zq'(z)\varphi(q(z)) \) be starlike. If \( p \) is analytic in \( \mathbb{D} \), \( p(0) = q(0) \) and \( zp'(z)\varphi(p(z)) < zq'(z)\varphi(q(z)) \), then \( p < q \) and \( q \) is the best dominant.

**Lemma 1.2.** [11, Theorem 3.4i, p.134] Let \( q \) be univalent in \( \mathbb{D} \) and let \( \varphi \) and \( \nu \) be analytic in a domain \( D \) containing \( q(\mathbb{D}) \) with \( \varphi(w) \neq 0 \) when \( w \in q(\mathbb{D}) \). Set \( Q(z) := zq'(z)\varphi(q(z)), \ h(z) := \nu(q(z)) + Q(z) \). Suppose that (i) either \( h \) is convex or \( Q(z) \) is starlike univalent in \( \mathbb{D} \) and (ii) \( \text{Re}(zh'(z)/Q(z)) > 0 \) for \( z \in \mathbb{D} \). Let \( p \) be analytic in \( \mathbb{D} \) with \( p(0) = q(0) \) and \( p(\mathbb{D}) \subset \mathbb{D} \). If \( p \) satisfies

\[
(1.2) \quad \nu(p(z)) + zp'(z)\varphi(p(z)) < \nu(q(z)) + zq'(z)\varphi(q(z)),
\]

then \( p < q \) and \( q \) is the best dominant.

2. Results associated with starlikeness

Let \( p \) be an analytic function in \( \mathbb{D} \) with \( p(0) = 1 \). In the first result, we find the conditions on \( \alpha \) and \( \beta \) so that \( p(z) \in \Omega_C \), whenever \((1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) < e^z\).

**Theorem 2.1.** Let the function \( p \) be analytic in \( \mathbb{D} \) with \( p(0) = 1 \). Let \( \alpha, \beta \in \mathbb{R} \) such that either (i) \( 3(e - 3)/(2e) < \alpha < (e - 3)/6, \beta > 9(e - 3 - 6\alpha)/8, \) or (ii) \( (e - 3)/6 \leq \alpha \leq 0, \beta > 0 \) holds. If the function \( p \) satisfies

\[
(1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) < e^z
\]

then \( p(z) < \varphi_C(z) \).

**Proof.** The function \( q : \mathbb{D} \to \mathbb{C} \) defined by \( q(z) = \varphi_C(z) = 1 + 2(z + z^2/2)/3 \) is univalent in \( \mathbb{D} \). Let \( h : \mathbb{D} \to \mathbb{C} \) be defined by

\[
h(z) := (1 - \alpha)q(z) + \alpha q^2(z) + \beta z q'(z)
\]

(2.1) \[ = (1 - \alpha) \left( 1 + \frac{4z}{3} + \frac{2z^2}{3} \right) + \alpha \left( 1 + \frac{4z}{3} + \frac{2z^2}{3} \right)^2 + \frac{4\beta z(1 + z)}{3 + 4z + 2z^2}. \]

The proof is by showing that (a)

\[
(2.2) \quad (1 - \alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) < (1 - \alpha)q(z) + \alpha q^2(z) + \beta z q'(z)/q(z)
\]
implies that \( p(z) \prec q(z) \) and (b) the subordination \( \psi(z) := e^z \prec h(z) \) holds.

(a) The subordination (2.2) is the same as (1.2) if we define the functions \( \nu, \varphi \) by \( \nu(w) = (1 - \alpha)w + \alpha w^2 \) and \( \varphi(w) = \beta/w \). The function \( \nu \) is analytic in \( \mathbb{C} \). Since \( \beta > 0 \), \( \varphi \) is analytic in \( \mathbb{C} \setminus \{0\} \) and \( \varphi(w) \neq 0 \). Consider the functions \( Q \) and \( h \) defined as follows:

\[
Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta z^q(z)}{q(z)} = \frac{4\beta z(1 + z)}{3 + 4z + 2z^2},
\]

and

\[
h(z) = \nu(q(z)) + Q(z) = (1 - \alpha)q(z) + \alpha q^2(z) + Q(z).
\]

The equation (2.3) gives

\[
\frac{zQ'(z)}{Q(z)} = \frac{z}{1 + z} + \frac{3 - 2z^2}{3 + 4z + 2z^2} =: K(z).
\]

Substituting \( x = \cos t \) (\( t \in [-\pi, \pi] \)), we have

\[
\text{Re}(K(e^{it})) = \frac{1}{2} + \frac{5 + 4\cos t}{29 + 40\cos t + 12\cos 2t} = \frac{1}{2} + \frac{5 + 4x}{24x^2 + 40x + 17} \geq \frac{11}{18} > 0.
\]

This together with the minimum principle for harmonic functions shows that the function \( Q \) is starlike univalent in \( \mathbb{D} \). Using (2.3) and (2.4), we get

\[
\frac{zh'(z)}{Q(z)} = \frac{1 - \alpha}{\beta} q(z) + \frac{2\alpha}{\beta} q^2(z) + \frac{zQ'(z)}{Q(z)} = M(z) + K(z),
\]

where

\[
M(z) = \left( (1 - \alpha)/\beta \right) q(z) + (2\alpha/\beta)q^2(z).
\]

We show that \( \text{Re}(zh'(z)/Q(z)) > 0, z \in \mathbb{D} \) when \( \alpha, \beta \in \mathbb{R} \) satisfy the conditions in (i) or (ii) in the hypothesis. For \( t \in [-\pi, \pi] \), we have

\[
\text{Re}(M(e^{it})) = (9 + 9\alpha + 12(1 + 3\alpha)\cos t + (6 + 50\alpha)\cos 2t + 32\alpha \cos 3t + 8\alpha \cos 4t)/9\beta =: H(\cos t).
\]

We need to prove that \( H(x) \geq 0 \) in the interval \(-1 \leq x \leq 1\) for cases (i) and (ii), where

\[
H(1) = (3 - 33\alpha + 132(1 - 5\alpha)x + 12(1 + 3\alpha)x^2 + 128\alpha x^3 + 64\alpha x^4)/9\beta.
\]

Since, \( H(1) = (3 + 15\alpha)/\beta \) and \( H(-1) = (3 - \alpha)/9\beta \), \( H(1) \) and \( H(-1) \) both are non-negative for \(-1/5 \leq \alpha \leq 3, \beta > 0 \). A calculation shows that \( H'(x) = 0 \) if

\[
x = x_0 = -\frac{1}{2} - \frac{1152\alpha - 5760\alpha^2}{4608(2\alpha(16\alpha^3 + \sqrt{2}\alpha^3 - 15\alpha^4 + 75\alpha^5 + 3\alpha^6))^3} - \frac{(16\alpha^3 + \sqrt{2}\alpha^3 - 15\alpha^4 + 75\alpha^5 + 3\alpha^6)^3}{4(2\alpha)}
\]

and

\[
H''(x) = (768x\alpha + 8(-16 + 96x^2)\alpha + 4(6 + 50\alpha))/9\beta.
\]
Therefore, $H(x) \geq \min(H(1), H(-1)) \geq 0$ for $-1 \leq x \leq 1$. This shows that $	ext{Re}(s h'(z)/Q(z)) > 0, z \in \mathbb{D}$ and therefore, $h(z) - 1$ is close-to-convex function and hence univalent in $\mathbb{D}$. If the subordination (2.2) holds, Lemma 1.2 shows that $p(z) \prec q(z)$.

(b) We now show that $\psi(z) := e^z \prec h(z)$ holds. The subordination $\psi(z) \prec h(z)$ holds if $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D}) = \{ w \in \mathbb{C} : |\log w| > 1 \}$, where $t \in [-\pi, \pi]$. Then, the inequality $|\log w| > 1$ reduces to

$$f(\theta) := (\log(u^2 + v^2))^2 + 4(\text{arg}(u + iv))^2 - 4 > 0.$$  

By definition of $h$ given in (2.1), we get

$$u = \frac{1}{9(29 + 40 \cos t + 12 \cos 2t)} \left(537 + 372 \alpha + 216 \beta \right. $$

$$+ 4(225 + 239 \alpha + 81 \beta) \cos t + 2(261 + 611 \alpha + 54 \beta) \cos 2t$$

$$+ 96(2 + 11 \alpha) \cos 3t + 4(9 + 142 \alpha) \cos 4t$$

$$+ 176 \alpha \cos 5t + 24 \alpha \cos 6t \right)$$

and

$$v = \frac{1}{9(29 + 40 \cos t + 12 \cos 2t)} \left(4(147 + 511 \alpha + 45 \beta) \right. $$

$$+ (225 + 907 \alpha + 54 \beta) \cos t + 8(12 + 77 \alpha) \cos 2t$$

$$+ 2(9 + 148 \alpha) \cos 3t + 88 \alpha \cos 4t + 12 \alpha \cos 5t \sin t \right).$$

Since $f(t)$ is an even function of $t$, it is enough to show that $f(t) > 0$ for $t \in [0, \pi]$. It can be easily verified that for both the cases (i) and (ii), the function $f(t)$ attains its minimum value either at $t = 0$ or $t = \pi$. So, we need to show that both $f(0)$ and $f(\pi)$ are positive in either cases. Note that

$$f(0) = -4 + 4(\text{arg}(27 + 54 \alpha + 8 \beta))^2 + (\log((27 + 54 \alpha + 8 \beta)^2/81))^2$$

and

$$f(\pi) = -4 + 4(\text{arg}(3 - 2 \alpha))^2 + (\log((3 - 2 \alpha)^2/81))^2.$$  

For the case (i), the relation $\beta > 9(e - 3 - 6a)/8$ gives $27 + 54 \alpha + 8 \beta > 9e$ so that $\text{arg}(27 + 54 \alpha + 8 \beta) = 0$ and $(\log((27 + 54 \alpha + 8 \beta)^2/81))^2 > (2 \log e)^2 = 4$. Thus, the use of (2.6) yields $f(0) > 0$. The conditions $\alpha < (e - 3)/6$ and $3(e - 3)/(2e) < \alpha$ lead to $3 - 2 \alpha > 4 - e/3 > 0$ and $(3 - 2 \alpha)^2/81 < 1/e^2$ respectively which further implies that $\text{arg}(3 - 2 \alpha) = 0$ and $(\log((3 - 2 \alpha)^2/81))^2 > 4$ respectively. Hence, by using (2.7), we get $f(\pi) > 0$.

For the case (ii), the condition $(e - 3)/6 \leq \alpha$ gives $27 + 54 \alpha + 8 \beta > 8 \beta + 9e > 9e$. So, proceeding as in the case (i), we get $f(0) > 0$. Using the fact that $\alpha \leq 0$, we get $3 - 2 \alpha > 0$ and hence $\text{arg}(3 - 2 \alpha) = 0$. Observe that
\( \alpha \geq (e - 3)/6 > 3(e - 3)/(2e) \). Thus, again proceeding as in the case (i), we get \( f(\pi) > 0 \). This completes the proof. □

By taking \( p(z) = zf'(z)/f(z), p(z) = z^2f'(z)/f^2(z) \) and \( p(z) = f'(z) \), the above theorem gives the following:

**Example 2.2.** Let \( \alpha, \beta \in \mathbb{R} \) such that either (i) \( 3(e - 3)/(2e) < \alpha < (e - 3)/6 \), \( \beta > 9(e - 3 - 6\alpha)/8 \), or (ii) \( (e - 3)/6 \leq \alpha \leq 0, \beta > 0 \) holds.

1. If the function \( f \in \mathcal{A} \) satisfies the subordination
   \[
   (1 - \alpha - \beta) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf'(z)}{f(z)} \right)^2 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec e^z
   \]
   then \( f \in S^\alpha_\Gamma \).

2. If the function \( f \in \mathcal{A} \) satisfies the subordination
   \[
   \left( (1 - \alpha) + \alpha \frac{zf'(z)}{f^2(z)} \right) z^2f'(z) + \beta \left( \frac{zf(z)}{f'(z)} \right) - \frac{2zf'(z)}{f(z)} \prec e^z
   \]
   then \( z^2f'(z)/f^2(z) \prec \varphi_C(z) \).

3. If the function \( f \in \mathcal{A} \) satisfies the subordination
   \[
   \left( (1 - \alpha) + \alpha f'(z) \right) f'(z) + \beta \frac{zf''(z)}{f'(z)} \prec e^z
   \]
   then \( f'(z) \prec \varphi_C(z) \).

In the next two theorems, we compute the conditions on \( \beta \) so that \( p(z) \in \Omega_L \), whenever
\[
(1 - \alpha)p(z) + \alpha p^2(z) + \beta zp'(z) \quad \text{or} \quad (1 - \alpha)p(z) + \alpha p^2(z) + \beta \frac{zp'(z)}{p(z)} \in \Omega_C,
\]
where \( p \) is an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \).

**Theorem 2.3.** Let \( \alpha, \beta \in \mathbb{R} \) satisfying \(-1/(2\sqrt{2} - 1) \leq \alpha \leq 1 \) and \( \beta > -2(2 - 3\sqrt{2} - 2\alpha + 2\sqrt{2}\alpha) \). If the function \( p \) is analytic in \( \mathbb{D} \) with \( p(0) = 1 \) and satisfies \( (1 - \alpha)p(z) + \alpha p^2(z) + \beta zp'(z) \prec \varphi_C(z) \) then \( p(z) \prec \sqrt{1 + z} \).

**Proof.** Let \( q \) be the convex univalent function defined by \( q(z) = \sqrt{1 + z} \). Then it is clear that \( \beta zq'(z) \) is starlike. We will prove the result by showing that (a)

\[
(1 - \alpha)p(z) + \alpha p^2(z) + \beta zp'(z) \prec (1 - \alpha)q(z) + \alpha q^2(z) + \beta zq'(z)
\]

implies that \( p(z) \prec q(z) \) and (b)

\[
\varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec (1 - \alpha)q(z) + \alpha q^2(z) + \beta zq'(z)
\]

\[
= (1 - \alpha)\sqrt{1 + z} + \alpha(1 + z) + \frac{\beta z}{2\sqrt{1 + z}} =: h(z).
\]
(a) To prove (2.8), define \( \nu(w) = (1 - \alpha)w + \alpha w^2 \) and \( \varphi(w) = \beta \). The function \( \nu \) is analytic in \( \mathbb{C} \). Since \( \beta > 0 \), \( \varphi \) is analytic in \( \mathbb{C} \setminus \{0\} \) and \( \varphi(w) \neq 0 \). The function \( Q \) defined by

\[
Q(z) := zq'(z)\varphi(q(z)) = \beta zq'(z) = \frac{\beta z}{2\sqrt{1 + z}}
\]
is starlike of order 3/4 and for the function \( h \) defined by

\[
h(z) := \nu(q(z)) + Q(z) = (1 - \alpha)q(z) + \alpha q^2(z) + Q(z),
\]
we have

\[
\frac{zh'(z)}{Q(z)} = \frac{1 - \alpha}{\beta} + \frac{2\alpha}{\beta} q(z) + \frac{zQ'(z)}{Q(z)}.
\]

Using the fact that \( 0 < \text{Re} q(z) < \sqrt{2}, z \in \mathbb{D} \), we have the following two cases:

Case 1: \( 0 \leq \alpha \leq 1 \). In this case, we have

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > \frac{1 - \alpha}{\beta} + \frac{3}{4} > 0.
\]

Case 2: \( -1/(2\sqrt{2} - 1) \leq \alpha < 0 \). In this case, we have

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > \frac{1 - \alpha}{\beta} + \frac{2\sqrt{2} \alpha}{\beta} + \frac{3}{4} > 0.
\]

This shows that \( \text{Re}(zh'(z)/Q(z)) > 0, z \in \mathbb{D} \) and therefore, \( h(z) - 1 \) is close-to-convex function and hence univalent in \( \mathbb{D} \). If the subordination (2.8) holds, Lemma 1.2 shows that \( p(z) \prec q(z) \).

(b) We now show that (2.9) holds. Clearly,

\[
\varphi_C(\mathbb{D}) = \{ w \in \mathbb{C} : | - 2 + \sqrt{6w - 2}| < 2 \}.
\]
The subordination \( \varphi_C(z) \prec h(z) \) holds if \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \varphi_C(\mathbb{D}) \). Thus, by using the definition of \( h \) as given in (2.9), the subordination \( \varphi_C(z) \prec h(z) \) holds if for \( t \in [-\pi, \pi] \), we have

\[
(2.12) \quad \sqrt{-2 + 6(1 - \alpha)\sqrt{1 + e^{\iota t}} + 6\alpha(1 + e^{\iota t}) + \frac{3\beta e^{\iota t}}{\sqrt{1 + e^{\iota t}}} - 2} > 2.
\]

By writing

\[
(2.13) \quad w = -2 + 6(1 - \alpha)\sqrt{1 + e^{\iota t}} + 6\alpha(1 + e^{\iota t}) + 3\beta e^{\iota t}(1 + e^{\iota t})^{-\frac{1}{2}},
\]
we see that the condition (2.12) holds if \( |\sqrt{w} - 2| > 2 \) or equivalently if \( |w| > 4 \text{Re}(\sqrt{w}) \). On further simplification after substituting \( w = u + iv \), (2.12) holds if

\[
(2.14) \quad (u^2 + v^2 - 8u)^2 - 64(u^2 + v^2) > 0.
\]
Using (2.13), we get
\[ u = -2 + 6(1 - \alpha) \sqrt{2} \cos(t/2) \cos(t/4) + 6\alpha (1 + \cos t) + 3\beta \cos(3t/4)(2 \cos(t/2))^{-\frac{1}{2}} \]
and
\[ v = 6(1 - \alpha) \sqrt{2} \cos(t/2) \sin(t/4) + 6\alpha \sin t \]
\[ + 3\beta \sin(3t/4)(2 \cos(t/2))^{-\frac{1}{2}}. \]
Using (2.15) and (2.16) in (2.14), we get
\[ g(t) := -64 \left( (-2 + 6\sqrt{2}(1 - \alpha) \cos(t/4) \sqrt{\cos(t/2)} + \frac{3\beta \cos(3t/4)}{\sqrt{2} \cos(t/2)}) \right. \\
+ 6\alpha (1 + \cos t)^2 + (6\sqrt{2}(1 - \alpha) \sqrt{\cos(t/2)} \sin(t/4) + \frac{3\beta \cos(3t/4)}{\sqrt{2} \cos(t/2)}) \\
\left. + 6\alpha \sin^2 t + 3\beta \cos(3t/4)(2 \cos(t/2))^{-\frac{1}{2}} + 6\alpha (1 + \cos t) \right) \\
\left. + 6\alpha (1 + \cos t)^2 + (6\sqrt{2}(1 - \alpha) \sqrt{\cos(t/2)} \sin(t/4) + \frac{3\beta \cos(3t/4)}{\sqrt{2} \cos(t/2)}) \\
+ 6\alpha \sin^2 t + 3\beta \sin(3t/4)(2 \cos(t/2))^{-\frac{1}{2}} + 6\alpha \sin t \right)^2 > 0. \]
Observe that \( g(t) = g(-t) \) for all \( t \in [-\pi, \pi] \) and \( g(t) \) attains its minimum value at \( t = 0 \). A calculation shows that
\[ g(0) = \frac{3}{16} (4 - 12\sqrt{2} + 12(-2 + \sqrt{2})\alpha - 3\sqrt{2}\beta)^3 \\
\times (12 - 4\sqrt{2} + 4(-2 + \sqrt{2})\alpha - \sqrt{2}\beta). \]
Note that the condition \( \beta > -2(2 - 3\sqrt{2} - 2\alpha + 2\sqrt{2}) \) is equivalent to \( 12 - 4\sqrt{2} + 4(-2 + \sqrt{2})\alpha - \sqrt{2}\beta < 0 \) and \( 4 - 12\sqrt{2} + 12(-2 + \sqrt{2})\alpha - 3\sqrt{2}\beta < 0 \). Thus, the use of (2.17) yields \( g(0) > 0 \) which implies that \( g(t) > 0 \) for all \( t \in [0, \pi] \). Hence the result follows.

**Theorem 2.4.** Let \( \alpha, \beta \in \mathbb{R} \) satisfying \( 0 \leq \alpha \leq 1 \) and \( \beta > 4(3 - \sqrt{2} + (\sqrt{2} - 2)\alpha) \). If \( p \) is an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \) satisfying
\[ (1 - \alpha)p(z) + \alpha p^2(z) + \beta \frac{p'(z)}{p(z)} \prec \varphi_C(z) \]
then \( p(z) \prec \sqrt{1 + z} \).
Proof. Define the function \( q : \mathbb{D} \rightarrow \mathbb{C} \) by \( q(z) = \sqrt{1 + z} \). Proceeding as in Theorem 2.3, the result is proved by showing that (a)

\[
(1 - \alpha)p(z) + \alpha q^2(z) + \frac{\beta z q'(z)}{q(z)} \prec (1 - \alpha)q(z) + \alpha q^2(z) + \frac{\beta z q'(z)}{q(z)}
\]

implies that \( p(z) \prec q(z) \) and (b)

\[
\varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec (1 - \alpha)q(z) + \alpha q^2(z) + \frac{\beta z q'(z)}{q(z)}
\]

\[
= (1 - \alpha)\sqrt{1 + z} + \alpha(1 + z) + \frac{\beta z}{2(1 + z)} =: h(z).
\]

(a) Let us define \( \nu(w) = (1 - \alpha)w + \alpha w^2 \) and \( \varphi(w) = \beta/w \). Clearly \( \beta > 0 \).

The functions \( \nu \) and \( \varphi \) are analytic in \( \mathbb{C} \setminus \{0\} \) which includes \( q(\mathbb{D}) \) and \( \varphi(w) \neq 0 \).

Next, define the functions \( Q \) and \( h \) by

\[
Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1 + z)}
\]

and

\[
h(z) := \nu(q(z)) + Q(z) = (1 - \alpha)q(z) + \alpha q^2(z) + Q(z).
\]

Since \( Q \) is a Möbius transformation, the function \( Q \) is convex. Further using (2.20) and (2.21), we get

\[
\frac{zh'(z)}{Q(z)} = \frac{1 - \alpha}{\beta} q(z) + \frac{2\alpha}{\beta} q^2(z) + \frac{z Q'(z)}{Q(z)}.
\]

Since \( 0 < \text{Re} q(z) < \sqrt{2} \) and \( 0 < \text{Re} q^2(z) < 2 \), \( z \in \mathbb{D} \), we have

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > \sqrt{2} \left( \frac{1 - \alpha}{\beta} \right) + \frac{4\alpha}{\beta} > 0.
\]

Therefore, \( h(z) - 1 \) is close-to-convex function and hence univalent in \( \mathbb{D} \). If the subordination (2.18) holds, Lemma 1.2 shows that \( p(z) \prec q(z) \).

(b) We now claim that (2.19) holds. Note that

\[
\varphi_C(\mathbb{D}) = \{ w \in \mathbb{C} : | -2 + \sqrt{6w - 2}| < 2 \}.
\]

The subordination \( \varphi_C(z) \prec h(z) \) holds if \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \varphi_C(\mathbb{D}) \). Using the definition of \( h \) given in (2.19), the subordination \( \varphi_C(z) \prec h(z) \) holds if for \( t \in [-\pi, \pi] \), the following condition holds

\[
(2.22) \quad \sqrt{-2 + 6(1 - \alpha)\sqrt{1 + e^{it}} + 6\alpha(1 + e^{it}) + \frac{3\beta e^{it}}{1 + e^{it}} - 2} > 2.
\]

Let

\[
(2.23) \quad w = u + iv = -2 + 6(1 - \alpha)\sqrt{1 + e^{it}} + 6\alpha(1 + e^{it}) + \frac{3\beta e^{it}}{1 + e^{it}}.
\]
Proceeding as in Theorem 2.3, the condition (2.22) holds if (2.14) holds. From (2.23), we get
\[ u = -2 + 6(1 - \alpha)\sqrt{2}\cos(t/2)\cos(t/4) + 6\alpha(1 + \cos t) + \frac{3\beta}{2} \]
and
\[ v = 6(1 - \alpha)\sqrt{2}\cos(t/2)\sin(t/4) + 6\alpha \sin t + \frac{3\beta}{2} \tan t/2. \]
Using these above expressions for \( u \) and \( v \), the condition (2.14) takes the following form
\[
k(t) := -64\left( (-2 + (3\beta)/2 + 6\sqrt{2}(1 - \alpha)\cos(t/4)\sqrt{\cos(t/2)} \\
+ 6\alpha(1 + \cos t)^2 + (6\sqrt{2}(1 - \alpha)\cos(t/2)\sin(t/4) + 6\alpha \sin t \\
+ (3/2)\beta \tan(t/2))^2 \right) + \left( -8(-2 + (3\beta)/2 \\
+ 6\sqrt{2}(1 - \alpha)\cos(t/4)\sqrt{\cos(t/2)} + 6\alpha(1 + \cos t) + (-2 + (3\beta)/2 \\
+ 6\sqrt{2}(1 - \alpha)\cos(t/2)\sin(t/4) + 6\alpha \sin t + (3/2)\beta \tan(t/2))^2 \right)^2 \]
\[ > 0. \]
Note that \( k(t) = k(-t) \), so it is enough to show that \( k(t) > 0 \) for \( t \in [0, \pi] \).
Also note that \( k(t) \) is an increasing function of \( t \). A calculation shows that
\[
k(0) = \frac{3}{16}(4 - 12\sqrt{2} + 12(-2 + \sqrt{2})\alpha - 3\beta)^3 \\
\times (12 - 4\sqrt{2} + 4(-2 + \sqrt{2})\alpha - \beta). \tag{2.24}\]
Consider the given relation \( \beta > 4(3 - \sqrt{2} - 2\alpha + \sqrt{2}\alpha) \) which is same as
\( 12 - 4\sqrt{2} + 4(-2 + \sqrt{2})\alpha - \beta < 0 \) and \( 4 - 12\sqrt{2} + 12(-2 + \sqrt{2})\alpha - 3\beta < 0 \). By using (2.24), we get \( k(0) \) is positive which implies that \( k(t) \) is positive for all \( t \in [0, \pi] \). This completes the proof. \[ \square \]

Next result depicts the condition on \( \beta \) so that \( p(z) \in \Omega_L \), whenever \( p(z) + \beta zp'(z)/p^2(z) \in \Omega_C \).

**Theorem 2.5.** Let \( \beta \in \mathbb{R} \) satisfying \( \beta > 4(-2 + 3\sqrt{2}) \). If \( p \) is an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \) satisfying
\[
p(z) + \beta zp'(z)/p^2(z) < \varphi_C(z) \]
then \( p(z) < \sqrt{1 + z} \).
Proof. Define the function \( q : \mathbb{D} \rightarrow \mathbb{C} \) by \( q(z) = \sqrt{1+z} \). Proceeding as in Theorem 2.3, we will prove the result by showing that (a)

\[
(2.25) \quad p(z) + \frac{\beta z p'(z)}{p^2(z)} < q(z) + \frac{\beta z q'(z)}{q^2(z)}
\]

implies that \( p(z) < q(z) \) and (b)

\[
(2.26) \quad \varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} < q(z) + \frac{\beta z q'(z)}{q^2(z)}
\]

\[
= \sqrt{1+z} + \frac{\beta z}{2(1+z)^{\frac{3}{2}}} =: h(z).
\]

(a) The subordination (2.25) is same as (1.2) if we define \( \nu(w) = w \) and \( \varphi(w) = \beta/w^2 \). Clearly, the functions \( \nu \) and \( \varphi \) are analytic in \( \mathbb{C} \setminus \{0\} \) which includes \( q(\mathbb{D}) \) and \( \varphi(w) \neq 0 \). Consider the functions \( Q \) and \( h \) defined as follows:

\[
(2.27) \quad Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta z q'(z)}{q^2(z)} = \frac{\beta z}{2(1+z)^{\frac{3}{2}}}
\]

and

\[
(2.28) \quad h(z) := \nu(q(z)) + Q(z) = q(z) + Q(z).
\]

Since \( z/(1-z)^{2-2\alpha} \in S^*(\alpha) \), the function \( Q \) is starlike in \( \mathbb{D} \). Using (2.27) and (2.28), we get

\[
\frac{zh'(z)}{Q(z)} = \frac{1}{\beta} q^2(z) + \frac{zQ'(z)}{Q(z)}
\]

which further gives

\[
\text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > \frac{1}{4} \quad > 0.
\]

Hence, \( h \) is univalent in \( \mathbb{D} \). If the subordination (2.25) holds, then from Lemma 1.2, it follows that \( p(z) < q(z) \).

(b) We now show that (2.26) holds. Proceeding as in Theorem 2.3 and by using the definition of \( h \) given in (2.26), the subordination \( \varphi_C(z) < h(z) \) holds if for \( t \in [-\pi, \pi] \), the following condition holds

\[
(2.29) \quad \left| \sqrt{-2 + 6\sqrt{1+e^{it}} + \frac{3\beta e^{it}}{(1+e^{it})^{\frac{3}{2}}} - 2} \right| > 2.
\]

Set

\[
w = u + iv = -2 + 6\sqrt{1+e^{it}} + \frac{3\beta e^{it}}{(1+e^{it})^{\frac{3}{2}}}
\]

so that

\[
(2.30) \quad u = -2 + 6\sqrt{2\cos(t/2)\cos(t/4)} + 3\beta \cos(t/4)(2\cos(t/2))^{-\frac{3}{2}}
\]

and

\[
(2.31) \quad v = 6\sqrt{2\cos(t/2)\sin(t/4)} + 3\beta \sin(t/4)(2\cos(t/2))^{-\frac{3}{2}}.
\]
Proceeding as in Theorem 2.3, the condition (2.29) holds if (2.14) holds. After using (2.30) and (2.31) in (2.14), we get
\[
g(t) := -64 \left( -2 + \frac{3\beta \cos(t/4)}{2\sqrt{2}(\cos(t/2))^{3/2}} + 6\sqrt{2} \cos(t/4) \sqrt{\cos(t/2)} \right)^2
\]
\[
+ \left( \frac{3\beta \sin(t/4)}{2\sqrt{2}(\cos(t/2))^{3/2}} + 6\sqrt{2} \cos(t/4) \sin(t/4) \right)^2
\]
\[
+ \left( -8 \left( -2 + \frac{3\beta \cos(t/4)}{2\sqrt{2}(\cos(t/2))^{3/2}} + 6\sqrt{2} \cos(t/4) \sqrt{\cos(t/2)} \right)^2
\]
\[
+ \left( \frac{3\beta \sin(t/4)}{2\sqrt{2}(\cos(t/2))^{3/2}} + 6\sqrt{2} \cos(t/4) \sin(t/4) \right)^2 \right)^2 > 0.
\]
Since \(g(t)\) is an even function of \(t\), we will consider \(g(t)\) for \(t \in [0, \pi]\). It can be easily seen that the function \(g(t)\) attains its minimum value at \(t = 0\). A simple calculation shows that
\[
(2.32) \quad g(0) = \frac{3}{256} (8(-3 + \sqrt{2}) + \sqrt{2}\beta)(-8 + 24\sqrt{2} + 3\sqrt{2}\beta)^3.
\]
The relation \(\beta > 4(-2 + 3\sqrt{2})\) gives \(8(-3 + \sqrt{2}) + \sqrt{2}\beta > 0\) and \(-8 + 24\sqrt{2} + 3\sqrt{2}\beta > 0\) so that (2.32) yields \(g(0) > 0\). Hence, we conclude that \(g(t) > 0\) for \(t \in [0, \pi]\).\[\square\]

By taking \(p(z) = zf'(z)/f(z)\) in Theorems 2.3, 2.4, and 2.5, we obtain the following example.

**Example 2.6.** Let \(f \in \mathcal{A}\). Then the following are sufficient conditions for \(f \in \mathcal{S}_L^1\):

1. Let \(-1/(2\sqrt{2} - 1) \leq \alpha \leq 1\) and \(\beta > -2(2 - 3\sqrt{2} - 2\alpha + 2\sqrt{2}\alpha)\). The function \(f\) satisfies the subordination
\[
\left( 1 - \alpha + (\alpha - \beta) \frac{zf'(z)}{f(z)} \right) \frac{zf'(z)}{f(z)} + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec \varphi_C(z).
\]

2. Let \(0 \leq \alpha \leq 1\) and \(\beta > 4(3 - \sqrt{2} + (\sqrt{2} - 2)\alpha)\). The function \(f\) satisfies the subordination
\[
\left( 1 - \alpha - \beta + \alpha \frac{zf'(z)}{f(z)} \right) \frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec \varphi_C(z).
\]

3. Let \(\beta > 4(-2 + 3\sqrt{2})\). The function \(f\) satisfies the subordination
\[
\frac{zf'(z)}{f(z)} - \beta + \beta \left( 1 + \frac{zf''(z)}{f'(z)} \right) / \left( \frac{zf'(z)}{f(z)} \right) \prec \varphi_C(z).
\]
By taking \( p(z) = f'(z) \) in Theorems 2.3, 2.4, and 2.5 respectively, we obtain the following example.

**Example 2.7.** Let \( f \in \mathcal{A} \). Then the following are sufficient conditions for \( f'(z) \prec \sqrt{1 + z} \).

1. Let \(-1/(2\sqrt{2} - 1) \leq \alpha \leq 1 \) and \( \beta > -2(2 - 3\sqrt{2} - 2\alpha + 2\sqrt{2}\alpha) \). The function \( f \) satisfies the subordination
   \[ (1 - \alpha)f'(z) + \alpha(f'(z))^2 + \beta zf''(z) < \varphi_C(z). \]

2. Let \( 0 \leq \alpha \leq 1 \) and \( \beta > 4(3 - \sqrt{2} + (\sqrt{2} - 2)\alpha) \). The function \( f \) satisfies the subordination
   \[ (1 - \alpha)f'(z) + \alpha(f'(z))^2 + \frac{\beta zf''(z)}{f'(z)} < \varphi_C(z). \]

3. Let \( \beta > 4(-2 + 3\sqrt{2}) \). The function \( f \) satisfies the subordination
   \[ f'(z) + \beta \frac{zf''(z)}{(f'(z))^2} < \varphi_C(z). \]

In the following theorem, condition on \( \beta \) is obtained so that \( 1 + \beta zp'(z) \in \Omega_C \) implies \( p(z) \in \Omega_P \), where \( p \) is an analytic function in \( D \) with \( p(0) = 1 \).

**Theorem 2.8.** Let \( \beta \in \mathbb{R} \) satisfying \( \beta < -2\pi \). If the function \( p \) is analytic in \( D \) with \( p(0) = 1 \) satisfies

\[ 1 + \beta zp'(z) \prec \varphi_C(z) \]

then \( p(z) \prec \varphi_{\text{PAR}}(z) \), where the function \( \varphi_{\text{PAR}}(z) \) is defined by (1.1).

**Proof.** Define the function \( q : D \to \mathbb{C} \) as \( q(z) = \varphi_{\text{PAR}}(z) \) with \( q(0) = 1 \). Let us define \( \varphi(w) = \beta \) and \( Q(z) = zq'(z)\varphi(q(z)) = \beta zq'(z) \). Since \( q \) is the convex univalent function, \( Q \) is starlike in \( D \). It follows from Lemma 1.1 that the subordination

\[ 1 + \beta zp'(z) \prec 1 + \beta zq'(z) \]

implies \( p(z) \prec q(z) \). The theorem is proved by showing that

\[ \varphi_C(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta zq'(z) \]

\[ = 1 - \frac{4\beta}{\pi^2} \frac{\sqrt{z}}{z - 1} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} =: h(z). \]

Proceeding as in Theorem 2.3 and by using the definition of \( h \) given in (2.33), the subordination \( \varphi_C(z) \prec h(z) \) holds if for \( t \in [-\pi, \pi] \), the following condition holds

\[ \sqrt{4 - \frac{24\beta}{\pi^2} \frac{e^{it}}{1 - e^{it}}} \log \frac{1 + e^{it}}{1 - e^{it}} - 2 > 2. \]
Set
\[ w = u + iv = 4 - \frac{24\beta}{\pi^2} \frac{e^{i t/2}}{e^{i t} - 1} \log \frac{1 + e^{i t/2}}{1 - e^{i t/2}} = 4 + \frac{12\beta i}{\pi^2} \csc \frac{t}{2} \log \left( i \cot \frac{t}{4} \right). \]

Clearly,
\[ u = 4 - \frac{1}{\pi^2} \left( 12\beta \csc \frac{t}{2} \arg \left( i \cot \frac{t}{4} \right) \right) \]
and
\[ v = \frac{1}{\pi^2} \left( 12\beta \csc \frac{t}{2} \log |\cot \frac{t}{4}| \right). \]

Proceeding as in Theorem 2.3, the condition (2.34) holds if (2.14) holds. Substituting the values of \( u \) and \( v \) given by (2.35) and (2.36) respectively in (2.14), we get
\[ f(t) := -16\pi^4((\pi^2 - 3\beta \arg(i \cot(t/4)) \csc(t/2))^2 \]
\[ + 9\beta^2 \csc^2(t/2)(\log |\cot(t/4)|)^2 + \csc^4(t/2)(\pi^4(-1 + \cos t) \]
\[ + 18\beta^2(\arg(i \cot(t/4)))^2 + (\log |\cot(t/4)|)^2)^2 > 0. \]

Note that \( f(t) \) is an even function of \( t \) so we will take \( t \in [0, \pi] \). Since for \( t \in [0, \pi] \), we have \( \arg(i \cot(t/4)) = \pi/2 \) and \( \log |\cot(t/4)| = \log \cot(t/4) \), the condition (2.37) further reduces to
\[ f(t) = -16\pi^4((\pi^2 - 3\beta(\pi/2) \csc(t/2))^2 + 9\beta^2 \csc^2(t/2)(\log \cot(t/4))^2) \]
\[ + \csc^4(t/2)(\pi^4(-1 + \cos t) + (9/2)\beta^2(\pi^2 + 4(\log \cot(t/4))^2))^2 > 0. \]

It can be easily verified that \( f \) is decreasing function of \( t \). The relation \( \beta < -2\pi \) implies \( 2\pi - 3\beta > 0 \) and \( 2\pi + \beta < 0 \) so that \( f(\pi) = -3\pi^4(2\pi - 3\beta)^3(2\pi + \beta)/4 > 0 \). Therefore, we conclude that \( f(t) > 0 \) for \( t \in [0, \pi] \). \( \square \)

We close this section by obtaining the conditions on \( \beta \) so that \( p(z) \in \Omega_p \), whenever \( 1 + \beta z p'(z) \in \Omega_L \).

**Theorem 2.9.** Let \( p \) be an analytic function defined on \( \mathbb{D} \) and \( p(0) = 1 \). Let \( |\beta - \pi| > \sqrt{2}\pi \). If the function \( p \) satisfies the subordination
\[ 1 + \beta z p'(z) \prec \sqrt{1 + z}, \]
then the function \( p \) satisfies the subordination
\[ p(z) \prec \varphi_{PAR}(z) \]
where the function \( \varphi_{PAR}(z) \) is defined by (1.1).

**Proof.** Let \( q \) be the convex univalent function \( \varphi_{PAR}(z) \) defined by (1.1). Proceeding as in Theorem 2.8, the result is proved by showing that
\[ \sqrt{1 + z} < 1 + \beta z q'(z) = 1 - \frac{4\beta}{\pi^2} \frac{\sqrt{z}}{z - 1} \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} =: h(z). \]
Set $\psi(z) = \sqrt{1+z}$. The subordination $\psi(z) \prec h(z)$ holds if $\partial h(D) \subset C \setminus \psi(D) = \{ w \in \mathbb{C} : |w^2 - 1| > 1 \}$. For $t \in [-\pi, \pi]$, let

$$w = u + iv = h(e^{it}) = 1 - \frac{4\beta}{\pi^2} e^{it/2} \frac{1 + e^{it/2}}{1 - e^{it/2}} \log 1 + e^{it/2} = 1 + \frac{2\beta i}{\pi^2} \frac{t}{2} \log \left( i \cot \frac{t}{4} \right).$$

The subordination $\psi(z) \prec h(z)$ holds if $|h^2(e^{it}) - 1| > 1$ which holds if

$$u^2 + v^2 - 2 > 0.$$ 

From (2.39), we get

$$u = 1 - \frac{2\beta}{\pi^2} \csc(t/2) \arg(i \cot(t/4)) \quad \text{and} \quad v = \frac{2\beta}{\pi^2} \csc(t/2) \log |\cot(t/4)|.$$ 

After substituting these values of $u$ and $v$ in (2.40), we get

$$f(t) := -2 + \left( 1 - \frac{2\beta}{\pi^2} \arg(i \cot(t/4)) \csc(t/2) \right)^2 + \frac{4\beta^2}{\pi^4} \csc^2(t/2)(\log |\cot(t/4)|)^2 > 0.$$ 

Since $f(t) = f(-t)$, we will consider $t \in [0, \pi]$. Therefore, the condition (2.41) further reduces to

$$f(t) := -2 + (-1 + \frac{\beta}{\pi} \csc(t/2))^2 + \frac{4\beta^2}{\pi^4} \csc^2(t/2)(\log|\cot(t/4)|)^2 > 0.$$ 

Note that

$$f'(t) = \frac{\beta \csc^3(t/2)}{2\pi^4} \left[ \pi^3 \sin t - \beta(8 \log|\cot(t/4)|) ight]$$

$$+ 2 \cos(t/2)(\pi^2 + 4(\log|\cot(t/4)|)^2)).$$ 

Since $f'(t) < 0$ and $f(\pi) = -2 + (-1 + \beta/\pi)^2 > 0$, we conclude that $f(t) > 0$ for $t \in [0, \pi]$. □

As applications of Theorems 2.8 and 2.9, we have the following examples.

**Example 2.10.** Let $f \in \mathcal{A}$. Then the following are sufficient conditions for $f \in \mathcal{S}_p$.

1. Let $\beta < -2\pi$. The function $f$ satisfies the subordination

$$1 + \beta \frac{zf'''(z)}{f'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \varphi_C(z).$$

2. Let $|\beta - \pi| > \sqrt{2}\pi$. The function $f$ satisfies the subordination

$$1 + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \sqrt{1+z}.$$
Example 2.11. Let \( f \in A \). Then the following are sufficient conditions for \( f'(z) \prec \varphi_{PAR}(z) \), where the function \( \varphi_{PAR}(z) \) is given by (1.1).

1. Let \( \beta < -2\pi \). The function \( f \) satisfies the subordination
   \[
   1 + \beta zf''(z) \prec \varphi_C(z).
   \]

2. Let \( |\beta - \pi| > \sqrt{2}\pi \). The function \( f \) satisfies the subordination
   \[
   1 + \beta zf''(z) \prec \sqrt{1 + z}.
   \]

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