ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 3, pp. 779–798

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 3, pp. 779–798
Online ISSN: 1735-8515

WEAK F-CONTRACTIONS AND SOME FIXED POINT RESULTS

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(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. In this paper we define weak F-contractions on a metric space into itself by extending F-contractions introduced by D. Wardowski (2012) and provide some fixed point results in complete metric spaces and in partially ordered complete generalized metric spaces. Some relationships between weak F-contractions and φ-contractions are highlighted. We also give some applications on fractal theory improving the classical Hutchinson-Barnsley's theory of iterated function systems. Some illustrative examples are provided.
Keywords: F-contraction, partially ordered metric space, generalized metric, iterated function system, fixed point theorem.
MSC(2010): Primary: 47H10; Secondary 47H09, 54H25, 28A80.

1. Introduction

Banach's contraction principle is one of the pivotal results of analysis. It establishes that, given a mapping f on a complete metric space (X, d) into itself and a constant $\lambda \in (0,1)$ such that

$$\mathrm{d}\big(f(x), f(y)\big) \leq \lambda \mathrm{d}(x,y),\ \forall\, x, y \in X, \tag{1.1}$$

there exists a unique $\xi \in X$ such that $f(\xi) = \xi$ and $\xi = \lim_n f^n(x)$, for every $x \in X$, where f^n, $n \geq 1$, denotes the n-times composition of f. A function f that satisfies (1.1) is called *Banach contraction*.

This result is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in various branches of mathematics. Many authors have provided several extensions of this result. In this regard, J. Matkowski [9] gives an extension of Banach's contraction principle to φ-contractions, where φ is a comparison function (see Definition 2.1).

Those functions $f : X \to X$ that satisfy the inequality

$$\mathrm{d}\big(f(x), f(y)\big) < \mathrm{d}(x,y),\ \forall\, x, y \in X,\ x \neq y,$$

Article electronically published on June 29, 2016.
Received: 24 October 2014, Accepted: 29 April 2015.

called *contractive* maps, have been considered. Niemytzki-Edelstein's Theorem (see e.g. [5, Th. 2.2, p. 34]) states that each contractive self-mapping f on a compact metric space X into itself has a unique fixed point ξ and $\xi = \lim_n f^n(x)$, for all x in X. Some particular contractive mappings on a complete metric space are investigated in fixed point theory. In this respect, starting from a function $F : (0, \infty) \to \mathbb{R}$ satisfying some suitable properties (see Definition 3.1), D. Wardowski [20] provided a new type of contractive mapping, namely *F-contraction*, and proved a fixed point theorem for F-contractions on a complete metric space. He showed that any Banach contraction is a particular case of F-contraction while there are F-contractions which are not Banach contractions.

We say that a self-function f on a metric space X is a *Picard mapping* if it has a unique fixed point ξ and $\xi = \lim_n f^n(x)$, for every $x \in X$, (the concept of Picard operator was introduced by I.A. Rus, see [15, 16]).

In this paper we consider the family $\mathcal{F}_1$ of functions $F : (0, \infty) \to \mathbb{R}$ satisfying only (F1) and (F2) and defined *weak F-contractions*. In Proposition 3.4 we provide a large class of functions F that fulfil (F1) and (F2) and do not satisfy (F3) such that every weak F-contraction on a complete metric space is a Picard mapping. Some sufficient conditions in which weak F-contractions are Picard mappings are given in Theorem 3.5 and Theorem 3.9.

In Section 3.1 we establish that, if $F \in \mathcal{F}_1$ is continuous, then it is a φ-contraction for a suitable function φ (Theorem 3.9). Also, in Theorem 3.12 a sufficient condition for which a φ-contraction is a weak F-contraction is described.

Fixed point theory in partially ordered metric spaces is of relatively recent origin. An early result in this direction is due to M. Turinici [19], in which fixed point problems were studied in partially ordered uniform spaces. Later, this branch of fixed point theory has been developed through a number of works. T.G. Bhaskar and V. Lakshmikantham [6] provided a fixed point theorem for a mixed monotone mapping in a partially ordered metric space using a weak contractivity type of assumption. This is generalized by V. Lakshmikantham and L. Ćirić in [8] where coupled coincidence and coupled common fixed point theorems for mixed g-monotone mapping in partially ordered complete metric spaces are presented. Some remarkable fixed point theorems for generalized contractions in ordered metric spaces can be found in [12].

Existence of fixed point in partially ordered sets has been considered recently by A.C.M. Ran and M.C.B. Reurings in [13]. Some fixed point results in ordered L-spaces that generalize and extend a result from [13] are proved by A. Petruşel and I.A. Rus in [11]. In [10] the authors extended the theoretical results of fixed points in a partially ordered complete metric space for Banach contractions. Very recently, in [7] the fixed point theorem in partially ordered complete metric space given in [10] is extended to the case of partially ordered complete generalized metric spaces and in [14] some multidimensional fixed point theorems in partially ordered complete metric spaces are proved.

In Section 4 of the present paper we improve the above mentioned results considering weak F-contractions instead of Banach contractions and provide two fixed point theorems in partially complete metric space and, respectively, in partially ordered complete generalized metric space (Theorem 4.4, resp. Theorem 4.10).

We apply our results in Section 3.2 and, respectively, in Section 4.1 to obtain the existence and uniqueness of the attractors of some iterated function systems and, respectively, of countable iterated function systems composed by weak F-contractions on a complete metric space. Iterated function systems play a crucial role in fractal theory.

Some other illustrative examples are presented.

2. Preliminaries

We recall here some notions and results used in the sequel.

Throughout this paper the symbols $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ will denote the set of real numbers, positive real numbers and positive integers, respectively.

2.1. φ-contractions.

Definition 2.1. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is said to be a *comparison function* if it is monotone increasing (i.e. $t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$) and $\varphi^n(t) \to 0$ as $n \to \infty$, for every $t \ge 0$, where $\varphi^n = \varphi \circ \varphi^{n-1}$ means the n-times composition of φ. A self-mapping f on a metric space (X, d) is called *φ-contraction* whenever it satisfies the following inequality

$$\mathrm{d}\big(f(x), f(y)\big) \le \varphi\big(\mathrm{d}(x,y)\big),\ \forall\, x, y \in X.$$

Remark 2.2. If φ is a comparison function, then $\varphi(t) < t$ for every $t > 0$, $\varphi(0) = 0$ and φ is continuous at 0.

Remark 2.3. If $\varphi : [0,\infty) \to [0,\infty)$ is an increasing and continuous map with $\varphi(t) < t$, then $\varphi^n(t) \underset{n}{\longrightarrow} 0$, for every $t \ge 0$. Consequently φ is a comparison function.

Proof. If $t = 0$, then clearly $\varphi(0) = 0$. Choose $t > 0$. Then $\varphi(t) < t$, hence $\varphi^2(t) \le \varphi(t)$ and so on. Thus the sequence $\big(\varphi^n(t)\big)_n$ is decreasing, so it converges to some $l \ge 0$. If $l > 0$, then $\varphi(l) < l$ while, by continuity of φ, one has $\varphi(l) = l$. Hence $l = 0$. □

Theorem 2.4. (Matkowski, [9]) *Let (X, d) be a complete metric space, φ a comparison function and $f : X \to X$ a φ-contraction. Then f is a Picard mapping.*

2.2. **Hausdorff-Pompeiu metric.** Given a metric space (X, d), we denote by $\mathcal{P}^*(X)$ the collection of all nonempty subsets of X and consider the mappings $\mathrm{D}, h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to [0, \infty]$ defined by

(2.1)
$$D(B, C) = \sup_{x\in B} \inf_{y\in C} \mathrm{d}(x, y) \text{ and, respectively, } h(B, C) = \max\big\{\mathrm{D}(B, C), \mathrm{D}(C, B)\big\}.$$

The function h is called the *Hausdorff-Pompeiu pseudo metric.*

Lemma 2.5. [18, Prop. 1.1, Th. 1.13] *Whenever $B, C \in \mathcal{P}^*(X)$, one has the following properties:*
(*a*) $B \subset C \Rightarrow \mathrm{D}(B, C) = 0$*;*
(*b*) $\mathrm{D}(B, C) = \mathrm{D}(\overline{B}, \overline{C})$*;*
(*c*) $\mathrm{D}(B, C) = 0 \Rightarrow B \subset \overline{C}$*;*
(*d*) *if* $(E_i)_{i\in\Im}$, $(F_i)_{i\in\Im}$ *are two classes of nonempty subsets of* X*, then*

$$h\Big(\overline{\bigcup_{i\in\Im} E_i}, \overline{\bigcup_{i\in\Im} F_i}\Big) = h\Big(\bigcup_{i\in\Im} E_i, \bigcup_{i\in\Im} F_i\Big) = \sup_{i\in\Im} h(E_i, F_i),$$

where the bar means the closure of the respective set.

When we consider the family $\mathcal{K}(X)$ of all nonempty compact subsets of X instead of $\mathcal{P}^*(X)$ in (2.1), one obtains the *Hausdorff-Pompeiu metric.*

3. Weak F-contractions

We consider a new type of contractive mappings, namely F-contractions, defined by D. Wardowski [20] and prove that, under certain conditions, the fixed point theorem can be improved.

Definition 3.1. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping and consider the following three conditions:
(F1) F is strictly increasing, i.e. for all $t, s \in \mathbb{R}_+$, $t < s$, one has $F(t) < F(s)$;
(F2) for each sequence of positive numbers $(t_n)_n$, $\lim_n t_n = 0$ if and only if $\lim_n F(t_n) = -\infty$;
(F3) there exists $\lambda \in (0, 1)$ such that $\lim_{t\searrow 0} t^\lambda F(t) = 0$.
We denote by $\mathcal{F}_1$ the family of mappings $F : (0, \infty) \to \mathbb{R}$ satisfying Conditions (F1) and (F2) and by $\mathcal{F}$ the class of those functions in $\mathcal{F}_1$ which satisfy (F3).
A mapping $f : X \to X$ is said to be an F-*contraction*, where $F \in \mathcal{F}$, if

(3.1)
$$\exists\, \tau > 0 \text{ such that } \tau + F\big(\mathrm{d}\big(f(x), f(y)\big)\big) \le F\big(\mathrm{d}(x, y)\big),\ \forall\, x, y \in X,\ f(x) \ne f(y).$$

When (3.1) holds for $F \in \mathcal{F}_1$ we say that f is a *weak F-contraction.*

Theorem 3.2. [20, Th. 2.1] *Assume that (X, d) is a complete metric space, $F \in \mathcal{F}$ and $f : X \to X$ is an F-contraction. Then f is a Picard mapping.*

Remark 3.3. [20, Ex. 2.1] It is easy to verify that every weak F-contraction is a contractive map and every Banach contraction with ratio $r \in (0,1)$ is an F-contraction with $F(t) = \ln t$ and $\tau = -\ln r$.

The next proposition states that there is a class of functions $F \in \mathcal{F}_1$ for which the result claimed in Theorem 3.2 holds.

Proposition 3.4. *Let (X, d) be a complete metric space and consider $F : (0,\infty) \to \mathbb{R}$, $F(t) = -t^{-\alpha}$, where $\alpha > 0$, and a weak F-contraction $f : X \to X$. Then f has a unique fixed point which is approximated by the sequence $\big(f^n(x)\big)_n$, for all $x \in X$. Moreover, if $\alpha \geq 1$, then $F \notin \mathcal{F}$.*

Proof. Assume that $\alpha \geq 1$. Then $\lim_{t\to 0} t^{\lambda} F(t) = -\infty$, for every $\lambda \in (0,1)$, hence (F3) does not occur. When $\alpha \in (0,1)$, Condition (F3) is verified, so the result follows from Theorem 3.2. Therefore $F \in \mathcal{F}_1 \setminus \mathcal{F}$.

From hypothesis, there is $\tau > 0$ such that (3.1) holds. That is, for every $x, y \in X$ with $f(x) \neq f(y)$,

$$-\frac{1}{\big(\mathrm{d}(f(x), f(y))\big)^{\alpha}} \leq -\frac{1}{\big(\mathrm{d}(x,y)\big)^{\alpha}} - \tau \Leftrightarrow \frac{1 + \tau\big(\mathrm{d}(x,y)\big)^{\alpha}}{\big(\mathrm{d}(x,y)\big)^{\alpha}} \leq \frac{1}{\big(\mathrm{d}(f(x), f(y))\big)^{\alpha}}$$

$$\Leftrightarrow \mathrm{d}\big(f(x), f(y)\big) \leq \frac{\mathrm{d}(x,y)}{\Big(1 + \tau\big(\mathrm{d}(x,y)\big)^{\alpha}\Big)^{\frac{1}{\alpha}}} \Leftrightarrow \mathrm{d}\big(f(x), f(y)\big) \leq \varphi\big(\mathrm{d}(x,y)\big),$$

where $\varphi(t) = \frac{t}{\left(1+\tau t^{\alpha}\right)^{\frac{1}{\alpha}}}$. Clearly φ is monotonically increasing and $\varphi^n(t) = \frac{t}{\left(1+n\tau t^{\alpha}\right)^{\frac{1}{\alpha}}} \underset{n}{\longrightarrow} 0$, for all $t > 0$. Therefore φ is a comparison function, so f is a φ-contraction. The conclusion now follows from Theorem 2.4. □

In many cases Condition (F3) is difficult to fulfil by a function F and there are bounded functions f which satisfy (3.1). Thus, in the examples given in [20, Ex.2.5], [17, Ex.3.1] and [4, Ex.3.1] bounded functions f are considered and the space (or a part of it) is a discrete set in $\mathbb{R}$.

The following theorem describes a sufficient condition for a weak F-contraction on a complete metric space to be a Picard mapping.

Theorem 3.5. *Let (X, d) be a complete metric space, $F \in \mathcal{F}_1$ and $f : X \to X$ be a weak F-contraction. Assume that there exists $x_0 \in X$ such that the sequence $\big(f^n(x_0)\big)_n$ is bounded (in particular f is bounded). Then f has a unique fixed point ξ. Moreover $\xi = \lim_n f^n(x)$, for every $x \in X$.*

Proof. Since $F\big(\mathrm{d}(f(x), f(y))\big) \leq F\big(\mathrm{d}(x,y)\big) - \tau < F\big(\mathrm{d}(x,y)\big)$, for all $x, y \in X$ with $f(x) \neq f(y)$, one deduces that $\mathrm{d}(f(x), f(y)) < \mathrm{d}(x,y)$ hence f is continuous. Next, if $\xi \neq \xi' \in X$ are two different fixed points of f, then

$$\mathrm{d}(\xi, \xi') = \mathrm{d}\big(f(\xi), f(\xi')\big) < \mathrm{d}(\xi, \xi')$$

which is a contradiction. Therefore f has at most one fixed point.

We intend to check the existence of the fixed point. For every $n, p \geq 1$ we have, according to (3.1),

$$F\big(\mathrm{d}\big(f^n(x_0), f^{n+p}(x_0)\big)\big) \leq F\big(\mathrm{d}\big(f^{n-1}(x_0), f^{n+p-1}(x_0)\big)\big) - \tau \leq \dots$$
$$\leq F\big(\mathrm{d}\big(x_0, f^p(x_0)\big)\big) - n\tau \leq F(M) - n\tau \underset{n}{\longrightarrow} -\infty,$$

where $M = \sup_{p\geq 1} \mathrm{d}\big(x_0, f^p(x_0)\big)$. Consequently, from (F2), it follows that the sequence $\big(f^n(x_0)\big)_n$ is Cauchy, so, the space (X, d) being complete, there is $\xi \in X$ such that $\xi = \lim_n f^n(x_0)$. Now using the continuity of f, we get

$$f(\xi) = f\big(\lim_n f^n(x_0)\big) = \lim_n f^{n+1}(x_0) = \xi.$$

Next, choose $x \in X$. If there is $N \in \mathbb{N}$ such that $f^N(x) = f^N(x_0)$, then $f^n(x) = f^n(x_0)$ for all $n \geq N$. So $\lim_n f^n(x) = \lim_n f^n(x_0)$. Assume that $f^n(x) \neq f^n(x_0)$ for every $n \geq 1$. Then

$$\begin{aligned} F\big(\mathrm{d}\big(f^n(x), f^n(x_0)\big)\big) &\leq F\big(\mathrm{d}\big(f^{n-1}(x), f^{n-1}(x_0)\big)\big) - \tau \\ &\leq \dots \\ &\leq F\big(\mathrm{d}\big(x, x_0\big)\big) - n\tau \underset{n}{\longrightarrow} -\infty, \end{aligned}$$

therefore $\mathrm{d}\big(f^n(x), f^n(x_0)\big) \underset{n}{\longrightarrow} 0$. Accordingly, $\xi = \lim_n f^n(x_0) = \lim_n f^n(x)$.

The proof is complete. □

Example 3.6. Let consider the metric space (X, d), where $X = [1, \infty)$ and $\mathrm{d}(x, y) = |\ln x - \ln y|$, a function $f : X \to X$ given by $f(x) = \frac{\alpha}{x} + \beta$, where $\alpha > 0$, $\beta \geq 1$, and $F : \mathbb{R}_+ \to \mathbb{R}$, $F(t) = \frac{1}{1-e^t}$. Then the metric space (X, d) is complete and the following assertions hold:

(a) $F \in \mathcal{F}_1 \setminus \mathcal{F}$;
(b) f is a weak F-contraction;
(c) f is a Picard mapping.

Proof. The completeness of the metric space (X, d) is a standard fact and it is easy to check.

(a) Clearly F fulfil (F1) and (F2). Since $\lim_{t\searrow 0} t^\lambda F(t) = -\infty$ for all $\lambda \in (0, 1)$, one deduces that (F3) does not occur.

(b) We shall show that (3.1) is fulfilled for each $\tau \in (0, \beta\alpha^{-1}]$. We observe that f is strictly decreasing. Choose $x, y \in X$ with $f(x) \neq f(y)$, that is $x \neq y$, and suppose that, for example, $x < y$. One has

$$F\big(\mathrm{d}(x, y)\big) - F\big(\mathrm{d}\big(f(x), f(y)\big)\big) = F\big(\ln \frac{y}{x}\big) - F\Big(\ln \frac{f(x)}{f(y)}\Big) =$$

$$\frac{1}{1 - \frac{y}{x}} - \frac{1}{1 - \frac{y(\alpha+\beta x)}{x(\alpha+\beta y)}} = \frac{\beta}{\alpha} \cdot \frac{xy}{y - x} \geq \frac{\beta}{\alpha} \geq \tau.$$

(c) The assertion follows from Theorem 3.5 using the fact that $f(X) \subset (\beta, \alpha+\beta]$. The fixed point of f is $\xi = \frac{\beta+\sqrt{\beta^2+4\alpha}}{2}$. □

Example 3.7. We consider the complete metric space $X = [0,\infty)$ endowed with the Euclidean metric $\mathrm{d}(x,y) = |x-y|$ and a map $f : X \to X$ defined by $f(x) = \frac{x}{kx+1} + \alpha$, where $k > 0$, $\alpha \geq 0$. Let us consider further a function $F : \mathbb{R}_+ \to \mathbb{R}$, $F(t) = \ln t - \frac{1}{t}$, for every $t > 0$. Then

(a) f is not a Banach contraction;

(b) $F \in \mathcal{F}_1 \setminus \mathcal{F}$;

(c) f is a weak F-contraction;

(d) f is a Picard mapping.

Proof. First of all note that f is strictly increasing.

(a) For each $n = 1, 2, \ldots$, we denote $x_n = \frac{1}{2n}$, $y_n = \frac{1}{n}$. Then

$$\lim_n \frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \lim_n \frac{\frac{y_n}{ky_n+1} - \frac{x_n}{kx_n+1}}{y_n - x_n} = \lim_n \frac{1}{\left(\frac{k}{2n}+1\right)\left(\frac{k}{n}+1\right)} = 1.$$

Therefore, the inequality (1.1) does not occur for every $\lambda \in (0,1)$.

(b) Clearly F fulfills Conditions (F1) and (F2). Since $\lim_{t\searrow 0} t^\lambda F(t) = -\infty$ for all $\lambda \in (0,1)$, it follows that (F3) does not hold.

(c) We show that Condition (3.1) is verified for every $\tau \in (0,k]$.

Choose $x, y \in [0,\infty)$ such that $f(x) \neq f(y)$. This means that $x \neq y$. Assume, for instance, that $x < y$, the other case can be treated analogously. One has

$$\begin{aligned} &F\big(|x-y|\big) - F\big(|f(x) - f(y)|\big) \\ &\quad = \ln(y-x) - \frac{1}{y-x} - \ln\Big(\frac{y}{ky+1} - \frac{x}{kx+1}\Big) + \Big(\frac{y}{ky+1} - \frac{x}{kx+1}\Big)^{-1} \\ &\quad = \ln(kx+1)(ky+1) + \frac{(kx+1)(ky+1)-1}{y-x} \\ &\quad \geq \frac{k^2xy + kx + ky}{y-x} \\ &\quad \geq k \\ &\quad \geq \tau, \end{aligned}$$

and hence $\tau + F\big(|f(x) - f(y)|\big) \leq F\big(|x-y|\big)$, that is f is a weak F-contraction.

(d) The assertion follows immediately from Theorem 3.5 taking into account that $f(X) \subset [\alpha, k^{-1} + \alpha)$. The fixed point is $\xi = \frac{\alpha k + \sqrt{\alpha^2 k^2 + 4\alpha k}}{2k}$. □

By following a known result concerning Banach's contraction principle, we can improve Theorem 3.5.

Theorem 3.8. *Let (X, d) be a complete metric space, $F \in \mathcal{F}_1$ and $f : X \to X$ a function. Suppose that there is an integer $p \geq 1$ such that f^p is a weak F-contraction and a point $x_0 \in X$ such that the sequence $\big(f^n(x_0)\big)_n$ is bounded. Then f is a Picard mapping.*

Proof. By applying Theorem 3.5 to the function f^p, one obtains a unique $\xi \in X$ such that $f^p(\xi) = \xi$. Hence $f^{p+1}(\xi) = f(\xi)$. Therefore $f(\xi)$ is also a fixed point for f^p, so $f(\xi) = \xi$. If ξ' is another fixed point for f, then it is a fixed point for f^p. Thus $\xi = \xi'$.

In order to check the last part of the theorem, choose $n \in \mathbb{N}$. There exist $m \geq 1$ and $r \in \{0, 1, \ldots, p-1\}$ such that $n = pm + r$. Since $\big(f^k(x_0)\big)_{k\geq 1}$ is bounded, it follows that $\big(f^k\big(f^r(x_0)\big)\big)_k$ is bounded too. Now, from Theorem 3.5, we deduce that $f^n(x_0) = f^{pm}\big(f^r(x_0)\big) \underset{m}{\longrightarrow} \xi$.

If $x \in X$, using the same argument as in the proof of Theorem 3.5, we get

$$\xi = \lim_n f^n(x_0) = \lim_n f^n(x),$$

as required. □

3.1. **The relationship between weak F-contractions and φ-contractions.** As we have seen above, every Banach contraction is a particular case of F-contraction. Also, Proposition 3.4 describes a class of weak F-contractions which are φ-contractions for some suitable comparison functions. In the following we provide some sufficient conditions to have implications between above mentioned kinds of contractions.

Theorem 3.9. *If $F \in \mathcal{F}_1$ is a continuous function, then every weak F-contraction on a metric space to itself is a φ-contraction. Therefore, if the metric space is complete, then every weak F-contraction is a Picard mapping.*

Proof. Since F is continuous and satisfies (F2) it follows that $F(\mathbb{R}_+) = (-\infty, \alpha)$, where $\alpha \in \mathbb{R} \cup \infty$. Furthermore, because F fulfil (F1), one deduces that F is injective. Thus $F : \mathbb{R}_+ \to (-\infty, \alpha)$ is invertible.

Let f be a weak F-contraction and $\tau > 0$ from (3.1). We define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = F^{-1}\big(F(t) - \tau\big)$ for $t > 0$ and $\varphi(0) = 0$. Since, from (F2), $t_n \searrow 0$ if and only if $F(t_n) \underset{n}{\longrightarrow} -\infty$, we get $F^{-1}\big(F(t_n) - \tau\big) \underset{n}{\searrow} 0$, for every sequence of positive real numbers $(t_n)_n$ converging to 0.

Thus φ is well defined and, further, it is continuous. Since F and so F^{-1} are strictly increasing we deduce that φ is monotonically increasing. Next, $F(t) - \tau < F(t)$ implies $\varphi(t) < t$ for all $t > 0$. Consequently, according to Remark 2.3, φ is a comparison function.

In order to prove that f is a φ-contraction we apply the function F^{-1} to the inequality,

$$F\big(\mathrm{d}\big(f(x), f(y)\big)\big) \leq F\big(\mathrm{d}(x, y)\big) - \tau$$

and obtain,

$$d\big(f(x), f(y)\big) \le F^{-1}\Big(F\big(d(x,y)\big) - \tau\Big) = \varphi\big(d(x,y)\big), \ \forall x, y \in X,$$

as required.

The last assertion of statement now follows from Theorem 2.4. □

Remark 3.10. The previous theorem offers a sufficient condition under which a weak F-contraction is a Picard mapping concerning the function F while Theorem 3.5 describes a sufficient condition for f.

Remark 3.11. The assertions of Examples 3.6 and 3.7 can also be proved using Theorem 3.9.

Theorem 3.12. *Let (X, d) be a metric space and $f : X \to X$ a map.*

(i) Let $F \in \mathcal{F}_1$ and assume that f is a weak F-contraction and there exists a comparison function φ such that $\sup_{t>0}\Big(F(t) - F\big(\varphi(t)\big)\Big) \le \tau$, where τ is the constant from (3.1). Then f is a φ-contraction.

(ii) If f is a φ-contraction and there is $F \in \mathcal{F}_1$ such that $\inf_{t>0}\Big(F(t) - F\big(\varphi(t)\big)\Big) > 0$, then f is a weak F-contraction.

Proof. (α) By hypothesis we have, from every $x, y \in X$ with $f(x) \neq f(y)$,

$$\begin{aligned} F\Big(d\big(f(x), f(y)\big)\Big) &\le F\big(d(x,y)\big) - \tau \\ &\le F\big(d(x,y)\big) + F\Big(\varphi\big(d(x,y)\big)\Big) - F\big(d(x,y)\big) \\ &= F\Big(\varphi\big(d(x,y)\big)\Big). \end{aligned}$$

Hence, F being increasing, $d\big(f(x), f(y)\big) \le \varphi\big(d(x,y)\big)$ for all $x, y \in X$.

(β) For every $x, y \in X$ with $f(x) \neq f(y)$, one has

$$d\big(f(x), f(y)\big) \le \varphi\big(d(x,y)\big)$$

hence, from (F1) and the hypothesis,

$$F\Big(d\big(f(x), f(y)\big)\Big) \le F\Big(\varphi\big(d(x,y)\big)\Big) \le F\big(d(x,y)\big) - \tau,$$

where $\tau = \inf_{t>0}\Big(F(t) - F\big(\varphi(t)\big)\Big)$. This means that f is a weak F-contraction. □

3.2. **Application: weak F-iterated function systems.** We provide here a more complex example from fractal theory. We first need some preparations (see details in [2] and [17]). In the following (X, d) will be a complete metric space and F denotes an element in $\mathcal{F}_1$.

Definition 3.13. A family of maps $(f_k)_{k=1}^N$, $N \in \mathbb{N}$, is called a *weak F-iterated function system* (weak F-IFS) whenever $f_k : X \to X$ is a weak F-contraction, for every $k = 1, \ldots, N$. When $F \in \mathcal{F}$ we say that $(f_k)_{k=1}^N$ is an *F-iterated function system* (F-IFS).

The set function $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $\mathcal{S}(B) = \bigcup_{k=1}^{K} f_k(B)$ is called the associated *Hutchinson operator*. A set $A \in \mathcal{K}(X)$ is said to be an *attractor* of the IFS whenever $\mathcal{S}(A) = A$.

Lemma 3.14. [17, L.4.1] *Let $f : X \to X$ be a weak F-contraction. Then the mapping $A \mapsto f(A)$ is a weak F-contraction too from $\mathcal{K}(X)$ into itself.*

In the following example we adapt [17, Th. 4.1] to the case of weak F-IFS.

Theorem 3.15. *Let $(f_k)_{k=1}^N$ be a weak F-iterated function system. Suppose that*

(a) F is continuous, or

(b) there exists a nonempty compact set $K \subset X$ such that $f_k(K) \subset K$ for every $k = 1, \dots, N$.

Then $(f_k)_{k=1}^N$ has a unique attractor A and $A = \lim_n \mathcal{S}^n(B)$, for all $B \in \mathcal{K}(X)$, the limit being taken with respect to the Hausdorff-Pompeiu metric.

Proof. For each $k = 1, \dots, N$, we denote by τ_k the constant from (3.1) associated to f_k.

Let $B, C \in \mathcal{K}(X)$ be such that $h\big(\mathcal{S}(B), \mathcal{S}(C)\big) > 0$. Lemma 2.5 implies

$$0 < h\big(\mathcal{S}(B), \mathcal{S}(C)\big) \le \sup_{1 \le k \le N} h\big(f_k(B), f_k(C)\big) = h\big(f_{k_0}(B), f_{k_0}(C)\big),$$

for some $k_0 \in \{1, \dots, N\}$. Next, from Lemma 3.14 we get

$$\min_{1 \le k \le N} \tau_k + F\Big(h\big(\mathcal{S}(B), \mathcal{S}(B)\big)\Big) \le \tau_{k_0} + F\Big(h\big(\omega_{k_0}(A), \omega_{k_0}(B)\big)\Big) \le F\big(h(A, B)\big),$$

which assures that $\mathcal{S}$ is a weak F-contraction on the complete metric space $\big(\mathcal{K}(X), h\big)$ into itself.

If (a) holds, then the conclusion follows from Theorem 3.9.

Now, from $f_k(K) \subset K$, for every $k = 1, \dots, N$, we deduce that $\mathcal{S}(K) \subset K$, so $\mathcal{S}^n(K) \subset K$ for all $n \ge 1$. Therefore the sequence $\big(\mathcal{S}^n(K)\big)_n$ is bounded. Next we apply Theorem 3.5. $\square$

Example 3.16. Assume that $X = [0, \infty)$ is endowed with the Euclidean metric $\mathrm{d}(x, y) = |x - y|$ and, for each $k = 1, \dots, N$, let $f_k : X \to X$ be a map defined by $f_k(x) = \frac{x}{kx+1} + \alpha_k$, where $\alpha_k \ge 0$. Let consider further the function $F : \mathbb{R}_+ \to \mathbb{R}$, $F(t) = \ln t - \frac{1}{t}$, for every $t > 0$. Then $(f_k)_{k=1}^N$ is a weak F-IFS and it has a unique attractor which is approximated with respect to the Hausdorff-Pompeiu metric by the sequence $\big(\mathcal{S}^n(B)\big)_n$, for every $B \in \mathcal{K}(X)$.

Moreover $(f_k)_{k=1}^N$ is not a classical Hutchinson IFS.

Proof. From Example 3.7 we deduce that $F \in \mathcal{F}_1$ and all f_k, $k = 1, \dots, N$, are weak F-contractions which are not Banach contractions. Both Conditions (a) and (b) from Theorem 3.15 are satisfied. Indeed, for (b), one has $f_k(X) \subset [\alpha_k, \alpha_k + k^{-1}) \subset K$, where $K = [\min_k \alpha_k, \max_k \alpha_k + 1]$. Hence $f_k(K) \subset K$, for all k. The conclusion now follows from Theorem 3.15. $\square$

4. Weak F-contractions on partially ordered complete metric spaces

Definition 4.1. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called *comparable* if $x \preceq y$ or $y \preceq x$ holds. If d is a metric on X, then we say that $(X, \mathrm{d}, \preceq)$ is a partially ordered metric space.

In [10] Banach's contraction principle in a partially ordered complete metric spaces is investigated and a fixed point theorem is proved. In [1] the following extension of the above result to φ-contractions is given.

Theorem 4.2. [1, Th.2.1] *Let $(X, \mathrm{d}, \preceq)$ be a partially ordered complete metric space and $f : X \to X$ a map. Assume that there exists a comparison function φ such that*

$$\mathrm{d}\big(f(x), f(y)\big) \le \varphi\big(\mathrm{d}(x, y)\big),\ \forall\, x \preceq y. \tag{4.1}$$

Also suppose either

f is continuous, or

if $(x_n)_n \subset X$ is a nondecreasing sequence with $x_n \to x$, then $x_n \preceq x$, for all n, holds. If there exists an $x_0 \in X$ with $x_0 \prec f(x_0)$, then f has a fixed point ξ and $\xi = \lim_n f^n(x_0)$.

By following the ideas from Section 3 and from [20], we provide in the next theorem a similar result as before for weak F-contractions which also generalizes [10, Th.2.1]. First we remind a classical result from real analysis.

Lemma 4.3. *Let $a \ge 0$ and $f : [a, \infty) \to \mathbb{R}_+$ be a decreasing map. Then, for every $k, p \in \mathbb{N}$, $k \ge a + 1$, one has*

$$\sum_{q=k}^{k+p} f(q) \le \int_{k-1}^{k+p} f(t)\,\mathrm{d}t.$$

Theorem 4.4. *Let $(X, \mathrm{d}, \preceq)$ be a partially ordered complete metric space, $x_0 \in X$ and $f : X \to X$ a mapping. We consider the following conditions:*

(a) for every comparable $x, y \in X$, $f(x)$ and $f(y)$ are comparable too;

(b) x_0 and $f(x_0)$ are comparable;

(c) there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F\big(\mathrm{d}\big(f(x), f(y)\big)\big) \le F\big(\mathrm{d}(x, y)\big), \tag{4.2}$$

for all comparable pairs $x, y \in X$ with $f(x) \ne f(y)$;

(d) there exist $F \in \mathcal{F}_1$ and $\tau > 0$ such that (4.2) holds for all comparable pairs $x, y \in X$ with $f(x) \ne f(y)$ and one of the following conditions is fulfilled:

(d_1) F is continuous, or

(d_2) the sequence $\big(f^n(x_0)\big)_n$ is bounded;

(e) one of the following statements occurs:

(e_1) f is continuous;

(e_2) for every sequence $(x_n)_n \subset X$ which converges to $x \in X$ and x_n is comparable to x_{n+1}, then x_n is comparable to x, for every $n \in \mathbb{N}$;

(f) for every $x \in X$, there exists $y \in X$ such that y is comparable to x and x_0;

Then we have the following conclusions:

1) *if Conditions (a), (b) and (c) hold, then the sequence $\big(f^n(x_0)\big)_n$ converges to a point ξ. More precisely, concerning the rate of convergence, one can find $N \in \mathbb{N}$ such that*

$$\mathrm{d}\big(f^n(x_0), \xi\big) \le \frac{\lambda}{1-\lambda}(n-1)^{\frac{\lambda-1}{\lambda}}, \ \forall\, n \ge N, \tag{4.3}$$

where $\lambda \in (0,1)$ is the constant from (F3)*;*

2) *if Conditions (a), (b) and (d) hold, then the sequence $\big(f^n(x_0)\big)_n$ converges;*

3) *if Conditions (a), (b), (c) and (e) or Conditions (a), (b), (d) and (e) hold, then f has a fixed point which is approximated by $\big(f^n(x_0)\big)_n$;*

4) *if Conditions (a), (b), (c), (e) and (f) or Conditions (a), (b), (d), (e) and (f) hold, then f is a Picard mapping.*

Proof. **1**) For each $n = 0, 1, \ldots$, we denote $\gamma_n = \mathrm{d}\big(f^n(x_0), f^{n+1}(x_0)\big)$. If there is $n_0 \ge 1$ such that $f^{n_0-1}(x_0) = f^{n_0}(x_0)$, then $\xi = f^{n_0-1}(x_0)$ is a fixed point of f.

Suppose now that $\gamma_n > 0$ for all $n \ge 0$. From Conditions (a) and (b), one deduces that $f^{n-1}(x_0)$ and $f^n(x_0)$ are comparable, for all $n \ge 1$. So, for every $n \in \mathbb{N}$ we have from (4.2)

$$F(\gamma_n) \le F(\gamma_{n-1}) - \tau \le \cdots \le F(\gamma_0) - n\tau. \tag{4.4}$$

It follows that $\lim_n F(\gamma_n) = -\infty$ hence, by (F2), $\gamma_n \to 0$. Next, from (F3), there is $\lambda \in (0,1)$ such that

$$\lim_n \gamma_n^\lambda F(\gamma_n) = 0. \tag{4.5}$$

According to (4.4) one has

$$\gamma_n^\lambda F(\gamma_n) - \gamma_n^\lambda F(\gamma_0) \le \gamma_n^\lambda\big(F(\gamma_0) - n\tau\big) - \gamma_n^\lambda F(\gamma_0) = -n\gamma_n^\lambda \tau \le 0$$

and hence, using (4.5), $\lim_n n\gamma_n^\lambda = 0$. So, one can find $N \in \mathbb{N}$ such that $n\gamma_n^\lambda \le 1$ for all $n \ge N$, hence $\gamma_n \le n^{-1/\lambda}$ that is $\mathrm{d}(f^n(x_0), f^{n+1}(x_0)) \le n^{-1/\lambda}$.

In order to show that $\big(f^n(x_0)\big)_n$ is a Cauchy sequence, choose $n \ge N$ and $p \in \mathbb{N}$. We get

$$\mathrm{d}\big(f^{n+p}(x_0), f^n(x_0)\big) \le \gamma_n + \gamma_{n+1} + \cdots + \gamma_{n+p-1} \le \sum_{k=n}^{k+p-1} \frac{1}{k^{\frac{1}{\lambda}}} < \sum_{k=n}^{\infty} \frac{1}{k^{\frac{1}{\lambda}}}. \tag{4.6}$$

Lemma 4.3 implies

$$\sum_{k \ge n} \frac{1}{k^{\frac{1}{\lambda}}} \le \int_{n-1}^{\infty} t^{-\frac{1}{\lambda}}\,\mathrm{d}t = \frac{1}{\big(\frac{1}{\lambda} - 1\big)(n-1)^{\frac{1}{\lambda}-1}} = \frac{\lambda}{1-\lambda}(n-1)^{\frac{\lambda-1}{\lambda}}. \tag{4.7}$$

From (4.6) and (4.7) it follows that the sequence $\big(f^n(x_0)\big)_n$ is Cauchy so, the space (X, d) being complete, it is convergent. Let ξ be its limit.

Letting in (4.6) $p \to \infty$, we obtain

$$\mathrm{d}\big(\xi, f^n(x_0)\big) \leq \sum_{q \geq k} \frac{1}{q^{\frac{1}{\lambda}}}, \ \forall n \geq N$$

and hence, using (4.7), the inequality (4.3) comes.

2) If Conditions (a), (b), (d) and (d_1) hold, then, using the same argument as in the proof of Theorem 3.9, for $x, y \in X$, x comparable to y, one obtains (4.1). Next we apply Theorem 4.2.

Under the hypotheses (a), (b), (d) and (d_2), the conclusion comes in the same manner as in the proof of Theorem 3.5 tacking into account that $f^{n-1}(x_0)$ and $f^n(x_0)$ are comparable, for all $n \geq 1$.

3) When (e_1) occurs, then

$$f(\xi) = f\big(\lim_n f^n(x_0)\big) = \lim_n f^{n+1}(x_0) = \xi.$$

We assume that Condition (e_2) is satisfied. Then, by hypothesis and **1**), ξ is comparable to $f^n(x_0)$, for all n. One has

$$\begin{aligned} F\Big(\mathrm{d}\big(f(\xi), f\big(f^n(x_0)\big)\big)\Big) &\leq F\Big(\mathrm{d}\big(\xi, f^n(x_0)\big)\Big) - \tau \\ &< F\Big(\mathrm{d}\big(\xi, f^n(x_0)\big)\Big), \ \forall n \in \mathbb{N}. \end{aligned}$$

So, F being increasing, $\mathrm{d}\big(f(\xi), f^{n+1}(x_0)\big) < \mathrm{d}\big(\xi, f^n(x_0)\big)$. Therefore

$$\begin{aligned} \mathrm{d}\big(f(\xi), \xi\big) &\leq \mathrm{d}\big(f(\xi), f^{n+1}(x_0)\big) + \mathrm{d}\big(f^{n+1}(x_0), \xi\big) \\ &< \mathrm{d}\big(\xi, f^n(x_0)\big) + \mathrm{d}\big(f^{n+1}(x_0), \xi\big) \underset{n}{\longrightarrow} 0 . \end{aligned}$$

4) Set $x \in X$ and let $y \in X$ be comparable to x and x_0. From hypothesis (a) it follows that $f^n(y)$ is comparable to both $f^n(x)$ and $f^n(x_0)$, for all $n \in \mathbb{N}$.

If there is $N_1 \in \mathbb{N}$ such that $f^{N_1}(y) = f^{N_1}(x_0)$, then $f^n(y) = f^n(x_0)$, for every $n \geq N_1$, so $\mathrm{d}\big(f^n(x_0), f^n(y)\big) \underset{n}{\longrightarrow} 0$. Suppose that $f^n(y) \neq f^n(x_0)$, for all $n \geq 1$. Then

$$F\Big(\mathrm{d}\big(f^n(x_0), f^n(y)\big)\Big) \leq F\Big(\mathrm{d}\big(f^{n-1}(x_0), f^{n-1}(y)\big)\Big) - \tau \leq \cdots \leq F\big(\mathrm{d}(x_0, y)\big) - n\tau.$$

Hence $\lim_n F\Big(\mathrm{d}\big(f^n(x_0), f^n(y)\big)\Big) = -\infty$ and, by (F2), $\mathrm{d}\big(f^n(x_0), f^n(y)\big) \underset{n}{\longrightarrow} 0$.

Using the same argument as before we get

$$\lim_n \mathrm{d}\big(f^n(y), f^n(x)\big) = 0.$$

Therefore

$$\mathrm{d}\big(f^n(x_0), f^n(x)\big) \leq \mathrm{d}\big(f^n(x_0), f^n(y)\big) + \mathrm{d}\big(f^n(y), f^n(x)\big) \underset{n}{\longrightarrow} 0.$$

Accordingly

$$\lim_n f^n(x) = \lim_n f^n(y) = \lim_n f^n(x_0) = \xi.$$

Finally, if $\xi' \in X$ is another fixed point of f, then $\xi' = f^n(\xi') \underset{n}{\longrightarrow} \xi$ and hence ξ is the unique fixed point of f. □

Remark 4.5. If, in the previous theorem, we consider $F(t) = \ln t$ and $\tau = -\ln r$, $r \in (0,1)$, f is nondecreasing and $x_0 \preceq f(x_0)$, then one obtains [10, Th.2.1,2.2], [13, Th.2.1]. The same result is also obtained when f is nonincreasing.

Remark 4.6. Because of the symmetry of the metric d, it is enough to impose that relation (4.2) to be verified for every $x, y \in X$ with $x \preceq y$ and $f(x) \neq f(y)$.

Remark 4.7. Following the proof of Theorem 4.4, it is easy to verify that the same results are obtained if we replace Conditions (a), (b) and (e_2) by

(a') f is nondecreasing (e.g. $x \preceq y \Rightarrow f(x) \preceq f(y)$),

(b') $x_0 \preceq f(x_0)$, and

(e_2') for every nondecreasing sequence $(x_n)_n \subset X$ which converges to $x \in X$, one has $x_n \preceq x$, for every $n \in \mathbb{N}$.

In the following we will provide some extensions of Theorem 4.4 and Theorem 3.15 which improve [7, Th. 2.1] and [7, Th. 2.2], respectively.

We first need to remind the notion of generalized metric.

Definition 4.8. Let X be a nonempty set. We say that a function $\mathrm{d} : X \times X \to [0, \infty]$ is a *generalized metric* on X whenever it satisfies the following properties:

(G1) $\mathrm{d}(x, y) = 0$ if and only if $x = y$;

(G2) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$, for all $x, y \in X$;

(G3) $\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$, for all $x, y, z \in X$.

In this event the pair (X, d) is called *generalized metric space*. If we further have

(G4) every Cauchy sequence in X is convergent,

then (X, d) is a *generalized complete metric space*.

If (X, d) is a generalized (complete) metric space endowed with a partial order "$\preceq$", we say that $(X, \mathrm{d}, \preceq)$ is a *partially ordered generalized (complete) metric space*.

Example 4.9. Let (X, d) be a metric space and $CL(X)$ the class of all nonempty closed subset of X. Then the mapping $h : CL(X) \times CL(X) \to [0, \infty]$, given by

$$h(B, C) = \max\left\{\mathrm{D}(B, C), \mathrm{D}(C, B)\right\}, \text{ where } \mathrm{D}(B, C) = \sup_{x \in B}\left(\inf_{y \in C} \mathrm{d}(x, y)\right)$$

is a generalized metric on $CL(X)$, namely the *generalized Hausdorff-Pompeiu metric* (see e.g. [3, §3.2]). Moreover $\big(CL(X), h, \preceq\big)$ is a partially ordered generalized metric space, where $\preceq$ represents the set-inclusion partial order.

According to [3, Th.3.2.4], the generalized metric space $\big(CL(X), h\big)$ is complete if and only if (X, d) is complete.

Theorem 4.10. *Let* $(X, \mathrm{d}, \preceq)$ *be a partially ordered generalized complete metric space and* $f : X \to X$ *a mapping which satisfies the following properties:*

(a) *for every comparable* $x, y \in X$, $f(x)$ *and* $f(y)$ *are comparable too;*

(b) *there is* $x_0 \in X$ *such that* x_0 *and* $f(x_0)$ *are comparable;*

(c) *there exist* $F \in \mathcal{F}_1$ *and* $\tau > 0$ *such that*

$$\tau + F\big(\mathrm{d}\big(f(x), f(y)\big)\big) \le F\big(\mathrm{d}(x, y)\big), \tag{4.8}$$

for all comparable pairs $x, y \in X$ *with* $\mathrm{d}(x, y) < \infty$ *and* $\mathrm{d}\big(f(x), f(y)\big) \in (0, \infty)$. *Furthermore one of the following assertions is satisfied:*

(c_1) $F \in \mathcal{F}$, *or*

(c_2) F *is continuous, or*

(c_3) *the sequence* $\big(f^n(x_0)\big)_n$ *is bounded;*

(d) *one of the following statements occurs:*

(d_1) f *is continuous, or*

(d_2) *for every sequence* $(x_n)_n \subset X$ *which converges to* $x \in X$ *and* x_n *is comparable to* x_{n+1}, *then* x_n *is comparable to* x, *for every* $n \in \mathbb{N}$;

(e) *for every* $x \in X$, *there exists an* $y \in X$ *comparable to* x *and* x_0 *and* $\mathrm{d}\big(f(x), f(y)\big) < \infty$, $\mathrm{d}\big(f(x_0), f(y)\big) < \infty$;

Then we have the followings:

(i) *if Conditions* $(a) - (d)$ *hold, then one of the following cases occurs:*

(α) $\mathrm{d}\big(f^n(x_0), f^{n+1}(x_0)\big) = \infty$, *for all* $n \in \mathbb{N}$, *or*

(β) $\big(f^n(x_0)\big)_n$ *converges to a fixed point of* f;

(ii) *if Conditions* $(a) - (e)$ *hold and* (α) *does not occur, then* f *is a Picard mapping.*

Proof. (i) Suppose that (α) does not occur and let N be the smallest positive integer such that $\mathrm{d}\big(f^N(x_0), f^{N+1}(x_0)\big) < \infty$. If there is $n_0 \ge N$ such that $f^{n_0}(x_0) = f^{n_0+1}(x_0) = 0$, then clearly $f^{n_0}(x_0) = f^{n_0+k}(x_0)$ for every $k \in \mathbb{N}$ and so $\big(f^n(x_0)\big)_n$ is convergent. Now suppose that $\mathrm{d}\big(f^n(x_0), f^{n+1}(x_0)\big) > 0$ for all $n \ge N$. Then $\mathrm{d}\big(f^n(x_0), f^{n+1}(x_0)\big) \in (0, \infty)$ and, by (4.8), one has

$$\begin{aligned} F\big(\mathrm{d}\big(f^n(x_0), f^{n+1}(x_0)\big)\big) &< F\big(\mathrm{d}\big(f^{n-1}(x_0), f^n(x_0)\big)\big) \\ &< \dots \\ &< F\big(\mathrm{d}\big(f^N(x_0), f^{N+1}(x_0)\big)\big) \end{aligned}$$

and hence $\mathrm{d}\big(f^n(x_0), f^{n+1}(x_0)\big) < \mathrm{d}\big(f^N(x_0), f^{N+1}(x_0)\big) < \infty$, for all $n > N$. Next, considering the sequence $\big(f^n(x_0)\big)_{n \ge N}$, we continue as in the proof of Theorem 4.4, **1)-3)**.

(ii) Choose $x \in X$ and let $y \in X$ be as in the statement. Then, from (4.8), it follows that $\mathrm{d}\big(f^n(x_0), f^n(y)\big) < \mathrm{d}\big(f(x_0), f(y)\big) < \infty$ and also $\mathrm{d}\big(f^n(x), f^n(y)\big) < \mathrm{d}\big(f(x), f(y)\big) < \infty$, for all $n \ge 1$. The conclusion now results in the same way as in the proof of Theorem 4.4, **4**). □

Remark 4.11. If we take $F(t) = \ln t$ in the previous theorem, we obtain [7, Th.2.1, Th.2.2].

4.1. **Application: weak F-countable iterated function systems.** In this section another example from fractal theory is described (more details of classical theory of countable iterated function systems can be found e.g. in [17]). In the following (X, d) will be a complete metric space and F an element in $\mathcal{F}_1$.

Definition 4.12. A countable family of weak F-contractions $(f_k)_{k\geq 1}$ on X into itself is called a *weak F-countable iterated function system* (abbreviated weak-F-CIFS). The set function $\mathcal{S} : CL(X) \to CL(X)$ defined by $\mathcal{S}(B) = \overline{\bigcup_{k\geq 1} f_k(B)}$ is called the associated *Hutchinson operator*. A set $A \in CL(X)$ is said to be an *attractor* of the weak-F-CIFS whenever $\mathcal{S}(A) = A$.

Lemma 4.13. *Assume that F is continuous and $f : X \to X$ is a weak F-contraction, its constant from (3.1) being 2τ. Then*

$$\tau + F\big(h\big(f(B), f(C)\big)\big) \leq F\big(h(B, C)\big),$$

for every $B, C \in CL(X)$, either $C \subset B$ or $B \subset C$, $h(B, C) < \infty$, $f(B) \neq f(C)$, where h denotes the Hausdorff-Pompeiu generalized metric (see Example 4.9).

Proof. Let $B, C \in CL(X)$ be such that $C \subset B$, $h(B, C) < \infty$ and $f(B) \neq f(C)$. So $f(C) \subset f(B)$ and, clearly, $\mathrm{D}(B, C) < \infty$ and, according to Lemma 2.5, $\mathrm{D}(C, B) = 0$.

By hypothesis, we deduce that $\mathrm{d}\big(f(x), f(y)\big) \leq \mathrm{d}(x, y)$, for all $x \in B$, $y \in C$. This implies

$$\mathrm{D}\big(f(B), f(C)\big) = \sup_{x\in B} \inf_{y\in C} \mathrm{d}\big(f(x), f(y)\big) \leq \sup_{x\in B} \inf_{y\in C} \mathrm{d}(x, y) = \mathrm{D}(B, C) < \infty.$$

Since $h\big(f(B), f(C)\big) > 0$ and $\mathrm{D}\big(f(C), f(B)\big) = 0$, one deduces that $\mathrm{D}\big(f(B), f(C)\big) > 0$.

Next, by the continuity of F, it follows that, for $\tau > 0$, there is $\varepsilon > 0$, $\varepsilon < \mathrm{D}\big(f(B), f(C)\big)$ such that

(4.9)
$$t \in \big(\mathrm{D}(f(B), f(C)) - \varepsilon, \mathrm{D}(f(B), f(C)) + \varepsilon\big) \;\Rightarrow\; F(t) > F\big(\mathrm{D}(f(B), f(C))\big) - \tau.$$

One can find $b \in B$ such that $\inf_{y\in C} \mathrm{d}\big(f(b), f(y)\big) + \varepsilon > \mathrm{D}\big(f(B), f(C)\big)$. Notice that $f(b) \notin f(C)$ because if that was not so, we would have $\varepsilon > \mathrm{D}\big(f(B), f(C)\big)$ contradicting the choice of b.

Now, taking $t = \inf_{y\in C} \mathrm{d}\big(f(b), f(y)\big)$ in (4.9), we get

$$F\big(\mathrm{D}(f(B), f(C))\big) < \tau + F\big(\inf_{y\in C} \mathrm{d}\big(f(b), f(y)\big)\big). \tag{4.10}$$

By hypothesis, $2\tau + F\big(\mathrm{d}(f(x), f(y))\big) \le F\big(\mathrm{d}(x,y)\big)$ for $x \in B$, $y \in C$, $f(x) \ne f(y)$. Therefore, using (4.10), one has

$$\begin{aligned}\tau + F\big(\mathrm{D}(f(B), f(C))\big) &< \tau + \tau + F\big(\inf_{y\in C} \mathrm{d}(f(b), f(y))\big)\\ &\le 2\tau + F\big(\mathrm{d}\big(f(b), f(y)\big)\big)\\ &\le F\big(\mathrm{d}(b,y)\big),\end{aligned}$$

for all $y \in C$. Accordingly, using again the continuity of F, we have

$$\tau + F\big(\mathrm{D}(f(B), f(C))\big) \le F\big(\inf_{y\in C} \mathrm{d}(b,y)\big) \le F\big(\mathrm{D}(B,C)\big) = F\big(h(B,C)\big). \tag{4.11}$$

Since $\mathrm{D}\big(f(C), f(B)\big) = 0$, it follows

$$\tau + F\big(h(f(B), f(C))\big) \le F\big(h(B,C)\big),$$

as required. □

Theorem 4.14. *Let $(f_k)_{k\ge1}$ be a weak F-CIFS on the complete metric space (X,d), where $F \in \mathcal{F}_1$ is continuous. Suppose that $\inf_{k\ge1} \tau_k > 0$, where $2\tau_k$ means the constant from (3.1) associated to f_k. If, for each $\emptyset \ne I \subset \mathbb{N}$, we put $C_I = \overline{\{\xi_k;\ k \in I\}}$, where ξ_k is the fixed point of f_k, then one of the following alternative may occur:*

(ii) $h\big(\mathcal{S}^n(C_I), \mathcal{S}^{n+1}(C_I)\big) = \infty$, for all $n \in \mathbb{N}$,
or

(ii) there exists an attractor $A \in CL(X)$ of the considered weak F-CIFS and $A = \lim_n \mathcal{S}^n(C_I)$, the limits being taken with respect to the generalized Hausdorff-Pompeiu metric.

If further $\mathcal{S}(X)$ is bounded, in particular when $\mathcal{S}(X)$ is compact, then the attractor A is unique and it is successively approximated by $\big(\mathcal{S}^n(B)\big)_n$, for every $B \in CL(X)$.

Proof. In order to apply Theorem 4.10, we check the conditions from the statement taking into account Remark 4.7.

Clearly $B, C \in CL(X)$, $B \subset C$ imply $\mathcal{S}(B) \subset \mathcal{S}(C)$, hence (a) holds.

Next, according to Theorem 3.9, for each $k \ge 1$, there exists a unique fixed point ξ_k of f_k. Hence, for every $I \subset \mathbb{N}$, $I \ne \emptyset$, $\{\xi_k;\ k \in I\} = \bigcup_{k\in I} f_k\big(\{\xi_k\}\big) \subset \bigcup_{k\ge1} f_k(C_I)$, so $C_I \subset \mathcal{S}(C_I)$, therefore (b) is fulfilled.

Now we intend to prove (c). Set $\tau = \inf_{k\ge1} \tau_k > 0$. Choose $B, C \in CL(X)$, $C \subset B$, $h(B,C) < \infty$, $h\big(\mathcal{S}(B), \mathcal{S}(C)\big) \in (0,\infty)$. Hence, one can find $k \in \mathbb{N}$ such that $f_k(B) \ne f_k(C)$, that is $\mathrm{D}(f_k(B), f_k(C)) > 0$. In view of the aforesaid, using Lemma 2.5, Lemma 4.13 and the fact that, F being continuous, $F(\sup M) \le \sup F(M)$ for every set $M \in CL(X)$, one has

$$\tau + F\big(\mathrm{D}(\mathcal{S}(B), \mathcal{S}(C))\big) \le \tau + F\big(\sup_{k\ge1} \mathrm{D}(f_k(B), f_k(C))\big)$$

$$\le \tau + \sup_{k\ge 1} F\big(\mathrm{D}(f_k(B), f_k(C))\big) \le \sup_{k\ge 1}\Big(\tau_k + F\big(\mathrm{D}(f_k(B), f_k(C))\big)\Big) \le F\big(h(B,C)\big),$$

where, at the last inequality, we used (4.11). Since $\mathcal{S}(C) \subset \mathcal{S}(B)$ implies $\mathrm{D}\big(\mathcal{S}(C), \mathcal{S}(B)\big) = 0$, we get

$$\tau + F\big(h(\mathcal{S}(B), \mathcal{S}(C))\big) \le F\big(h(B,C)\big).$$

In view of Remark 4.7, it is enough to prove (e_2') instead of (e_2). Let $(B_n)_n$ be a nondecreasing sequence of closed subsets of C which converges in the generalized Hausdorff-Pompeiu metric to a set $B \in CL(X)$, that is $B = \lim_n(B_n)$, so $\mathrm{D}(B_n, B) \underset{n}{\longrightarrow} 0$. We need to show that $B_n \subset B$, for all $n \ge 1$. Let $n_0 \in \mathbb{N}$ and $b \in B_{n_0}$ be fixed. Since $B_{n_0} \subset B_n$, one has $b \in B_n$, for every $n \ge n_0$. Then, for each $n \ge n_0$, we get

$$\inf_{x\in B} \mathrm{d}(b,x) \le \sup_{y\in B_n} \inf_{x\in B} \mathrm{d}(y,x) = \mathrm{D}(B_n, B) \underset{n}{\longrightarrow} 0.$$

Thus $b \in \overline{B} = B$. This means $B_{n_0} \subset B$. The first conclusion now follows from Theorem 4.10, (i).

Assuming that $\mathcal{S}(X)$ is bounded, it follows that (α) does not occur. Since, for every $B, C \in CL(X)$, one has $B \cup C \in CL(X)$, Condition (e) from Theorem 4.10 is obviously fulfilled. In conclusion, there exists a unique attractor $A \in CL(X)$ and $A = \lim_n \mathcal{S}^n(B)$, for all $B \in CL(X)$, completing the proof. □

Remark 4.15. In the particular case when (X, d) is a compact metric space and, for each $k \ge 1$, f_k is a weak F-contraction with $F(t) = \ln t$ and $\inf_k \tau_k > 0$, we obtain the theorem concerning the existence, uniqueness and approximation of the attractor of the classical CIFS consisting of Banach contractions whose ratios satisfy $\sup_k r_k < 1$ (see e.g. [18, Th. 3.2]). Indeed, in view of Remark 3.3, $F \in \mathcal{F}$ and f_k is a Banach contraction with ratio $r_k = e^{-\tau_k}$, for all $k \in \mathbb{N}$. Since $\inf_k \tau_k > 0$ is equivalent to $\sup_k r_k < 1$, the conclusion comes.

Example 4.16. Let us consider $X = [0, \infty)$ endowed with the Euclidean metric and, for each $k = 1, 2, \ldots$, a map $f_k : X \to X$ given by $f_k(x) = \frac{x}{kx+1} + \alpha_k$, where $\alpha_k \ge 0$ (see Example 3.16). We consider further the mapping $F : \mathbb{R}_+ \to \mathbb{R}$, $F(t) = \ln t - \frac{1}{t}$, for every $t > 0$. Then:

(i) f_k is not a Banach contraction, for all $k \ge 1$;

(ii) $F \notin \mathcal{F}$;

(iii) $(f_k)_{k\ge 1}$ is a weak F-CIFS and, for each $k \ge 1$, $\tau_k \in (0, k]$, τ_k being the constant associated to f_k from (3.1). Moreover, if we denote $C_I = \overline{\left\{\frac{k\alpha_k + \sqrt{k^2\alpha_k^2 + 4\alpha k}}{2k};\ k \in I\right\}}$, for every $\emptyset \ne I \subset \mathbb{N}$, two cases may occur:

(i) $h\big(\mathcal{S}^n(C_I), \mathcal{S}^{n+1}(C_I)\big) = \infty$, for all $n \in \mathbb{N}$, or

(ii) there exists an attractor $A \in CL(X)$ and $A = \lim_n \mathcal{S}^n(C_I)$, the limits being taken with respect to the generalized Hausdorff-Pompeiu metric;

(iv) whenever $\sup_k \alpha_k < \infty$, there is a unique attractor A of the considered weak F-CIFS and $A = \lim_n \mathcal{S}^n(B)$, for every $B \in CL(X)$.

Proof. From Example 3.7 one deduces that $F \in \mathcal{F}_1 \setminus \mathcal{F}$ and, for each $k \geq 1$, f_k is a weak F-contraction with $\tau_k \in (0, k]$ and its fixed point is $\xi_k = \frac{k\alpha_k + \sqrt{k^2\alpha_k^2 + 4\alpha k}}{2k}$. In addition, f_k is not a Banach contraction. Notice that we can take $\tau_k = 1$ for all k, so $\inf_k \tau_k > 0$.

On the other hand, using again Example 3.7 (d), one has $f_k(X) \subset [\alpha_k, \alpha_k + k^{-1})$, for every $k \in \mathbb{N}$, so $\mathcal{S}(X) \subset [0, \sup_k \alpha_k + 1]$. Therefore, if $\sup_k \alpha_k < \infty$, then $\mathcal{S}(X)$ is bounded.

The assertions of statement now follows immediately from Theorem 4.14. □

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