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ON WEAKLY CO-HOPFIAN MODULES

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ABSTRACT. A quasi-injective weakly co-Hopfian right module over a right Noetherian ring is characterized. Those right semi-Artinian rings and right FBN rings over which completely weakly co-Hopfian right modules are precisely right modules of finite uniform dimension, are characterized. In addition, some applications are provided.

1. Weakly co-Hopfian modules over some special rings

Throughout the paper rings will have a nonzero identity element and modules will be unitary. Recall from [4] that a module M is weakly co-Hopfian if any injective endomorphism f of M is essential, i.e., $f(M) \leq_e M$. Now suppose that R is a right Noetherian ring, and M a quasi-injective R-module. Then M has a decomposition into indecomposable quasi-injective modules by [5, Theorem 3.48, Proposition 6.73(1), and Exercises 32 and 37 of §6] and the endomorphism ring of each direct summand of such a decomposition is a local ring. We call the direct sum of all isomorphic direct summands of such a decomposition, a homogeneous component of that decomposition. In addition, recall from [4, Proposition 1.4] that notions co-Hopfian, weakly co-Hopfian and Dedekind-finite are identified for a quasi injective module. From these facts we obtain the following theorem.

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Theorem 1.1. *Let R be a right Noetherian ring and let M be a quasi-injective right R-module. Then M is weakly co-Hopfian if and only if every homogeneous component of a decomposition of M into indecomposable quasi-injective submodules is a finite direct sum.*

Proof. ($\Rightarrow$). As M is weakly co-Hopfian so is every direct summand of M [4, Corollary (1.3)]. Moreover, any homogenous component of M is isomorphic to $N^{(\Lambda)}$, for some indecomposable quasi-injective submodule N. If Λ were infinite, then the shift map on $N^{(\Lambda)}$ would be a monomorphism that is not essential. Thus Λ is finite and any homogenous component is a finite direct sum.

($\Leftarrow$). It suffices to show that $f(M) = M$ for every injective endomorphism f of M. Because of the quasi-injectivity of M, $f(M)$ is a direct summand of M, say $M = f(M) \bigoplus K$. Assume $K \neq 0$. Then for quasi-injective modules M and K we have $M = \bigoplus_{\alpha \in A} M_\alpha$ and $K = \bigoplus_{\beta \in B} K_\beta$, where M_α's and K_β's are indecomposable. Now we have

$$M = \bigoplus_{\alpha \in A} M_\alpha \qquad (*)$$

and also

$$M = \bigoplus_{\alpha \in A} f(M_\alpha) \bigoplus (\bigoplus_{\beta \in B} K_\beta). \qquad (**)$$

As the endomorphism ring of each direct summand of these two decompositions is a local ring, they are equivalent [1, Theorem 12.6]. Now, for a $\beta_1 \in B$, $K_{\beta_1} \cong M_{\alpha_1}$ for some $\alpha_1 \in A$. Suppose that the homogenous component of M corresponding to M_{α_1} has n direct summands $M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}$. Then in $(**)$ there is a homogenous component with at least $n+1$ direct summands, i.e., $f(M_{\alpha_1}), f(M_{\alpha_2}), \ldots, f(M_{\alpha_n}), K_{\beta_1}$ which are all isomorphic to M_{α_1}. This contradicts the equivalence of $(*)$ and $(**)$. Therefore, $K = 0$, and $f(M) = M$. $\square$

Proposition 1.2. *Let R be a right Noetherian ring. Then the following are equivalent.*

(i) *In the class of quasi-injective right R-modules being weakly co-Hopfian is the same as having finite uniform dimension.*
(ii) *Up to isomorphisms, there are only finitely many indecomposable injective right R-modules.*

Proof. (i) $\Rightarrow$ (ii). Let $\{M_\lambda : \lambda \in \Lambda\}$ be any set of nonisomorphic indecomposable injective right R-modules. Then $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is injective and by Theorem 1.1, is weakly co-Hopfian, so by (i) its uniform dimension is finite. Thus Λ must be finite.

(ii) $\Rightarrow$ (i). It suffices to show that a quasi-injective weakly co-Hopfian module M is of finite uniform dimension. Since R is right Noetherian, $E(M) = M_1^{(\Lambda_1)} \oplus M_2^{(\Lambda_2)} \oplus \cdots \oplus M_n^{(\Lambda_n)}$, where $M_1, M_2, \ldots, M_n$ are nonisomorphic indecomposable injective modules and $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are some sets [5, Theorem 3.48]. Now as M is weakly co-Hopfian, so is $E(M)$ [4, Corollary 1.5] , hence each direct summand $M^{(\Lambda_i)}$ is weakly co-Hopfian. Therefore, Λ_i must be finite. Consequently, the uniform dimension of M is finite. □

Corollary 1.3. *Let R be a right Artinian ring and let M be a quasi-injective right R-module. Then M is weakly co-Hopfian if and only if* u.dim(M) $< \infty$.

Proof. This is an immediate consequence of Proposition 1.2, since the number of isomorphism classes of indecomposable injective right modules over a right Artinian ring R is finite by [5, Theorem 3.61]. □

Proposition 1.4. *Let R be a right Artinian ring. Then the following are equivalent on a right R-module M.*

(i) *Every submodule of M is weakly co-Hopfian.*
(ii) $\mathrm{Soc}(M)$ *is weakly co-Hopfian.*
(iii) $\mathrm{u.dim}(M) < \infty$.

Moreover, if M is quasi-injective the above statements are equivalent to

(iv) *M is weakly co-Hopfian.*

Proof. (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are obvious.

(ii) $\Rightarrow$ (iii). As $\mathrm{Soc}(M)$ is weakly co-Hopfian and there are only finitely many simple right R-modules up to isomorphism, we conclude that $\mathrm{u.dim}(\mathrm{Soc}(M)) < \infty$ [4, Corollary 1.12]. On the other hand, $\mathrm{Soc}(M) \leq_e M$, so $\mathrm{u.dim}(M) < \infty$.

Now let M be quasi-injective. Then by Corollary 1.3, (iii) $\Leftrightarrow$ (iv). □

2. Completely weakly co-Hopfian modules

In view of [4, Corollary 1.5] every submodule of a quasi-injective weakly co-Hopfian module is weakly co-Hopfian, moreover, every submodule of a module of finite uniform dimension is weakly co-Hopfian. These motivate to investigate modules with all submodules weakly co-Hopfian. We call such modules *completely weakly co-Hopfian.* In Proposition 1.4, we characterized such right modules over right Artinian rings. In addition, by [4, Theorem 1.1], we see that a module M is completely weakly co-Hopfian if and only if M contains no infinite direct sum of nonzero pairwise isomorphic modules. Thus, if M is a module in which there exists a nonzero submodule that embeds in every nonzero submodule (e.g., M is a nonzero compressible module (see the paragraph before Lemma 2.3)), then M is completely weakly co-Hopfian if and only if u.dim$(M) < \infty$. Note that in Proposition 1.2 we characterized right Noetherian rings R possessing the property that among quasi-injective right R-modules, the only ones which are completely weakly co-Hopfian are those having finite uniform dimension. In the following we characterize those right semi-Artinian rings and right FBN rings over which completely weakly co-Hopfian right modules are precisely modules of finite uniform dimension.

Theorem 2.1. *Let R be a right semi-Artinian ring. Then the following are equivalent.*

(i) *In the class of right R-modules being completely weakly co-Hopfian is equivalent to having finite uniform dimension.*
(ii) *In the class of quasi-injective right R-modules being weakly co-Hopfian is equivalent to having finite uniform dimension.*
(iii) *Up to isomorphisms, there are only finitely many simple right R-modules.*
(iv) *Up to isomorphisms, there are only finitely many indecomposable injective right R-modules.*

Proof. (i) $\Rightarrow$ (ii). It is evident since a quasi-injective weakly co-Hopfian right R-module is completely weakly co-Hopfian.

(ii) $\Rightarrow$ (iii). If $\{S_\lambda : \lambda \in \Lambda\}$ is any nonempty set of nonisomorphic simple R-modules, then $\bigoplus_{\lambda\in\Lambda} S_\lambda$ is a quasi-injective weakly co-Hopfian R-module, so by (ii), its uniform dimension is finite. Hence Λ must be finite.

(iii) $\Rightarrow$ (i). It suffices to show that any completely weakly co-Hopfian right module M is of finite uniform dimension. Since M is completely weakly co-Hopfian, $\mathrm{Soc}(M)$ is weakly co-Hopfian. Consequently, by (iii), $\mathrm{Soc}(M)$ must be finitely generated. Because R is right semi-Artinian, $\mathrm{Soc}(M)$ is an essential submodule in M, hence we can conclude that the uniform dimension of M is finite.

Finally, it is easy to see that over a right semi-artinian ring, (iii) $\Leftrightarrow$ (iv). $\square$

A module M is called *retractable* (resp. *compressible*) if there exists a non-zero homomorphism (resp. monomorphism) $f : M \to N$ for any non-zero submodule N of M. The following Lemma is due to M. R. Vedadi.

Lemma 2.2 *Let R be a right FBN ring. Then every non-zero right R-module M contains an essential submodule $L = \oplus_{i\in I} U_i$ where each U_i is a uniform compressible right R-module.*

Proof. First we show that every nonzero submodule N of M has a uniform compressible submodule. Clearly, N contains a cyclic uniform submodule U. For $P \in \mathrm{Ass}(U)$, there exists a nonzero submodule V in U such that $P = \mathrm{ann}_R(V)$. Now by [2, Corollary 8.3], $Z_{R/P}(V)$ is zero. It follows that there exists a right ideal A of R/P such that A can be embedded in $V_{R/P}$ (see [2, Ex. 3W]). However, any nonzero right ideal of a prime ring is a retractable module. Therefore, A is a retractable nonsingular uniform R/P-module. Hence A is compressible as an R/P-module. So A is also compressible as an R-module. Consequently, N contains the uniform compressible submodule A. Now let $L = \oplus_{i\in I} U_i$ be a maximal direct sum of uniform compressible submodules of M. From what we showed first, it follows that L is essential in M. $\square$

Theorem 2.3. *Let R be a right FBN ring. Then the following statements are equivalent.*

(i) *In the class of right R-modules being completely weakly co-Hopfian is equivalent to having finite uniform dimension.*

(ii) *Up to isomorphisms, there are only finitely many indecomposable injective R-modules.*

Proof. (i) $\Rightarrow$ (ii). Let $\{M_\lambda : \lambda \in \Lambda\}$ be any set of nonisomorphic indecomposable injective right R-modules and $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. Since R is right Noetherian, M is injective, so by Theorem 1.1, M is weakly co-Hopfian. Thus M is completely weakly co-Hopfian. Hence by (i), the uniform dimension of M is finite. It follows that Λ is indeed finite.

(ii) $\Rightarrow$ (i). Let M be a completely weakly co-Hopfian right R-module. We show that the uniform dimension of $E(M)$ is finite. By Lemma 2.2, M contains an essential submodule $L = \oplus_{i\in I} U_i$ where each U_i is a uniform compressible right R-module. Consequently, $E(M) = E(\oplus_{i\in I} U_i)$. Because R is a right Noetherian ring, we have $E(M) = \oplus_{i\in I} E(U_i)$. Since $E(U_i)$ is indecomposable injective, if we show that every homogenous component of $E(M)$ is a finite direct sum, then we have the finite uniform dimension of $E(M)$ by (ii). Assume that U_i and U_j are uniform compressible right modules with $E(U_i)$ and $E(U_j)$ are two direct summands of a homogenous component of $E(M)$ (i.e., $E(U_i) \cong E(U_j)$), hence there exist monomorphisms $f_{(i,j)} : U_i \to U_j$ and $f_{(j,i)} : U_j \to U_i$. On the other hand, L is a weakly co-Hopfian R-module by our assumption, so is every direct summand of L, especially $\oplus_{k\in K} U_k$ which corresponds to a homogenous component of $E(M)$. However, by monomorphisms $f_{(i,j)}$ we can have a shift map on $\oplus_{k\in K} U_k$ which is a monomorphism if K is infinite, moreover, it is not essential which is a contradiction. Therefore, K is finite and hence every homogeneous component of E(M) has finite uniform dimension. From (ii) we see that E(M) itself has finite uniform dimension. $\square$

3. Some applications

Recall that a ring R is said to satisfy the *rank condition* (resp. *strong rank condition*) if a right R-epimorphism (resp. R-monomorphism) $R^{(k)} \to R^{(l)}$ can exist only when $k \geq l$ (resp. $k \leq l$). It is well known that the strong rank condition implies the rank condition, see [5, Proposition 1.21]. Moreover, recall from [4] that a ring R is right strong stably finite if $R_R^{(n)}$ is weakly co-Hopfian, for all $n \geq 1$. Every commutative ring is strong stably finite, and R being right strong stably finite is a Morita invariant property. A right strong stably finite ring satisfies the strong rank condition. In the following result we replace the regular bimodule ${}_RR_R$ by an arbitrary bimodule and deduce a more general conclusion.

Proposition 3.1. *Let ${}_SM_T$ be a non-zero (S,T)-bimodule such that $M_T^{(n)}$ is weakly co-Hopfian for all $n \geq 1$. Then one of the rings S or T satisfies the rank condition.*

Proof. Form the triangular matrix ring $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ and let $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. Using the hypothesis, we conclude that $I_R^{(n)}$ is weakly co-Hopfian for all $n \geq 1$. Now we show that the ring R satisfies the strong rank condition, hence the rank condition. The result will then follow from [3, Proposition 4.1]. Thus suppose that R does not satisfy the strong rank condition and $f : R^{(m)} \longrightarrow R^{(n)}$ is an R-monomorphism where $m > n$. Then $f(I^{(m)}) = f(R^{(m)}I) = f(R^{(m)})I \leq (R^{(n)})I = I^{(n)}$. So f maps monomorphicly $I^{(n)} \bigoplus I^{(m-n)}$ to $I^{(n)}$, hence by [4, Theorem 1.1], $I^{(m-n)} = 0$, so $I = 0$, which is a contradiction. $\square$

Proposition 3.2. *Let ${}_SM_T$ be a (S,T)-bimodule such that* $\mathrm{ann}_S(M) = 0$ *and $M_T^{(n)} \oplus T_T^{(n)}$ is weakly co-Hopfian for all $n \geq 1$. Then $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ is right strong stably finite.*

Proof. Let $I = \begin{pmatrix} 0 & M \\ 0 & T \end{pmatrix}$. It is easy to see that I is an ideal and as a right ideal is essential, moreover, by hypothesis $I_R^{(n)}$ is weakly co-Hopfian for all $n \geq 1$. Thus $R_R^{(n)}$ is weakly co-Hopfian for all $n \geq 1$ by [4, Theorem 1.1], hence R is right strong stably finite. $\square$

At last, as a corollary of Proposition 3.2, we construct more examples of right strong stably finite rings. On the other hand, in [4, Proposition 2.2] it is shown that every right strong stably finite ring is stably finite, i.e., the matrix ring $M_n(R)$ is Dedekind-finite for all $n \geq 1$. By applying this we show that every subring of a right strong stably finite ring is stably finite. $\square$

Corollary 3.3. (i) *If T is a right strong stably finite ring, then so is $R = \begin{pmatrix} T & T \\ 0 & T \end{pmatrix}$.*
(ii) *Let S be a subring of a right strong stably finite ring T. Then S is*

stably finite.

Proof. (i). Apply Proposition 3.2 to the bimodule ${}_TT_T$.

(ii). By applying Proposition 3.2 to the bimodule ${}_ST_T$, the ring $R = \begin{pmatrix} S & T \\ 0 & T \end{pmatrix}$ is right strong stably finite, hence stably finite. Therefore S is stably finite by [3, Proposition 4.2]. $\square$

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