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AUTOMATIC CONTINUITY OF HOMOMORPHISMS BETWEEN BANACH ALGEBRAS AND FRÉCHET ALGEBRAS

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Abstract. In 1989, T. J. Ransford presented a short proof of Johnson's uniqueness of norm theorem. We follow the same method to show that if A and B are Fréchet algebras, B is semisimple and $T : A \to B$ is a surjective homomorphism with a certain condition, then T is continuous. In particular, when A is a Banach algebra we conclude that every epimorphism $T : A \longrightarrow B$ is automatically continuous and hence every semisimple Banach algebra has a unique topology as a Fréchet algebra, which is an extension of Johnson's uniqueness of norm theorem.

If A and B are Banach algebras, B is semisimple and $T : A \to B$ is a dense range homomorphism, then the continuity of T is a long-standing open question. In this work we give a positive answer to this open question with an extra condition on B and then present a partial answer to the well-known Michael's problem. We also obtain similar results for dense range homomorphisms of Fréchet algebras. Finally, we show that if the above question on the continuity of dense range homomorphisms of Banach algebras has a positive answer then the same question has a positive answer for Fréchet algebras.

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1. **Introduction**

In this section we give a brief outline of the definitions and known results. For further details one can refer to [2] and [4].

Definition 1.1. A normed algebra A is an algebra (over the complex field) with an algebra norm $\|\cdot\|$, i.e., $\|x.y\| \leq \|x\|.\|y\|$ for all $x, y \in A$. A complete normed algebra is called a Banach algebra.

Remark 1.2. In this paper we assume that all algebras are unital.

Definition 1.3. The *(Jacobson) radical* of an algebra A, denoted by rad$A$, is the intersection of all maximal left (right) ideals in A. The algebra A is called *semisimple* if rad$A = \{0\}$.

Definition 1.4. For the algebra A the spectrum $\sigma_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda.1 - x$ is not invertible in A. The spectral radius $r_A(x)$ of an element $x \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$.

If $(A, \|\cdot\|)$ is a Banach algebra (not necessarily commutative) then $r_A(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\|$. However, if $(A, \|\cdot\|)$ is a normed algebra which is not complete, then the inequality $r_A(x) \leq \|x\|$ may fail to hold.

It is known that for any algebra A we have

$$\text{rad}A = \{x \in A : r_A(xy) = 0, \text{ for every } y \in A\}.$$

The following lemma is due to T. J. Ransford [8].

Lemma 1.5. *Let A be a Banach algebra, $P(z)$ a polynomial with coefficients in A, and $R > 0$. Then*

$$r_A^2(P(1)) \leq \sup_{|z|=R} r_A(P(z)). \sup_{|z|=\frac{1}{R}} r_A(P(z)).$$

In the next sections we will extend the following known result to Fréchet algebras under certain conditions on A and B.

Theorem 1.6. [11; 11.10] *Let A and B be Banach algebras such that B is commutative and semisimple. Then every homomorphism $T : A \longrightarrow B$ is automatically continuous.*

Definition 1.7. Let A and B be metrizable topological linear spaces and let $T : A \longrightarrow B$ be a linear mapping. The *separating space* of T is defined by

$$\mathfrak{S}(T) = \{y \in B : \text{there exists } (x_n)_n \text{ in } A \; s.t. \; x_n \longrightarrow 0 \; and \; Tx_n \longrightarrow y\}.$$

The separating space $\mathfrak{S}(T)$ is a closed linear subspace of B and moreover, by the Closed Graph Theorem, T is continuous if and only if $\mathfrak{S}(T) = \{0\}$ [2, 5.1.2]. The following celebrated result is due to B. E. Johnson [6].

Theorem 1.8. [2, 5.1.5] *If A and B are Banach algebras and $T : A \longrightarrow B$ is an epimorphism then $\mathfrak{S}(T) \subseteq \mathrm{rad}B$. If, furthermore, B is semisimple then T is automatically continuous.*

By the above theorem, if A is a semisimple Banach algebra then A has a unique complete norm. This result is known as the Johnson's uniqueness-of-norm theorem.

In 1989 T. J. Ransford presented a short proof of Johnson's uniqueness of norm theorem in [8] by applying Lemma 1.5.

Definition 1.9. A Frećhet algebra is an algebra which is a complete metrizable topological linear space and has a neighbourhood basis $(V_n)_n$ of zero consisting of convex sets V_n such that $V_n.V_n \subseteq V_n$ for all $n \in \mathbb{N}$.

The topology of a Fréchet algebra A can be generated by a sequence $(p_n)_n$ of separating submultiplicative seminorms, i.e., $p_n(xy) \leq p_n(x)p_n(y)$ for all n and every $x, y \in A$, such that $p_n(x) \leq p_{n+1}(x)$ for all $x \in A$ and $n \in \mathbb{N}$. If A is unital then p_n can be chosen such that $p_n(1) = 1$. For further information one may refer to [4].

The Fréchet algebra A with the above generating sequence of seminorms is denoted by $(A, (p_n))$. Note that a sequence (x_k) in the Fréchet algebra $(A, (p_n))$ converges to $x \in A$ if and only if $p_n(x_k - x) \longrightarrow 0$, for each $n \in \mathbb{N}$, as $k \to \infty$.

If A is a commutative Fréchet algebra then $\mathrm{rad}A$ is the intersection of all closed maximal ideals, i.e., $\mathrm{rad}A = \bigcap_{\varphi \in M_A} \ker \varphi$ [4; Proposition 8.1.2].

In 1971 R. L. Carpenter extended the Johnson's Theorem by proving a similar result for the uniqueness of topology for commutative semisimple Fréchet algebras [1], [2, 4.10.17].

Definition 1.10. Let A be a Fréchet algebra. An element $x \in A$ is called quasi-invertible if there exists $y \in A$ such that $x + y - x.y = y + x - y.x = 0$. The Fréchet algebra A is called a *Q-algebra* if the set G of all quasi-invertible elements of A is open in A, or equivalently, if G has an interior point in A [7; Lemma E2].

Note that A is a *Q-algebra* if and only if the set of invertible elements of A is open in A. If $(A, (p_n))$ is a Fréchet algebra and A_n is the completion of the quotient algebra $A/\ker p_n$ with respect to the norm $p'_n(x + \ker p_n) = p_n(x)$, $x \in A$, then A_n is a Banach algebra.

Definition 1.11. A Fréchet algebra A is called *functionally continuous* if every non-zero complex homomorphism on A is continuous.

Clearly every Banach algebra is functionally continuous. It is also known that Q-algebras are functionally continuous [7; Lemma E4]. Actually, E. A. Michael posed the question in 1952 as whether each multiplicative linear functional on a commutative Fréchet algebra is automatically continuous [7]. This question, known as the Michael's problem, have been intensively studied, but only partial answers have been obtained so far. For example, it is a result of R. Arens that if A is finitely generated then every character (multiplicative linear functional) on A is continuous.

Recently Mustapha Laayouni presented a positive answer to the Michael's problem, which has been published in Bull. Belg. Math. Soc. Simon Stevin 8(2001), No. 1, 105-108. But a few months later it was announced that the proof of Laayouni breaks down somewhere and so the Michael's Problem still remains unanswered. There are several outstanding open automatic continuity problems for homomorphisms from C^*-algebras. For example:

1) Let A be a C^*-algebra, and B a semisimple Banach algebra. Is every dense range homomorphism $T : A \longrightarrow B$ continuous?
2) Let A be a C^*-algebra, and B a Banach algebra. Is every epimorphism $T : A \longrightarrow B$ continuous?

In 1993, V. Runde has shown that in both situations $T|_{Z(A)}$ is continuous, where $Z(A)$ is the centre of A, and so the answers to both questions are positive under certain conditions [10]. In 1995, A. Rodriguez Palacio proved that if A is a Banach algebra and B is an H^*-algebra with zero annihilator, then any dense range homomorphism $T : A \longrightarrow B$ is continuous [9]. The following is a long-standing open question in Banach algebras.

Question 1.12. [2, 5.1.A] Let A, B be Banach algebras, B semisimple and $T : A \longrightarrow B$ a dense range homomorphism, i.e., $\overline{T(A)} = B$. Is T automatically continuous? However, if we define $\mathfrak{Q}(B) = \{y \in B : r_B(y) = 0\}$, then the above question is equivalent to the following.

Question 1.13. [2, 5.1.A] Let A, B be Banach algebras and $T : A \longrightarrow B$ a homomorphism. Is it true that $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$?

We may now raise similar questions for Fréchet algebras.

Question 1.14. Let A be a Banach algebra and B be a semisimple Fréchet algebra. Let $T : A \longrightarrow B$ be a dense range homomorphism, i.e., $\overline{T(A)} = B$. Is T automatically continuous?

Question 1.15. Let A be a Banach algebra, B a Fréchet algebra, and $T : A \longrightarrow B$ a homomorphism. Is it true that $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$?

2. Uniqueness of topology for certain semisimple Fréchet algebras

In this section we obtain some results on the automatic continuity of surjective homomorphism on certain Fréchet algebras and then discuss the uniqueness of topology for particular classes of Fréchet algebras. The uniqueness of topology for some classes of semisimple Fréchet algebras, including Fréchet Q-algebras, has been discussed in [3].

First we present the following general result on Fréchet algebras. For the proof we follow the same technique as Ransford [8]. One may also refer to [12; 3.1] for similar results.

Theorem 2.1. *Let $(A, (p_k))$ and $(B, (q_k))$ be Fréchet algebras, B semisimple and B_k the completion of $B/\ker q_k$ with respect to the norm $q'_k(y + \ker q_k) = q_k(y)$, $y \in B$. If $T : A \longrightarrow B$ is a surjective homomorphism such that for each k and every $x \in A$, $r_{B_k}(Tx + \ker q_k) \leq p_k(x)$, then T is continuous.*

Proof. By the Closed Graph Theorem, for the continuity of T it is enough to show that for any sequence (a_n) in A if $a_n \longrightarrow 0$ in A and $Ta_n \longrightarrow b$ in B, then $b = 0$.

By surjectivity of T, there exists $a \in A$ such that $Ta = b$. We define $P_n(z) = zTa_n + T(a - a_n)$. Since for each $y \in B$, $r_{B_k}(y + \ker q_k) \leq q'_k(y + \ker q_k) = q_k(y)$, we have

$$r_{B_k}(P_n(z) + \ker q_k) \leq q_k(P_n(z)) \leq |z| q_k(Ta_n) + q_k(b - Ta_n).$$

On the other hand by the hypothesis we have

$$\begin{aligned} r_{B_k}(P_n(z) + \ker q_k) &= r_{B_k}(T(za_n + a - a_n) + \ker q_k) \\ &\leq p_k(za_n + a - a_n) \leq |z| p_k(a_n) + p_k(a - a_n). \end{aligned}$$

By Lemma 1.5 we can write

$$\begin{aligned} r^2_{B_k}(b + \ker q_k) &= r^2_{B_k}(P_n(1) + \ker q_k) \\ &\leq \sup_{|z|=R} r_{B_k}(P_n(z) + \ker q_k). \sup_{|z|=\frac{1}{R}} r_{B_k}(P_n(z) + \ker q_k) \\ &\leq (Rp_k(a_n) + p_k(a - a_n)(\frac{1}{R} q_k(Ta_n) + q_k(b - Ta_n)). \end{aligned}$$

We fix k and let $n \longrightarrow \infty$ to get

$$r^2_{B_k}(b + \ker q_k) \leq p_k(a). \frac{1}{R} q_k(b).$$

Now let $R \longrightarrow \infty$ to obtain $r_{B_k}(b + \ker q_k) = 0$ for each k and hence $r_B(b) = \sup_{k \in \mathbb{N}} r_{B_k}(b + \ker q_k) = 0$ [7; Corollary 5.3].

Let d be an arbitrary element of B and $Tc = d$ for some $c \in A$. Since $ca_n \longrightarrow 0$ in A and $T(ca_n) \longrightarrow db$ in B, by the same argument as above we conclude that $r_B(db) = 0$. Hence $b \in \mathrm{rad}B$ and so $b = 0$. This completes the proof of the theorem. □

Corollary 2.2. *Let $(A, \|\cdot\|)$ be a Banach algebra and $(B, (q_k))$ a semisimple Fréchet algebra, which is not necessarily commutative. If $T : A \longrightarrow B$ is a surjective homomorphism then T is automatically continuous.*

Proof. Since $\sigma_B(Tx) \subseteq \sigma_A(x)$, for each $x \in A$, we have

$$r_{B_k}(Tx + \ker q_k) \leq r_B(Tx) \leq r_A(x) \leq \|x\|,$$

for all k$\geq$ 1. Hence the conditions of the above theorem are satisfied and so T is automatically continuous. □

The following result is a generalization of the Johnson's Theorem on the uniqueness of norm for semisimple Banach algebras .

Corollary 2.3. *If B is a semisimple Banach algebra then it has a unique topology as a Fréchet algebra.*

Proof. It is an immediate consequence of Corollary 2.2. □

There may be classes of Fréchet algebras, which are more general than Banach algebras, and satisfy the conditions of the theorem. For example, it has been shown in [3] that if A is a semisimple Fréchet Q-algebra then it has a unique topology as a Fréchet algebra.

Note that if A is a Q-algebra and B is a semisimple commutative Fréchet algebra, then every homomorphism $T : A \longrightarrow B$, (not necessarily surjective) is continuous. In fact, we have the following more general result, which can be found in [12; Theorem 3.1.1]. For the proof one can use the Closed Graph Theorem.

Theorem 2.4. *Let A be a functionally continuous Fréchet algebra and B a semisimple commutative Fréchet algebra. Then every homomorphism $T : A \longrightarrow B$ is automatically continuous.*

3. **Continuity of dense range homomorphisms**

If A and B are Banach algebras and $T : A \longrightarrow B$ is a dense range homomorphism, i.e., $\overline{T(A)} = B$, then it is known that $\mathfrak{S}(T)$ is a closed (two sided) ideal in B [2, 5.1.3]. However, this result can be extended to Fréchet algebras.

Theorem 3.1. *Let A and B be Fréchet algebras and $T : A \longrightarrow B$ a dense range homomorphism. Then the separating space $\mathfrak{S}(T)$ is a closed (two sided) ideal in B.*

Proof. Clearly $\mathfrak{S}(T)$ is a closed linear subspace of B. We first show that it is an ideal in $D = T(A)$. Let $y \in \mathfrak{S}(T)$ and $z \in D$. There exists a sequence $(x_n)_n$ in A such that $x_n \longrightarrow 0$ and $Tx_n \longrightarrow y$. Moreover, $z = Tx$ for some $x \in A$. Since in Fréchet algebras the operation of multiplication of elements is continuous, we conclude that $xx_n \longrightarrow 0$ and

$T(xx_n) = TxTx_n \longrightarrow zy$ and so $zy \in \mathfrak{S}(T)$. Similarly $yz \in \mathfrak{S}(T)$. Hence $\mathfrak{S}(T)$ is an ideal in D.

Now we show that $\mathfrak{S}(T)$ is, in fact, an ideal in $\overline{D} = B$. For $y \in \mathfrak{S}(T)$ and $z \in B = \overline{D}$ there exist sequences $(x_n)_n$, and $(z_n)_n$ in A and D, respectively, such that $x_n \longrightarrow 0$ in A, $z_n \longrightarrow z$, and $Tx_n \longrightarrow y$ in B. Since yz_n and $z_n y$ are both in $\mathfrak{S}(T)$ and $yz_n \longrightarrow yz$, $z_n y \longrightarrow zy$, it follows that $yz, zy \in \overline{\mathfrak{S}(T)} = \mathfrak{S}(T)$. Thus $\mathfrak{S}(T)$ is an ideal in B. □

Theorem 3.2. *Let A and B be Fréchet algebras such that B is semisimple, the spectral radius r_B is continuous on B and the spectral radius r_A is continuous at zero. If $T : A \longrightarrow B$ is a dense range homomorphism, i.e., $\overline{T(A)} = B$, then T is automatically continuous.*

Proof. For every $y \in \mathfrak{S}(T)$ there exists a sequence $(x_n)_n$ in A such that $x_n \longrightarrow 0$ in A and $Tx_n \longrightarrow y$ in B. Since $r_B(Tx) \leq r_A(x)$ for every $x \in A$ and $r_A(x_n) \longrightarrow 0$, we have $r_B(Tx_n) \longrightarrow 0$. On the other hand, $r_B(Tx_n) \longrightarrow r_B(y)$. Hence $r_B(y) = 0$.

Since $\overline{T(A)} = B$, by Theorem 3.1, $\mathfrak{S}(T)$ is an ideal in B. Thus for every $z \in B$, $yz \in \mathfrak{S}(T)$. By the above argument we conclude that $r_B(yz) = 0$. Since $\text{rad}B = \{y \in B : r_B(yz) = 0 \text{ for every } z \in B\}$, we conclude that $y \in \text{rad}B$. Therefore, $\mathfrak{S}(T) \subseteq \text{rad}B$. Since B is semisimple, $\mathfrak{S}(T) = \{0\}$ and so T is continuous. □

Corollary 3.3. *If A is a Fréchet algebra such that the spectral radius r_A is continuous at zero and if B is a semisimple commutative Banach algebra, then any surjective homomorphism $T : A \longrightarrow B$ is automatically continuous.*

Proof. Since in commutative Banach algebras the spectral radius is uniformly continuous, it is clear that T is continuous. □

Corollary 3.4. *If A is a Fréchet algebra such that r_A is continuous at zero, then every complex homomorphism $T : A \longrightarrow \mathbb{C}$ is continuous, i.e., A is functionally continuous. Therefore, the Michael's problem (question) has a positive answer for those Fréchet algebras whose spectral radii are continuous functions at zero.*

Remark 3.5. (i) In Banach algebras the spectral radius is always continuous at zero, but it may be discontinuous at other points. See an example due to Dixon in [2, 2.3.15]. However, if A is a commutative

Banach algebra, then the spectral radius r_A is continuous at all points of A.
(ii) In commutative Fréchet algebras, the spectral radius may be discontinuous at zero. An interesting example is the commutative Fréchet algebra $C(\mathbb{R})$, whose spectral radius is not continuous at zero.

However, if A is a commutative Fréchet algebra then $r_A(x+y) \leq r_A(x) + r_A(y)$ for all $x, y \in A$ and so $|r_A(x) - r_A(y)| \leq r_A(x-y)$ for all $x, y \in A$. Hence the continuity of r_A at zero implies the continuity of r_A at all points of A.

Theorem 3.6. *Let A and B be Banach algebras and $T : A \longrightarrow B$ a homomorphism such that r_B is continuous on $\mathfrak{S}(T)$. Then $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$, where $\mathfrak{Q}(B) = \{y \in B : r_B(y) = 0\}$.*

Proof. For $y \in \mathfrak{S}(T)$ there exists a sequence $(x_n)_n$ in A such that $x_n \longrightarrow 0$ in A and $Tx_n \longrightarrow y$ in B. By the continuity of r_B on $\mathfrak{S}(T)$, $r_B(Tx_n) \longrightarrow r_B(y)$. On the other hand, $r_B(Tx_n) \leq r_A(x_n) \leq \|x_n\|$ for all $n \in \mathbb{N}$. Hence $r_B(Tx_n) \longrightarrow 0$ and so $r_B(y) = 0$. Therefore, $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$. □

Corollary 3.7. *If A and B are Banach algebras and $T : A \longrightarrow B$ is a dense range homomorphism such that r_B is continuous on $\mathfrak{S}(T)$, then $\mathfrak{S}(T) \subseteq \mathrm{rad}B$. Therefore, T is continuous if B is semisimple.*

Proof. Since $\mathfrak{S}(T)$ is an ideal and $\mathrm{rad}B = \{y \in B : r_B(yz) = 0$ for all $z \in B\}$, the result follows from the theorem. □

Theorem 3.8. *Questions 1.14 and 1.15 are equivalent.*

Proof. Let Question 1.14 has a positive answer. Let A be a Banach algebra, B a Fréchet algebra, and $T : A \longrightarrow B$ a homomorphism. Clearly $D = \overline{T(A)}$ is a Fréchet subalgebra of B and $D/\mathrm{rad}D$ is a semisimple Fréchet algebra. If $\pi : D \longrightarrow D/\mathrm{rad}D$ is the canonical mapping, defined by $\pi(y) = y + \mathrm{rad}D$, then $S = \pi \circ T : A \longrightarrow D/\mathrm{rad}D$ is a dense range homomorphism. To see this let $z = y + \mathrm{rad}D \in D/\mathrm{rad}D$. Since $y \in D$, there exists a sequence $(x_n)_n$ in A such that $Tx_n \longrightarrow y$ in D. Since π is continuous, $Sx_n = \pi(Tx_n) \longrightarrow \pi(y) = y + \mathrm{rad}D$. Hence $\overline{S(A)} = D/\mathrm{rad}D$ and so by the hypothesis S is continuous.

For every $y \in \mathfrak{S}(T)$ there exists a sequence in A such that $x_n \longrightarrow 0$ in A and $Tx_n \longrightarrow y$ in B. Since $y \in D$ and π is continuous on D,

$Sx_n = \pi(Tx_n) \longrightarrow \pi(y)$. Thus $\pi(y) = 0$ and so $y \in \mathrm{rad}D$. Since $r_B(z) \leq r_D(z)$ for every $z \in D$, we have $\mathfrak{Q}(D) \subseteq \mathfrak{Q}(B)$. Therefore, $\mathfrak{S}(T) \subseteq \mathrm{rad}D \subseteq \mathfrak{Q}(D) \subseteq \mathfrak{Q}(B)$, and consequently Question 1.15 has a positive answer.

Conversely, let Question 1.15 have a positive answer, and let A be a Banach algebra, B a semisimple Fréchet algebra and $T : A \longrightarrow B$ a dense range homomorphism. By hypothesis $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$, and by Theorem 3.1 $\mathfrak{S}(T)$ is an ideal in B. Hence $\mathfrak{S}(T) \subseteq \mathrm{rad}B = \{0\}$ and so T is continuous. □

Theorem 3.9. *If Question 1.13 has a positive answer then Question 1.15 has also a positive answer.*

Proof. Let A be a Banach algebra, $(B, (q_k))$ a Fréchet algebra with the generating sequence of seminorms $(q_k)_k$ in B and $T : A \longrightarrow B$ a homomorphism. Let B_k be the completion of $B/\ker q_k$ with the norm $q'_k(y + \ker q_k) = q_k(y)$. Let $\pi_k : B \longrightarrow B_k$ be the canonical mapping, defined by $\pi_k(y) = y + \ker q_k$, which is a continuous homomorphism. Clearly $T_k = \pi_k \circ T : A \longrightarrow B_k$ is a homomorphism and $\mathfrak{S}(T_k) \subseteq \mathfrak{Q}(B_k)$, by the hypothesis.

Let $y \in \mathfrak{S}(T)$. There exists a sequence $(x_n)_n$ in A such that $x_n \longrightarrow 0$ in A and $Tx_n \longrightarrow y$ in B. Hence $T_k x_n = \pi_k(Tx_n) \longrightarrow \pi_k(y) = y + \ker q_k \in B_k$ and so $y + \ker q_k \in \mathfrak{Q}(B_k)$, i.e., $r_{B_k}(y + \ker q_k) = 0$. By [7, Corollary 5.3] $r_B(y) = \sup_{k\in\mathbb{N}} r_{B_k}(y + \ker q_k) = 0$ and so $y \in \mathfrak{Q}(B)$, which implies that $\mathfrak{S}(T) \subseteq \mathfrak{Q}(B)$. This completes the proof of the theorem. □

As a consequence of Theorems 3.8 and 3.9 we also obtain the following interesting result.

Theorem 3.10. *If Question 1.12 has a positive answer then Question 1.14 has also a positive answer.*

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