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THE GEOMETRIC PROPERTIES OF A DEGENERATE PARABOLIC EQUATION WITH PERIODIC SOURCE TERM

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Abstract. In this paper, we discuss the geometric properties of solution and lower bound estimate of Δu^{m-1} of the Cauchy problem for a degenerate parabolic equation with periodic source term $u_t = \Delta u^m + u^p \sin t$. Our objective is to show that: (1) with continuous variation of time t, the surface $\phi = [u(x,t)]^{\frac{m\delta}{q}}$ is a complete Riemannian manifold floating in space $\mathbb{R}^{N+1}$ and is tangent to the space $\mathbb{R}^N$ at $\partial H_0(t)$; (2) the surface $u = u(x,t)$ is tangent to the hyperplane $W(t)$ at $\partial H_u(t)$.
Keywords: Degenerate parabolic equation, Riemannian manifold, periodic source term.
MSC(2010): Primary: 35A30; Secondary: 35K45, 35K55.

1. **Introduction**

Consider the Cauchy problem of nonlinear degenerate parabolic equation with periodic source term:

$$\begin{cases} u_t = \Delta u^m + u^p \sin t & (x,t) \in Q, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

where $Q = \mathbb{R}^N \times \mathbb{R}^+$, $N \geqslant 1$, $m > 1$, $p \geqslant m+2$, $0 \leqslant u_0(x) \leqslant M$, and

$$0 < M < [2(p-1)]^{-\frac{1}{p-1}},\ 0 < \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x < \infty. \tag{1.2}$$

We know that degenerate parabolic equation has a wide range of applications in mathematics, physics, chemistry, biology, and many other fields. Especially the well-known seepage equation $u_t = \Delta u^m$ ($m > 1$) has received a wide and deep study in the past few decades and also obtained abundant results in existence, uniqueness, regularity, stability, blow-up property, the continuous dependence of nonlinear properties of solutions and large-time behavior of

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the solutions, etc([1–4, 7–11, 13]). With the deepening of the research, many scholars began to study the case with source term, namely

$$u_t = \Delta u^m + f(u).$$

In the progress of studying such problem with the term $f(u) = u^p$ $(p > 0)$, they found that the range of parameter p plays an important role in portraying the asymptotic behavior of the solutions, such as the large time behavior, blow-up property and extinguishing of the solutions. Among those, the pioneering work is established by Fujita in 1961 ([5]). He studied the Cauchy problem of the equation

$$u_t = \Delta u + u^p$$

in $\mathbb{R}^N$ and obtained the following results:

(i) If $1 < p < 1 + \frac{2}{N}$, there is no existence of non-negative and nontrivial global solution;

(ii) If $p > 1 + \frac{2}{N}$, there is the non-negative global solution when the initial value is sufficiently small.

If $f(u) = -u^p$ $(p > 0)$, we may visually foresee that with the increase of time, the solution of equation should be decayed till it is extinguished. But as to the singular diffusion equation and degenerate parabolic equation, there is a big difference between them. In case of singular diffusion equation $(0 < m < 1)$, Raual Ferreira and Juan Vazquez ([6]) discussed the extinguishing properties of solution for Cauchy problem

$$\begin{cases} u_t = (u^m)_{xx} - u^p & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

on one dimension, where $p < 1$. The result they got is

$$u(x,t) \le [(1-p)(T-t)]^{\frac{1}{1-p}}.$$

In case of degenerate parabolic equation $(m > 1)$, Pan ([12]) discussed the existence, uniqueness and large-time behavior of the non-negative solutions of following problem

$$\begin{cases} u_t = \Delta u^m - \theta u^p & (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R}^N. \end{cases}$$

He proved that the solution will not be extinguished even if with absorbing term and he also gained the lower bound estimate of solution.

In this paper, we discuss the Cauchy problem for a degenerate parabolic equation with periodic source term (1.1), (1.2). We say $u(x,t)$ is the solution of the problem (1.1), (1.2), if $u \in C\left([0,\infty) : L^1\left(\mathbb{R}^N\right)\right)$ and satisfying:

(i) $u_t \in L^1_{loc}\left((0,\infty) : L^1\left(\mathbb{R}^N\right)\right)$;

(ii) $u_t = \Delta u^m + u^p \sin t$ in the sense of distributions in Q;

(iii) $\|u(x,t) - u_0(x)\|_{L^1(\mathbb{R}^N)} \to 0,\ t \to 0.$

With the condition (1.2), it is easy to know $M^{1-p}-(p-1)(1-\cos t)>0$. Considering the function

$$\overline{u}(x,t)=\left[M^{1-p}-(p-1)(1-\cos t)\right]^{-\frac{1}{p-1}},$$

we can easily verify that $\overline{u}(x,t)$ satisfies the equation (1.1) and

$$\overline{u}(x,0)=M\geqslant u_0(x).$$

Therefore $\overline{u}(x,t)$ is an upper solution of the problem (1.1), (1.2). In addition, because of $\sin t\geqslant -1$, we get $\underline{u}(x,t)=0$ is a lower solution of the problem (1.1), (1.2) by using literature [12]. Consequently from [14] we can obtain the problem (1.1), (1.2) has global solution in Q and

$$0\leqslant u(x,t)\leqslant\left[M^{1-p}-(p-1)(1-\cos t)\right]^{-\frac{1}{p-1}}\leqslant L. \tag{1.3}$$

where $L=\left[M^{1-p}-2\,(p-1)\right]^{-\frac{1}{p-1}}$. Especially, by the standard parabolic equation theory, we know that $u(x,t)$ can be regarded as the limitation of classical solutions of the problem

$$\begin{cases} u_t=\Delta u^m+u^p\sin t & (x,t)\in Q,\\ u(x,0)=u_0(x)+\eta & x\in\mathbb{R}^N\end{cases} \tag{1.4}$$

$u_\eta(x,t)$: $u(x,t)=\lim\limits_{\eta\to0^+}u_\eta(x,t)$. Hypothesizing u, $\widetilde{u}$ are two solutions of (1.1), (1.2) in Q_T, by taking use of the same method used in [12] Lemma 2.1, we get

$$\begin{aligned}\int_{\mathbb{R}^N}\left[u\left(x,t\right)-\tilde{u}\left(x,t\right)\right]_+\zeta_n dx &\leqslant \int_{\mathbb{R}^N}\left[u\left(x,t_0\right)-\tilde{u}\left(x,t_0\right)\right]_+\zeta_n dx\\ &+\frac{\gamma}{n^2}\int_{t_0}^{t}\int_{\mathbb{R}^N}|w|dxdt\\ &+\int_{t_0}^{t}\int_{\mathbb{R}^N}\left[u^p-\tilde{u}^p\right]_+\zeta_n dxdt.\end{aligned} \tag{1.5}$$

and

$$\int_{\mathbb{R}^N}|u^p-\tilde{u}^p|dx\leqslant\left(L^{p-1}\int_{\mathbb{R}^N}|u_0-\tilde{u}_0|\,dx\right)e^{pL^{p-1}t}. \tag{1.6}$$

We immediately obtain the uniqueness and continuous dependence of solution about initial value from (1.5), (1.6). Because $u\left(x,t\right)\in C^\infty\left(Q_+\right)$ ([1, 2, 3, 4, 7]), where

$$Q_+=\left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}^+\,|u(x,t)>0\right\},$$

and $u(x,t)$ is nonnegative, we divide the space $Q=\mathbb{R}^N\times\mathbb{R}^+$ into two parts: $Q=Q_+\cup Q_0$,

$$Q_0=\left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}^+\,|u(x,t)=0\right\}.$$

Further, if Q_0 contains open set Q_1, we can still get $u\left(x,t\right)\in C^\infty\left(Q_1\right)$ because of $u\left(x,t\right)\equiv0$ in Q_1. Hence, we expect that the solution of degenerate parabolic equation is smooth in everywhere except zero measure set in Q.

For any $t\in\mathbb{R}^+$, we define hyperplane $W(t)$ which fluctuates periodically in $\mathbb{R}^{N+1}$ with continuous change of the time t:

$$W(t):\begin{cases} x_i=x_i,\ i=1,2,3,...,N,\\ x_{N+1}=\left[M^{1-p}-(p-1)\left(1-\cos t\right)\right]^{-\frac{1}{1-p}},\ t\in\mathbb{R}^+\end{cases}$$

and the surface $S(t)$ which floats in space $\mathbb{R}^{N+1}$ with continuous change of the time t:

$$S\left(t\right):\begin{cases} x_i=x_i,\ i=1,2,3,...,N,\\ x_{N+1}=\phi(x,t),\ x\in\mathbb{R}^N.\end{cases}$$

where $\phi=\left[u(x,t)\right]^{\frac{m\delta}{q}},\ \delta>1$. Take

$$\begin{cases} g_1=\left(1,0,...,\frac{\partial\phi}{\partial x_1}\right),\\ g_2=\left(0,1,...,\frac{\partial\phi}{\partial x_2}\right),\\ ......,\\ g_n=\left(0,0,...,n,\frac{\partial\phi}{\partial x_n}\right).\end{cases}$$

and define Riemann measure $(ds)^2=\sum\limits_{i,j=1}^{n}g_{ij}dx_idx_j$ on $S(t)$, where $g_{ij}=g_i\cdot g_j$. Furthermore, we set

$$H_u(t)=\left\{x\left|x\in\mathbb{R}^N,\ u(x,t)<\left[M^{1-p}-(p-1)\left(1-\cos t\right)\right]^{-\frac{1}{1-p}}\right.\right\}\ (t>0),$$

$$H_0(t)=\left\{x\left|x\in\mathbb{R}^N,\ u(x,t)>0\right.\right\}\ (t>0).$$

If $u\left(x,t\right)\in C\left(\mathbb{R}^N\right)$, it is easy to konw

$$\partial H_u(t)=\left\{x\left|x\in\mathbb{R}^N,\ u(x,t)=\left[M^{1-p}-(p-1)\left(1-\cos t\right)\right]^{-\frac{1}{1-p}}\right.\right\}\ (t>0),$$

$$\partial H_0(t)=\left\{x\left|x\in\mathbb{R}^N,\ u(x,t)=0\right.\right\}\ (t>0).$$

We state our main conclusions below:

Theorem 1.1. *If $u(x,t)$ is the solution of (1.1), (1.2) in Q, then for any given $t>0$, $u\left(x,t\right)\in C\left(\mathbb{R}^N\right)$; setting $v=\frac{m}{m-1}u^{m-1}$, there exists the constant β_0, β_1, β_2 which are irrelevant to the time t such that*

$$\Delta v\geqslant-\left[\beta_1+\beta_2\left(e^{\beta_0 t}-1\right)^{-1}\right]. \tag{1.7}$$

Theorem 1.2. *(1) For any $t\in\mathbb{R}^+$, $\delta>1$, the surface $S(t)$ is the complete Riemannian manifold in space $\mathbb{R}^{N+1}$, and it is tangent to the space $\mathbb{R}^N$ at lower-dimension maniflod $\partial H_0(t)$; (2) The surface $u=u(x,t)$ is tangent to the hyperplane $W(t)$ at $\partial H_u(t)$, the maximum value of $u(x,t)$.*

In this paper, we use the symbols:

$$Q = \mathbb{R}^N \times \mathbb{R}^+;\ Q_T = \mathbb{R}^N \times (0, T)\,.$$

2. **Preliminary lemmas**

Lemma 2.1. *If the function $u(x,t)$ is the solution of (1.1), (1.2) in Q, then there exists constants C_1, C_2 which are only relevant to m, L such that*

$$\left|\nabla u^{\frac{m}{q}}\right| \leqslant \sqrt{\frac{C_2}{C_1}\frac{1}{1-e^{-C_2 t}}},\ (x,t) \in Q \tag{2.1}$$

in the sense of distributions, where $q \in \left(1, \frac{m}{m-1}\right)$.

Proof. For arbitrary positive number T, set

$$V = {u_\eta}^{\frac{m}{q}},\ (x,t) \in Q_T,$$

where u_η is the solution of (1.4) and $q \in \left(1, \frac{m}{m-1}\right)$. Then

$$V_t = m{u_\eta}^{m-1}\Delta V + m(q-1){u_\eta}^{m-1}\frac{|\nabla V|^2}{V} + \frac{m\sin t}{q}V^{\frac{pq-q+m}{m}}.$$

Let $h_j = \frac{\partial V}{\partial x_j}$, $j = 1, 2, ..., N$. We differentiate the above formula respectively to x_j and multiply by h_j, then we have

$$\begin{aligned} \tfrac{1}{2}\left({h_j}^2\right)_t &= m{u_\eta}^{m-1}h_j\Delta h_j + q\,(m-1)\,{u_\eta}^{m-2}V^{\frac{q}{m}-1}{h_j}^2\Delta V \\ &+ h_j m(q-1)\tfrac{\left[(m-1)\frac{q}{m}{u_\eta}^{m-2}V^{\frac{q}{m}-1}h_j|\nabla V|^2 + 2{u_\eta}^{m-1}\nabla V\cdot\nabla h_j\right]}{V} \\ &- h_j m(q-1)\tfrac{{u_\eta}^{m-1}|\nabla V|^2 h_j}{V^2} + \tfrac{\theta(pq-q+m)\sin t}{q}{u_\eta}^{p-1}{h_j}^2. \end{aligned}$$

for $j = 1, 2, ..., N$. Letting $H^2 = \sum\limits_{j=1}^{N}{h_j}^2$, we get

$$\begin{aligned} \left(H^2\right)_t &= m{u_\eta}^{m-1}\Delta H^2 + 2q\,(m-1)\,{u_\eta}^{m-1-\frac{m}{q}}H^2\Delta V \\ &+ 2m(q-1)u_\eta^{m-1-\frac{m}{q}}\nabla V\cdot\nabla H^2 \\ &+ 2\,(q-1)\,[q\,(m-1)-m]\,u^{m-1-\frac{2m}{q}}H^4 \\ &+ \frac{2\theta\,(pq-q+m)\sin t}{q}{u_\eta}^{p-1}H^2 - 2m{u_\eta}^{m-1}\sum_{j=1}^{N}\left|\nabla {h_j}^2\right|. \end{aligned} \tag{2.2}$$

From $p > 1,\ m > 1,\ q \in \left(1, \frac{m}{m-1}\right)$, we get $2\,(q-1)\,[q\,(m-1)-m] < 0,\ m - 1 - \frac{2m}{q} < 0,\ \frac{2(pq-q+m)}{q} > 0$. Thus through (1.3) we obtain

$$2\,(q-1)\,[q\,(m-1)-m]\,u^{m-1-\frac{2m}{q}} \leqslant -C_1,\ \frac{2\,(pq-q+m)}{q}u_\eta^{p-1}\,|\sin t| \leqslant C_2,$$

where

$$C_1 = 2(q-1)\left|q(m-1)-m\right| L^{m-1-\frac{2m}{q}},\ C_2 = \frac{2(pq-q+m)}{q} L^{p-1}.$$

Putting above formulas into (2.2), we get

$$\begin{aligned}(2.3)\qquad \left(H^2\right)_t &\leqslant mu_\eta^{m-1}\Delta H^2 + 2q(m-1)\,u_\eta^{m-1-\frac{m}{q}} H^2\Delta V \\ &+ 2m(q-1)u_\eta^{m-1-\frac{m}{q}}\nabla V\cdot\nabla H^2 - C_1H^4 + C_2H^2.\end{aligned}$$

Considering the ordinary differential equation(ODE)

$$\frac{d}{dt}\tilde{h}^2 = -C_1\tilde{h}^4 + C_2\tilde{h}^2,$$

it is easy to verify that the function $\tilde{h}^2 = \frac{C_2}{C_1}\frac{1}{1-e^{-C_2t}}$ is a solution to the equation with initial condition $\tilde{h}^2(t)\,|_{t=0} = +\infty$. Setting operator

$$\begin{aligned}L(H^2) &= mu_\eta^{m-1}\Delta H^2 + 2q(m-1)\,u_\eta^{m-1-\frac{m}{q}} H^2\Delta V \\ &+ 2m(q-1)u_\eta^{m-1-\frac{m}{q}}\nabla V\cdot\nabla H^2 - C_1H^4 + C_2H^2.\end{aligned}$$

then from (2.3) we obtain

$$\left(H^2\right)_t \leqslant L(H^2).$$

On the other hand, function $\tilde{h}^2 = \frac{C_2}{C_1}\frac{1}{1-e^{-C_2t}}$ is satisfied to the equation $\left(\tilde{h}^2\right)_t = L(\tilde{h}^2)$. Therefor, by comparison principle, we get

$$H^2 \leqslant \tilde{h}^2 = \tilde{h}^2 = \frac{C_2}{C_1}\frac{1}{1-e^{-C_2t}},$$

namely

$$\left|\nabla u_\eta^{\frac{m}{q}}\right| \leqslant \sqrt{\frac{C_2}{C_1}\frac{1}{1-e^{-C_2t}}}.$$

Thus (2.1) holds about u_η. Finally, letting $\eta \to 0$ gives the result of our lemma. □

3. **Proof of Theorem 1.2**

Proof. For arbitrary given $\tau > 0$, in (2.1) we get

$$\begin{aligned}(3.1)\qquad \left|\left[u(x_1,t)\right]^{\frac{m}{q}} - \left[u(x_2,t)\right]^{\frac{m}{q}}\right| &\leqslant \sqrt{\frac{C_2}{C_1}\frac{1}{1-e^{-C_2t}}}\,|x_1-x_2| \\ &\leqslant \sqrt{\frac{C_2}{C_1}\frac{1}{1-e^{-C_2\tau}}}\,|x_1-x_2|\,.\end{aligned}$$

for arbitrary $(x_1,t)\,,\ (x_2,t) \in \mathbb{R}^N\times[\tau,T)$. Hence, $u^{\frac{m}{q}} \in C\left(\mathbb{R}^N\right)$ holds for arbitrary $t>0$. In addition, because the function $y = s^{\frac{m}{q}}$ is strictly monotone increasing respect to $s>0$, its inverse function is also strictly monotone increasing respect to $s>0$. Thus we obtain that $u\in C\left(\mathbb{R}^N\right)$ holds for arbitrary $t>0$.

Owing to u_η satisfying (1.4) in the sense of classics, letting $v_\eta=\frac{m}{m-1}{u_\eta}^{m-1}$ and putting v_η into (1.1), we have

$$(v_\eta)_t=(m-1)v_\eta\Delta v_\eta+|\nabla v_\eta|^2+m(\frac{m-1}{m})^{\frac{m+p-2}{m-1}}{v_\eta}^{\frac{m+p-2}{m-1}}\sin t.$$

Let $q=\Delta v_\eta$, then

$$\begin{aligned}(3.2)\qquad q_t &= (m-1)v_\eta\Delta q+2m\nabla v_\eta\nabla q+(m-1)q^2\\ &+ 2\sum_{i,j=1}^{N}\left(\frac{\partial^2 v_\eta}{\partial x_i\partial x_j}\right)^2+\left[(m+p-2)\,u_\eta^{p-1}\theta\sin t\right]q\\ &+ \frac{(p-1)\,(m+p-2)}{m}u_\eta^{p-m}|\nabla v_\eta|^2\theta\sin t.\end{aligned}$$

Because of

$$\sum_{i,j=1}^{N}(a_{ij})^2\geqslant\sum_{i=1}^{N}(a_{ii})^2\geqslant\frac{1}{N}\left(\sum_{i=1}^{N}a_{ii}\right)^2,$$

and from (1.3), we get that there is a constant C_4 irrelevant to t such that

$$\left|(m+p-2)\,u_\eta^{p-1}\sin t\right|\leqslant C_4.$$

Moreover, it is easy to know $\frac{\sin t}{1-e^{-C_2t}}\to\frac{1}{C_2}$ when $t\to0^+$. Thus from (1.3), (2.1) we obtain a constant C_5 irrelevant to t such that

$$\begin{aligned}&\left|\frac{(p-1)\,(m+p-2)}{m}u_\eta^{p-m}|\nabla v_\eta|^2\sin t\right|\\ \leqslant\ &\frac{C_2q^2}{C_1}{u_\eta}^{m+p-2-\frac{2m}{q}}\frac{(p-1)\,(m+p-2)}{m}\left|\frac{\sin t}{1-e^{-C_2t}}\right|\leqslant C_5.\end{aligned}$$

Finally, from we get

$$(3.3)\qquad q_t\geqslant(m-1)v_\eta\Delta q+2m\nabla v_\eta\nabla q+C_3q^2-C_4q-C_5,$$

where $C_3=m-1+\frac{2}{N}$. Considering the ODE

$$(3.4)\qquad \frac{dQ}{dt}=C_3Q^2-C_4Q-C_5.$$

Clearly, the function $Q\left(t\right)=\frac{C_4\left(1-e^{\sqrt{\Lambda}t}\right)+\sqrt{\Lambda}\left(1+e^{\sqrt{\Lambda}t}\right)}{2C_3\left(1-e^{\sqrt{\Lambda}t}\right)}=-\left[\beta_1+\beta_2\left(e^{\beta_0t}-1\right)^{-1}\right]$ is a solution to (3.4) and satisfies the initial condition $Q\left(t\right)|_{t=0}=-\infty$, where $\Lambda={C_4}^2+4C_3C_5$, $\beta_0=\sqrt{\Lambda}$, $\beta_1=\frac{\sqrt{\Lambda}-C_4}{2C_3}$, $\beta_2=\frac{\sqrt{\Lambda}}{C_3}$. Setting operator

$$L\left(q\right)=(m-1)v\Delta q+2m\nabla v\nabla q+C_3q^2-C_4q-C_5.$$

Then from (3.3) we obtain

$$q_t \geqslant L(q)\,.$$

On the other hand, the function $Q(t) = -\left[\beta_1 + \beta_2\left(e^{\beta_0 t} - 1\right)^{-1}\right]$ is satisfied to the equation $Q_t = L(Q)$. Therefor, by comparison principle, we get

$$q \geqslant Q(t) = -\left[\beta_1 + \beta_2\left(e^{\beta_0 t} - 1\right)^{-1}\right],$$

namely, (1.7) holds with respect to v_η. Finally, letting $\eta \to 0$ gives the result of our theorem. □

4. **Proof of Theorem 1.2**

We prove the theorem 1.2 by two steps.

Proof. For arbitrary given $t > 0,\ \delta > 1$, setting

$$\phi(x,t) = [u(x,t)]^{\frac{m\delta}{q}},$$

from (2.1) we get

$$|\nabla\phi(x,t)| \leqslant \delta u^{\frac{m(\delta-1)}{q}}\sqrt{\frac{C_2}{C_1}\frac{1}{1-e^{-C_2 t}}}. \tag{4.1}$$

For arbitrary given $t > 0$, we define Riemannian manifold $S(t)$ in space $\mathbb{R}^{N+1}$ as above, then we obtain

$$(d\rho)^2 \leqslant (ds)^2 \leqslant \left(1 + \max_{i=1,2,\dots,n}\left|\frac{\partial\phi}{\partial x_i}\right|^2\right)(d\rho)^2,$$

where $(d\rho)^2 = \sum\limits_{i=1}^{n}(dx_i)^2$ is Euclidean metric in $\mathbb{R}^N$.
From (1.3) and (4.1), we know there exist the constant K_0, K_1 such that $\max\limits_{i=1,2,\dots,n}\left|\frac{\partial u}{\partial x_i}\right|^2 \leqslant K_1\left(1 - e^{-K_0 t}\right)^{-1}$. Thus we have

$$(d\rho)^2 \leqslant (ds)^2 \leqslant \left[1 + K_1\left(1 - e^{-K_0 t}\right)^{-1}\right](d\rho)^2. \tag{4.2}$$

For $\mathbb{R}^N$ is complete, we get $S(t)$ is complete from (4.2), namely, the surface $\phi = [u(x,t)]^{\frac{m\delta}{q}}$ is complete Riemannian manifold in $\mathbb{R}^{N+1}$ for arbitrary given $t > 0$.

Besides, for arbitrary $x_* \in \partial H_0(t)$, we obtain $\nabla\phi(x_*,t)=0$ from (4.1). Combining (3.1) with (4.1), we have

$$\begin{aligned}
|\nabla\phi(x,t)-\nabla\phi(x_*,t)| &= |\nabla\phi(x,t)| \\
&\leqslant \delta\sqrt{\frac{C_2}{C_1}\frac{1}{1-e^{-C_2 t}}}\left|[u(x,t)]^{\frac{m}{q}}\right|^{\delta-1} \\
&\leqslant \delta\left(\frac{C_2}{C_1}\frac{1}{1-e^{-C_2 t}}\right)^{\frac{\delta}{2}}|x-x_*|^{\delta-1}.
\end{aligned} \tag{4.3}$$

Then we can get $\nabla\phi(x,t)\in C(\partial H_0(t))$ from (4.3). Moreover, we can obtain $u(x,t)\in C^\infty(H_0(t))$ through the standard parabolic regularity theory. Thus we have $\phi(x,t)\in C^1(\mathbb{R}^N)$. In addition, because of the continuity of $u(x,t)$, we get that $H_0(t)$ is open set for arbitrary given $t>0$. Hence when $x_*\in\partial H_0(t)$, we have

$$\phi(x_*,t)=\nabla\phi(x_*,t)=0.$$

Therefore, the surface $\phi=[u(x,t)]^{\frac{m\delta}{q}}$ is tangent to $\mathbb{R}^N$ at $\partial H_0(t)$. In other words, $\mathbb{R}^N$ is just the tangent plane of $S(t)$ at $\partial H_0(t)$. □

Proof. At first, we can get $u(x,t)\in C^1(\mathbb{R}^N)$ from the conclusion $\phi(x,t)\in C^1(\mathbb{R}^N)$. By (4.1) and $u\in L^1(\mathbb{R}^N)$, we can obtain $u(x,t)\to 0$ when $x\to\infty$ through the conclusions well known in functional analysis. Therefore, for arbitrary $T>0$ and every given $t\in(0,T)$, there exists a point $x(t)\in\mathbb{R}^N$ such that $u(x,t)$ reaches the maximum $u(x(t),t)$ at the point $(x(t),t)$, and

$$\nabla u(x(t),t)=0,\ \Delta u(x(t),t)\leqslant 0,\ t\in(0,T). \tag{4.4}$$

Setting $U(t)=\max_{x\in\mathbb{R}^N}u(x,t)$, putting it into the equation (1.1), we have

$$U_t\leqslant U^p\sin t.$$

Note that $p>1$, we have

$$\left(U^{1-p}\right)_t\geqslant(1-p)\sin t.$$

Integrating above formula from 0 to t, we get

$$U(t)\leqslant\left[M^{1-p}-(p-1)(1-\cos t)\right]^{-\frac{1}{p-1}}.$$

In addition, because of the continuity of $u(x,t)$, we obtain that $H_u(t)$ is open set for arbitrary given $t>0$, thus when $x_*\in\partial H_u(t)$, we have

$$u(x_*,t)=\left[M^{1-p}-(p-1)(1-\cos t)\right]^{-\frac{1}{p-1}}, \tag{4.5}$$

namely, $u(x,t)$ reaches the maximum $\left[M^{1-p}-(p-1)(1-\cos t)\right]^{-\frac{1}{p-1}}$ at the point (x_*,t). Therefore

$$\nabla u(x_*,t)=0. \tag{4.6}$$

Finally, we obtain that the surface $u = u(x,t)$ is tangent to $W(t)$ at $\partial H_u(t)$ through (4.5), (4.6). In other words, the manifold $W(t)$ is just the tangent plane of the surface $u = u(x,t)$ at $\partial H_u(t)$. □ □

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