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CONSTRUCTING VERTEX DECOMPOSABLE GRAPHS

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Abstract. Recently, some techniques such as adding whiskers and attaching graphs to vertices of a given graph, have been proposed for constructing a new vertex decomposable graph. In this paper, we present a new method for constructing vertex decomposable graphs. Then we use this construction to generalize the result due to Cook and Nagel.
Keywords: Finite graph, well-covered graph, independence complex, edge ideal, vertex decomposable graph.
MSC(2010): Primary: 13H10; Secondary: 05E45, 13F55, 05E40.

1. Introduction

Adding a whisker to a graph G at one of its vertices, x, means adding a new vertex y and edge xy to G. In [9] Villarreal showed that if a whisker is added to every vertex of G, then the resulting graph denoted by G^W is Cohen-Macaulay. It is not difficult to show that this graph is very well-covered. Furthermore, Dochtermann and Engstom [3, Theorem 4.4] showed that the independence complex of G^W is vertex-decomposable. Also, Cook and Nagel [2] showed that if we choose a partition $\pi = \{W_1, \ldots, W_t\}$ of cliques for $V(G)$, add new vertices $y_1, \ldots, y_t$, and connect y_i to every vertices in the clique W_i for $1 \leq i \leq t$, then the new graph denoted by G^π will be well-covered and vertex decomposable. G^π is called clique-whiskering of G.

Let G be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. Also, let $G_1, \ldots, G_n$ be simple graphs with disjoint vertex sets and $v_i \in V(G_i)$ for $1 \leq i \leq n$. If we define H with $V(H) = V(G) \cup (\bigcup_{i=1}^n (V_i \setminus \{v_i\}))$ and $E(H) = E(G) \cup (\bigcup_{i=1}^n E(G_i \setminus \{v_i\})) \cup \{x_i v | v_i v \in E(G_i)\}$, then H is the graph obtained by attaching G_i to G on the vertex v_i for all $i = 1, \ldots, n$. Recently, Hibi et al. [5, Theorem 1.1] showed that by attaching any complete graph to the vertices of G, one can construct a well-covered and vertex decomposable graph. Mousivand et al. [6, proposition 3.2] generalized this result by proving that

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attaching vertex decomposable graphs to each vertices of a given graph on one of its shedding vertices, a new vertex decomposable graph is constructed. They also showed that this new graph is well-covered if and only if for any graph G' which is attached to G in its shedding vertex v, both G' and $G' \setminus \{v\}$ are well-covered.

This paper is organized as follows: in Section 2, we refer to some definitions and notions which will be needed later. In Section 3, a new approach is presented for constructing a vertex decomposable graph. We show that if u and v are respectively shedding vertices of vertex decomposable graphs G and $G \setminus N_G[u]$, then $G \cup \{uv\}$ is again a vertex decomposable graph. Also, if G is well-covered, then $G \cup \{uv\}$ is a well-covered graph, see Theorem 3.3.

As a special case, for a given graph G with $V(G) = \{x_1, \dots, x_n\}$ and the vertex decomposable graphs $G_1, \dots, G_n$ with disjoint vertex sets, if u_i is a shedding vertex of G_i, then $H = G_1 \cup G_2 \cup \dots \cup G_n \cup \{u_i u_j | x_i x_j \in E(G)\}$ is a vertex decomposable graph. One can see that H is isomorphic to the graph that obtained from G by attaching G_i to the vertices of G in shedding vertices u_i.

As a result of our paper we prove the following result, see Theorem 3.8:

Let G be a graph and $V(G) = \{W_1, \dots, W_t\}, W_i = \{w_{i_1}, \dots, w_{i_{m_i}}\}$ be a clique partition for its vertex set. Let $G_1, G_2, \dots, G_t$ be in the class $\mathcal{SQC}$ such that G_i $(1 \leq i \leq t)$ contains a simplicial vertex y_i with the property that $N_{G_i}(y_i) = \{u_{i_1}, \dots, u_{i_{n_i}}\}$ which $n_i \geq m_i$ (see Section 2 for the definition). Then $H = G_1 \cup \dots \cup G_t \cup \{u_{i_j} u_{k_l} | w_{i_j} w_{k_l} \in E(G)\}$ is well-covered and vertex decomposable.

This is a generalisation of a result of Cook and Nagel [2, Theorem 3.3], see Corollary 3.10.

2. **Preliminaries**

In this section we recall some concepts and notations on simplicial complexes and graphs that will be used in this article.

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The *edge ideal* of G is defined as the ideal $I(G) \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ generated by all monomials $x_i x_j$ whenever $x_i x_j \in E(G)$, where $\mathbb{K}$ is a field. Also, the *edge ring* of G is $\mathbb{K}[G] = R/I(G)$. A graph G is called *(sequentially) Cohen-Macaulay*, if its edge ring is a (sequentially) Cohen-Macaulay ring.

A subgraph H of a graph G is said to be *induced* if, for any pair of vertices u and v of H, $uv \in E(H)$ if and only if $uv \in E(G)$. The induced subgraph of G restricted to $A \subseteq V(G)$ is denoted by G_A. The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u \mid u \in V(G),\, vu \in E(G)\}$. Similarly, for $A \subseteq V(G)$, we have $N_G(A) = \cup_{v \in A} N_G(v)$ and $N_G[A] = A \cup N_G(A)$.

For $u, v \in V(G)$, the graph with the vertex set $V(G)$ and the edge set $E(G) \cup \{uv\}$ is denoted by $G \cup \{uv\}$. Also the induced subgraph of G on the vertex set $V(G) \setminus A$ is denoted by $G \setminus A$. If the induced subgraph of G on W is a complete subgraph, then $W \subseteq V(G)$ is called *clique*. For two graphs G and H, the graph on vertex set $V(G \cup H) = V(G) \cup V(H)$ with edge set $E(G \cup H) = E(G) \cup E(H)$ is denoted by $G \cup H$.

An *independent* set of G is a set of pairwise non-adjacent vertices of $V(G)$. An independent set of maximum size will be referred to as a maximum independent set of G. Also, the independence number of G, denoted by $\alpha(G)$, is the cardinality of a maximum independent set of G. Moreover, the graph G is called *well-covered* (or equivalently *unmixed*) if all its maximal independent sets are of the same size. Furthermore, if G has no isolated vertices and $|V(G)| = 2\alpha(G)$, then G is *very well-covered*.

A 5-cycle C as a subgraph of a graph G is called basic, if it does not contain two adjacent vertices of degree three or more in G. A vertex x is called *simplicial vertex* if $N_G[x]$ is a clique and it is called a simplex of G. A 4-cycle Q as a subgraph of a graph G is called basic, if it contains two adjacent vertices of degree two, and the remaining two vertices belong to a simplex or a basic 5-cycle of G. G is in class $\mathcal{SQC}$ if there are simplicial vertices $x_1, \ldots, x_m$; basic 5-cycles $C_1, \ldots, C_s$; and basic 4-cycles $Q_1, \ldots, Q_t$ such that $N_G[x_1], \ldots, N_G[x_m]$, $V(C_1), \ldots, V(C_s)$, and $B(Q_1), \ldots, B(Q_t)$ be a partition of $V(G)$, where $B(Q_j)$ is the set of two vertices of degree 2 of the basic 4-cycle Q_j. The class $\mathcal{SQC}$ is a subclass of the class of well-covered graphs [7, Theorem 3.1] and every graph in this class is vertex decomposable [4, Theorem 2.3].

A *simplicial complex* Δ, on a finite set V, is a set of subsets of V closed under inclusion. The elements of Δ are called faces of Δ and the maximum faces with respect to inclusion are called facets of Δ.

Let $\sigma \in \Delta$, the *link* and the *deletion* of σ from Δ are given by $\operatorname{link}_\Delta \sigma := \{\tau \in \Delta | \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$ and $\operatorname{del}_\Delta \sigma := \{\tau \in \Delta | \sigma \nsubseteq \tau\}$. The *independence complex* of a graph G, denoted by $\operatorname{Ind}(G)$, is the simplicial complex whose faces are the independent sets of G.

3. **Vertex decomposable graphs**

A simplicial complex Δ is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex v so that (1) both $\Delta \setminus v$ and $\operatorname{link}_\Delta v$ are vertex decomposable, and (2) no face of $\operatorname{link}_\Delta v$ is a facet of $\Delta \setminus v$. The vertex decomposable complexes were introduced in the pure case by Provan and Billera [8] and extended to non-pure complexes by Bjöner and Wachs [1].

We call a graph G vertex decomposable if the independence complex $\operatorname{Ind}(G)$ is vertex decomposable. In [10] Woodroofe translated the definition of vertex decomposability for graphs as follow.

Definition 3.1. A graph G is called vertex decomposable if either it is an edgeless graph or it has a vertex x such that:

(i) $G \setminus \{x\}$ and $G \setminus N_G[x]$ are both vertex-decomposable.

(ii) For every independent set S of $G\setminus N_G[x]$, there is some vertex $y \in N_G(x)$ such that $S \cup \{y\}$ is an independent set of $G \setminus \{x\}$.

The vertex x of G that satisfies conditions (i) and (ii) is called a *shedding* vertex of G.

Remark 3.2. *Our definition of shedding vertex is similar to that one in [6]. This definition is slightly different from the definition in [10], where a shedding vertex is a vertex which satisfies only condition (ii).*

The first main result of this section is the following theorem which provides a new method for constructing vertex decomposable graphs.

Theorem 3.3. *Let G be a vertex decomposable graph and u one of its shedding vertex. If v is a shedding vertex of $G \setminus N_G[u]$, then*

(a) *$G \cup \{uv\}$ is again a vertex decomposable graph.*

(b) *If G is well-covered, then $G \cup \{uv\}$ is well-covered too, and*

$$\alpha(G \cup \{uv\}) = \alpha(G).$$

Proof. (a) Let $G' = G \cup \{uv\}$, we show that u is a shedding vertex of G'. Since $G' \setminus \{u\} = G \setminus \{u\}$, $G' \setminus \{u\}$ is vertex decomposable. Since v is a shedding vertex of $G \setminus N_G[u]$ and $G' \setminus N_{G'}[u] = G \setminus (N_G[u] \cup \{v\}) = (G \setminus N_G[u]) \setminus \{v\}$, $G' \setminus N_{G'}[u]$ is vertex decomposable. So, u satisfies the condition (i) of Definition 3.1.

Now, let S be an independent set of $G' \setminus N_{G'}[u]$. It is clear that $G' \setminus N_{G'}[u] = G \setminus (N_G[u] \cup \{v\})$, so, S is independent set of $G \setminus N_G[u]$. Since u is a shedding vertex of G, there exists $y \in N_G(u)$ such that $S \cup \{y\}$ is an independent set of $G \setminus \{u\}$. Since $y \neq v$, $S \cup \{y\}$ is an independent set of $G' \setminus \{u\}$. Therefore, u is a shedding vertex of G' and so, G' is vertex decomposable.

b) Let S be a maximal independent set of G'. We show that S is a maximal independent set of G. Consider the following three cases:

Case 1 ($v \in S$ and $u \notin S$): By contrary assume that S is not a maximal independent set of G. Therefore $S \cup \{u\}$ must be an independent set of G, so, $S \cap N_G[u] = \emptyset$ and it means that S is an independent set of $G \setminus N_G[u]$. Since u is shedding vertex of G, there exists a vertex $x \in N_G(u)$ such that $S \cup \{x\}$ is an independent set of G and hence is an independent set of G', which contradicts the maximality of S.

Case 2 ($v \notin S$ and $u \in S$): As in the case 1, by contrary assume that $S \cup \{v\}$ is an independent set of G, so, $S \cap N_G(v) = \emptyset$. On the other hand $u \in S$, so, $S \cap N_G(u) = \emptyset$. Therefor, $S \setminus \{u\}$ is independent set of $(G \setminus N_G[u]) \setminus N_G[v]$. Since v is shedding vertex of $G \setminus N_G[u]$, there is a vertex $y \in N_G(v)$ such that $(S \setminus \{u\}) \cup \{y\}$ and so, $S \cup \{y\}$ is independent set of G and G', which again contradicts the maximality of S.

Case 3 ($u \notin S$ and $v \notin S$): It is clear that S is a maximal independent of G. So, G' is well-covered too.

In addition, we have $\alpha(G \cup \{uv\}) = \alpha(G)$. □

The following example is remarkable.

Example 3.4. *In Figure 3.1, G is a chordal graph and hence vertex decomposable. The vertex u is a shedding vertex of G and v is a shedding vertex of $G \setminus N_G[u]$. As it can be seen, $G \cup \{uv\}$ is a vertex decomposable graph which contains C_6. Note that by adding an edge between x and y, the graph is converted to C_8, which is not vertex decomposable.*

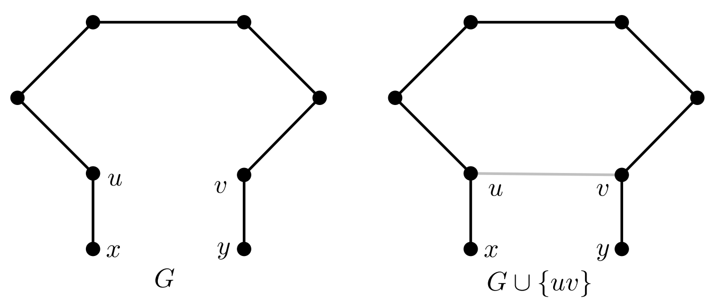


FIGURE 3.1.

Let $G_1, G_2, \ldots, G_n$ be connected component of a vertex decomposable graph G, by [10, Lemma 20] G_i is vertex decomposable and so, it has a shedding vertex u_i for $1 \leq i \leq n$. In the proof of Theorem 3.3 we showed that u_i is shedding vertex of $G \cup \{u_i u_j\}$. Hence, By applying Theorem 3.3 $H = G_1 \cup G_2 \cup \ldots \cup G_n \cup \{u_i u_j | i \neq j\}$, which is a vertex decomposable graph. In addition, if G is a well-covered graph, then H will be a well-covered graph, too. As a special case, for a given graph G with $V(G) = \{x_1, \ldots, x_n\}$ and the vertex decomposable graphs $G_1, \ldots, G_n$ with disjoint vertex sets, if u_i is a shedding vertex of G_i, then $H = G_1 \cup G_2 \cup \ldots \cup G_n \cup \{u_i u_j | x_i x_j \in E(G)\}$ is a vertex decomposable graph. One can see that H is isomorphic to the graph that obtained from G by attaching G_i to the vertices of G in shedding vertices u_i. It means that attaching G_i and G_j to the vertices x_i and x_j of G, which $x_i x_j \in E(G)$, in shedding vertices $u_i \in V(G_i)$ and $u_j \in V(G_j)$ is the same as saying that adding the edge $u_i u_j$ to the graph $\cup_{i=1}^{n} G_i$. As an example consider the following one.

Example 3.5. *In the Figure 3.2, there are four vertex decomposable graphs $G_1, \ldots, G_4$ which contain the shedding vertices $u_1, \ldots, u_4$, respectively. These graphs have been connected according to edges of the graph G. To this end, we add the edge $u_i u_j$ if $x_i x_j \in E(G)$. Using this method, the vertex decomposable graph $H = G_1 \cup G_2 \cup G_3 \cup G_4 \cup \{u_1 u_2, u_2 u_3, u_3 u_4, u_2 u_4\}$ have been constructed. One can see, H, as well as G_1, is not well-covered. If G_1 is replaced with a* *well-covered graph, then H will convert to a well-covered graph, too. One can see, H is isomorphic to the graph obtained by attaching vertex decomposable graphs $G_1, \ldots, G_4$ to each vertices of G.*

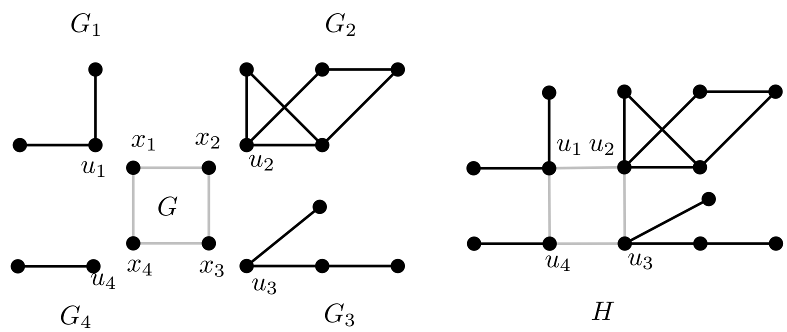


Figure 3.2.

Let G be a graph and $V(G) = \{x_1, \ldots, x_n\}$. If we set $H = (\cup_{i=1}^{i=n}\{u_i v_i\}) \cup \{u_i u_j | x_i x_j \in E(G)\}$, then H is isomorphic to G^W. Since K_2 is a well-covered vertex decomposable graph and its vertices are shedding vertices, so we have the following result.

Corollary 3.6. *[3, Theorem 4.4] For every graph G, G^W is v well-covered and vertex decomposable.*

Since the complete graphs are well-covered and vertex decomposable, and their vertices are shedding vertices, we have the following corollary:

Corollary 3.7. *[5, Theorem 1.1] Let G be a finite simple graph on a vertex set $V = \{x_1, \ldots, x_n\}$ and Let $k_1, \ldots, k_n \geq 2$ be integers. Then the graph G' obtained from G by attaching the complete graph K_{k_i} to x_i for $i = 1, \ldots, n$ is well-covered and vertex decomposable.*

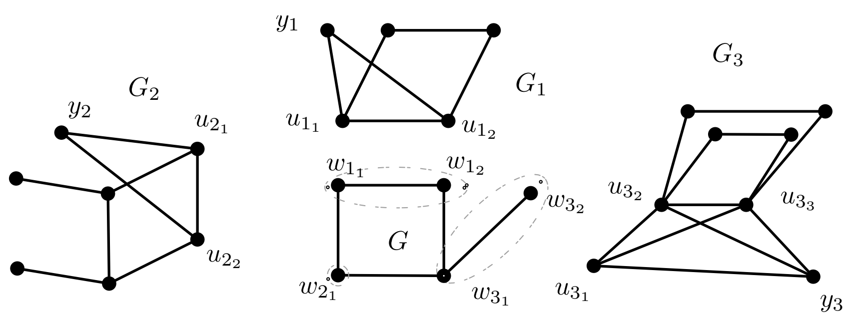
In the following theorem, a new construction is justified to generalize clique-whiskering of a graph.

Theorem 3.8. *Let G be a graph and $V(G) = \{W_1, \ldots, W_t\}, W_i = \{w_{i_1}, \ldots, w_{i_{m_i}}\}$ be a clique partition for its vertex set. Let $G_1, G_2, \ldots, G_t$ be in class $\mathcal{SQC}$ such that G_i $(1 \leq i \leq t)$ contains a simplicial vertex y_i with the property that $N_{G_i}(y_i) = \{u_{i_1}, \ldots, u_{i_{n_i}}\}$ which $n_i \geq m_i$. Then $H = G_1 \cup \ldots \cup G_t \cup \{u_{i_j} u_{k_l} | w_{i_j} w_{k_l} \in E(G)\}$ is well-covered and vertex decomposable.*

Proof. Let u belongs to the neighborhood of a simplicial vertex x_i of G_i. It is not difficult to see that $G_i \setminus \{u\}$ and $G \setminus N_G[u]$ belong to $\mathcal{SQC}$ and for any independent set of $G \setminus N_G[u]$, $S \cup \{x_i\}$ is independent of G. So, u is a shedding vertex of G_i. On the other hand, if v belongs to the neighborhood of a simplicial vertex of G_j for $i \neq j$, then v is shedding vertex of $G \setminus N_G[u]$. Hence by Theorem 3.3, we can add an edge between these two vertices and the resulting graph is well-covered and vertex decomposable. □

The above theorem is illustrated using following example.

Example 3.9. *In the Figure 3.3* (a), $\pi = \{\{w_{1_1}, w_{1_2}\}, \{w_{2_1}\}, \{w_{3_1}, w_{3_2}\}\}$ *is a partition for* $V(G)$ *and* $y_1 \in G_1$, $y_2 \in G_2$, *and* $y_3 \in G_3$ *are simplicial vertices. It is easy to see that* $N_{G_1}(y_1) = \{u_{1_1}, u_{1_2}\}$; $N_{G_2}(y_2) = \{u_{2_1}, u_{2_2}\}$; $N_{G_3(y_3)} = \{u_{3_1}, u_{3_2}, u_{3_2}\}$. *In the Figure 3.3* (b), *as can be seen,* $H = G_1 \cup G_2 \cup G_3 \cup \{u_{1_2}u_{3_1}, u_{1_1}u_{2_1}, u_{2_1}u_{3_1}\}$. *In fact,* H *is isomorphic to the graph obtained by attaching vertex decomposable graphs* G_1, G_2, G_3 *to the cliques of* G *in the neighborhoods of their simplicial vertices.*



(a)

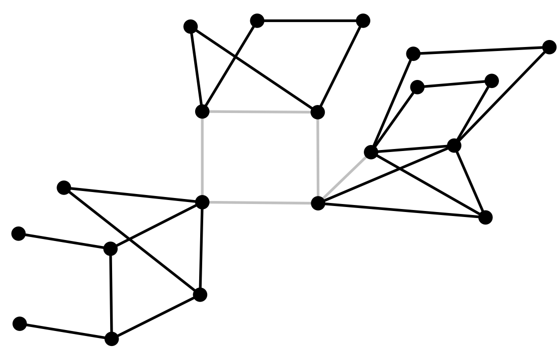


Figure 3.3. (b)

As a corollary of our result we have the following:

Corollary 3.10. *[2, Theorem 3.3] For every graph G and any partition $\pi = \{W_1, \ldots, W_t\}$ of cliques for $V(G)$, G^π is well-covered and vertex decomposable.*

Proof. Let $W_i = \{w_{i_1}, \ldots, w_{i_{m_i}}\}$, corresponding to any W_i we consider a complete graph $K_{|W_i|+1}$ with vertex set $\{u_{i_1}, \ldots, u_{i_{m_i}}, w_i\}$ that is a well-covered and vertex decomposable, so, $H = \bigcup_{i=1}^{i=t} K_{|W_i|+1} \cup \{u_{i_j}u_{k_l} | w_{i_j}w_{k_l} \in E(G)\} \cong G^\pi$ is well-covered and vertex decomposable. □

Remark 3.11. (a). *Theorem 3.8 remains true if the clique $\{u_{i_1}, \ldots, u_{i_{n_i}}\}$ is a subset of $N_{G_i}(\{y_i^1, \ldots, y_i^{t_i}\})$ where y_i^j is a simplicial vertex for $1 \leq j \leq t_i$.*

(b). *If W_i have simplicial vertex for some i, then G_i can be the subgraph of G induced by W_i. In fact, we attach vertex decomposable graphs just at the cliques that have not simplicial vertex and construct well-covered vertex decomposable graphs.*

The above remark is illustrated using the following example:

Example 3.12. *In the Figure 3.4, $\pi = \{\{w_{1_1}, w_{1_2}\}, \{w_{2_1}, w_{2_2}\}, \{w_{3_1}, w_{3_2}\}\}$ is a partition of $V(G)$. G_1 and G_2 are two vertex decomposable graphs which $u_{2_1} \in N_{G_2}(y_{2_1})$ and $u_{2_2} \in N_{G_2}(y_{2_2})$. Also, G_3 is the subgraph of G induced by W_3(because $w_{3_2} \in W_3$ is a simplicial vertex). So, $H = G_1 \cup G_2 \cup G_3 \cup \{u_{1_1}u_{2_1}, u_{1_2}u_{2_2}, w_{3_1}u_{2_2}\}$ which it means that we attach vertex decomposable graphs just at W_1 and W_2.*

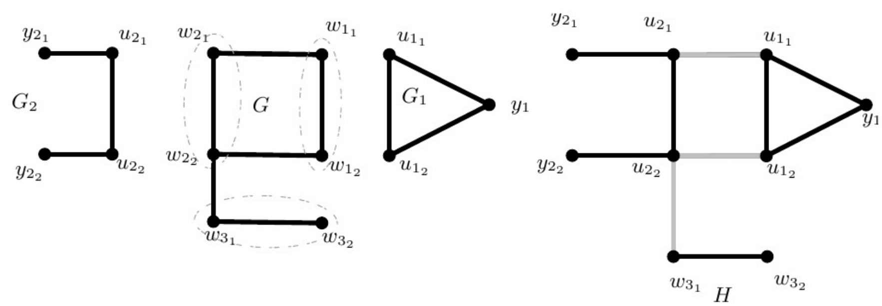


Figure 3.4.

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