Title:
Constructing vertex decomposable graphs

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CONSTRUCTING VERTEX DECOMPOSABLE GRAPHS

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Abstract. Recently, some techniques such as adding whiskers and attaching graphs to vertices of a given graph, have been proposed for constructing a new vertex decomposable graph. In this paper, we present a new method for constructing vertex decomposable graphs. Then we use this construction to generalize the result due to Cook and Nagel.

Keywords: Finite graph, well-covered graph, independence complex, edge ideal, vertex decomposable graph.


1. Introduction

Adding a whisker to a graph $G$ at one of its vertices, $x$, means adding a new vertex $y$ and edge $xy$ to $G$. In [9] Villarreal showed that if a whisker is added to every vertex of $G$, then the resulting graph denoted by $G^W$ is Cohen-Macaulay. It is not difficult to show that this graph is very well-covered. Furthermore, Dochtermann and Engstrom [3, Theorem 4.4] showed that the independence complex of $G^W$ is vertex-decomposable. Also, Cook and Nagel [2] showed that if we choose a partition $\pi = \{W_1, \ldots, W_t\}$ of cliques for $V(G)$, add new vertices $y_1, \ldots, y_t$, and connect $y_i$ to every vertices in the clique $W_i$ for $1 \leq i \leq t$, then the new graph denoted by $G^\pi$ will be well-covered and vertex decomposable. $G^\pi$ is called clique-whiskering of $G$.

Let $G$ be a simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$. Also, let $G_1, \ldots, G_n$ be simple graphs with disjoint vertex sets and $v_i \in V(G_i)$ for $1 \leq i \leq n$. If we define $H$ with $V(H) = V(G) \cup (\bigcup_{i=1}^{n} (V_i \setminus \{v_i\}))$ and $E(H) = E(G) \cup (\bigcup_{i=1}^{n} E(G_i \setminus \{v_i\})) \cup \{x_iv_i, v_i \in E(G_i)\}$, then $H$ is the graph obtained by attaching $G_i$ to $G$ on the vertex $v_i$ for all $i = 1, \ldots, n$. Recently, Hibi et al. [5, Theorem 1.1] showed that by attaching any complete graph to the vertices of $G$, one can construct a well-covered and vertex decomposable graph. Mousivand et al. [6, proposition 3.2] generalized this result by proving that...
attaching vertex decomposable graphs to each vertices of a given graph on one of its shedding vertices, a new vertex decomposable graph is constructed. They also showed that this new graph is well-covered if and only if for any graph $G'$ which is attached to $G$ in its shedding vertex $v$, both $G'$ and $G' \setminus \{v\}$ are well-covered.

This paper is organized as follows: in Section 2, we refer to some definitions and notions which will be needed later. In Section 3, a new approach is presented for constructing a vertex decomposable graph. We show that if $u$ and $v$ are respectively shedding vertices of vertex decomposable graphs $G$ and $G \setminus N_G[u]$, then $G \cup \{uv\}$ is again a vertex decomposable graph. Also, if $G$ is well-covered, then $G \cup \{uv\}$ is a well-covered graph, see Theorem 3.3.

As a special case, for a given graph $G$ with $V(G) = \{x_1, \ldots, x_n\}$ and the vertex decomposable graphs $G_1, \ldots, G_n$ with disjoint vertex sets, if $u_i$ is a shedding vertex of $G_i$, then $H = G_1 \cup G_2 \cup \ldots \cup G_n \cup \{u_i, u_j | x_i, x_j \in E(G)\}$ is a vertex decomposable graph. One can see that $H$ is isomorphic to the graph that obtained from $G$ by attaching $G_i$ to the vertices of $G$ in shedding vertices $u_i$.

As a result of our paper we prove the following result, see Theorem 3.8:

Let $G$ be a graph and $V(G) = \{W_1, \ldots, W_t\}$, $W_i = \{w_{i1}, \ldots, w_{in}\}$ be a clique partition for its vertex set. Let $G_1, G_2, \ldots, G_t$ be in the class $\mathcal{SQC}$ such that $G_i$ (1 $\leq i \leq t$) contains a simplicial vertex $y_i$ with the property that $N_{G_i}(y_i) = \{u_{i1}, \ldots, u_{in}\}$ which $n_i \geq m_i$ (see Section 2 for the definition). Then $H = G_1 \cup \ldots \cup G_t \cup \{u_i, w_{i1} | w_{i1}, w_{i1} \in E(G)\}$ is well-covered and vertex decomposable.

This is a generalisation of a result of Cook and Nagel [2, Theorem 3.3], see Corollary 3.10.

2. Preliminaries

In this section we recall some concepts and notations on simplicial complexes and graphs that will be used in this article.

Let $G$ be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The edge ideal of $G$ is defined as the ideal $I(G) \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ generated by all monomials $x_ix_j$ whenever $x_ix_j \in E(G)$, where $\mathbb{K}$ is a field. Also, the edge ring of $G$ is $\mathbb{K}[G] = R/I(G)$. A graph $G$ is called (sequentially) Cohen-Macaulay, if its edge ring is a (sequentially) Cohen-Macaulay ring.

A subgraph $H$ of a graph $G$ is said to be induced if, for any pair of vertices $u$ and $v$ of $H$, $uv \in E(H)$ if and only if $uv \in E(G)$. The induced subgraph of $G$ restricted to $A \subseteq V(G)$ is denoted by $G_A$. The neighborhood of a vertex $v \in V$ is the set $N_G(v) = \{u \mid u \in V(G), vu \in E(G)\}$. Similarly, for $A \subseteq V(G)$, we have $N_G(A) = \cup_{v \in A} N_G(v)$ and $N_G[A] = A \cup N_G(A)$.

For $u, v \in V(G)$, the graph with the vertex set $V(G)$ and the edge set $E(G) \cup \{uv\}$ is denoted by $G \cup \{uv\}$. Also the induced subgraph of $G$ on the
vertex set \( V(G) \setminus A \) is denoted by \( G \setminus A \). If the induced subgraph of \( G \) on \( W \) is a complete subgraph, then \( W \subseteq V(G) \) is called clique. For two graphs \( G \) and \( H \), the graph on vertex set \( V(G \cup H) = V(G) \cup V(H) \) with edge set \( E(G \cup H) = E(G) \cup E(H) \) is denoted by \( G \cup H \).

An independent set of \( G \) is a set of pairwise non-adjacent vertices of \( V(G) \). An independent set of maximum size will be referred to as a maximum independent set of \( G \). Also, the independence number of \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum independent set of \( G \). Moreover, the graph \( G \) is called well-covered (or equivalently unmixed) if all its maximal independent sets are of the same size. Furthermore, if \( G \) has no isolated vertices and \( |V(G)| = 2\alpha(G) \), then \( G \) is very well-covered.

A 5-cycle \( C \) as a subgraph of a graph \( G \) is called basic, if it does not contain two adjacent vertices of degree three or more in \( G \). A vertex \( x \) is called simplicial vertex if \( N_G[x] \) is a clique and it is called a simplex of \( G \). A 4-cycle \( Q \) as a subgraph of a graph \( G \) is called basic, if it contains two adjacent vertices of degree two, and the remaining two vertices belong to a simplex or a basic 5-cycle of \( G \). \( G \) is in class \( \mathcal{SQC} \) if there are simplicial vertices \( x_1, \ldots, x_m \); basic 5-cycles \( C_1, \ldots, C_s \); and basic 4-cycles \( Q_1, \ldots, Q_t \) such that \( N_G[x_1], \ldots, N_G[x_m], V(C_1), \ldots, V(C_s), \) and \( B(Q_1), \ldots, B(Q_t) \) be a partition of \( V(G) \), where \( B(Q_j) \) is the set of two vertices of degree 2 of the basic 4-cycle \( Q_j \). The class \( \mathcal{SQC} \) is a subclass of the class of well-covered graphs \( \mathcal{G} \), Theorem 3.1] and every graph in this class is vertex decomposable \( \mathcal{G} \), Theorem 2.3].

A simplicial complex \( \Delta \), on a finite set \( V \), is a set of subsets of \( V \) closed under inclusion. The elements of \( \Delta \) are called faces of \( \Delta \) and the maximum faces with respect to inclusion are called facets of \( \Delta \).

Let \( \sigma \in \Delta \), the link and the deletion of \( \sigma \) from \( \Delta \) are given by link\( \Delta \sigma := \{ \tau \in \Delta | \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta \} \) and del\( \Delta \sigma := \{ \tau \in \Delta | \sigma \not\subseteq \tau \} \). The independence complex of a graph \( G \), denoted by \( \text{Ind}(G) \), is the simplicial complex whose faces are the independent sets of \( G \).

3. Vertex decomposable graphs

A simplicial complex \( \Delta \) is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex \( v \) so that (1) both \( \Delta \setminus v \) and link\( \Delta v \) are vertex decomposable, and (2) no face of link\( \Delta v \) is a facet of \( \Delta \setminus v \). The vertex decomposable complexes were introduced in the pure case by Provan and Billera [8] and extended to non-pure complexes by Björner and Wachs [1].

We call a graph \( G \) vertex decomposable if the independence complex \( \text{Ind}(G) \) is vertex decomposable. In [10] Woodroofe translated the definition of vertex decomposability for graphs as follow.

**Definition 3.1.** A graph \( G \) is called vertex decomposable if either it is an edgeless graph or it has a vertex \( x \) such that:

(i) \( G \setminus \{x\} \) and \( G \setminus N_G[x] \) are both vertex-decomposable.
our definition of shedding vertex is similar to that one in \cite{G10}.

Let \( S \) be a maximal independent set of \( G \) and \( x \in S \) is a shedding vertex of \( G \).

Remark 3.2. Our definition of shedding vertex is similar to that one in \cite{G10}. This definition is slightly different from the definition in \cite{G10}, where a shedding vertex is a vertex which satisfies only condition (ii).

The first main result of this section is the following theorem which provides a new method for constructing vertex decomposable graphs.

**Theorem 3.3.** Let \( G \) be a vertex decomposable graph and \( u \) one of its shedding vertex. If \( v \) is a shedding vertex of \( G \setminus N_G[u] \), then

(a) \( G \cup \{uv\} \) is again a vertex decomposable graph.

(b) If \( G \) is well-covered, then \( G \cup \{uv\} \) is well-covered too, and

\[ \alpha(G \cup \{uv\}) = \alpha(G). \]

**Proof.** (a) Let \( G' = G \cup \{uv\} \), we show that \( u \) is a shedding vertex of \( G' \). Since \( G' \setminus \{u\} = G \setminus \{u\} \), \( G' \setminus \{u\} \) is vertex decomposable. Since \( v \) is a shedding vertex of \( G \setminus N_G[u] \) and \( G' \setminus N_{G'}[u] = G \setminus (N_G[u] \cup \{v\}) = (G \setminus N_G[u]) \setminus \{v\}, \)

\( G' \setminus N_{G'}[u] \) is vertex decomposable. So, \( u \) satisfies the condition (i) of Definition 3.1.

Now, let \( S \) be an independent set of \( G' \setminus N_{G'}[u] \). It is clear that \( G' \setminus N_{G'}[u] = G \setminus (N_G[u] \cup \{v\}) \), so, \( S \) is independent set of \( G \setminus N_G[u] \). Since \( u \) is a shedding vertex of \( G \), there exists \( y \in N_G(u) \) such that \( S \cup \{y\} \) is an independent set of \( G \setminus \{u\} \). Since \( y \neq v \), \( S \cup \{y\} \) is an independent set of \( G' \setminus \{u\} \). Therefore, \( u \) is a shedding vertex of \( G' \) and so, \( G' \) is vertex decomposable.

(b) Let \( S \) be a maximal independent set of \( G' \). We show that \( S \) is a maximal independent set of \( G \). Consider the following three cases:

**Case 1** \((v \in S \text{ and } u \notin S)\): By contrary assume that \( S \) is not a maximal independent set of \( G \). Therefore \( S \cup \{u\} \) must be an independent set of \( G \), so, \( S \cap N_G[u] = \emptyset \) and it means that \( S \) is an independent set of \( G \setminus N_G[u] \). Since \( u \) is shedding vertex of \( G \), there exists a vertex \( x \in N_G(u) \) such that \( S \cup \{x\} \) is an independent set of \( G \) and hence is an independent set of \( G' \), which contradicts the maximality of \( S \).

**Case 2** \((v \notin S \text{ and } u \in S)\): As in the case 1, by contrary assume that \( S \cup \{v\} \) is an independent set of \( G \), so, \( S \cap N_G(v) = \emptyset \). On the other hand \( u \in S \), so, \( S \cap N_G(u) = \emptyset \). Therefore, \( S \setminus \{u\} \) is independent set of \( (G \setminus N_G[u]) \setminus N_G[v] \). Since \( v \) is shedding vertex of \( G \setminus N_G[u] \), there is a vertex \( y \in N_G(v) \) such that \( (S \setminus \{u\}) \cup \{y\} \) and so, \( S \cup \{y\} \) is independent set of \( G \) and \( G' \), which again contradicts the maximality of \( S \).

**Case 3** \((u \notin S \text{ and } v \notin S)\): It is clear that \( S \) is a maximal independent of \( G \). So, \( G' \) is well-covered too.
In addition, we have $\alpha(G \cup \{uv\}) = \alpha(G)$. \hfill \Box

The following example is remarkable.

\textbf{Example 3.4.} In Figure 3.1, $G$ is a chordal graph and hence vertex decomposable. The vertex $u$ is a shedding vertex of $G$ and $v$ is a shedding vertex of $G \setminus N_G[u]$. As it can be seen, $G \cup \{uv\}$ is a vertex decomposable graph which contains $C_6$. Note that by adding an edge between $x$ and $y$, the graph is converted to $C_8$, which is not vertex decomposable.

\begin{center}
\begin{tikzpicture}
\node[draw, circle] (u) at (0,0) {$u$};
\node[draw, circle] (v) at (2,0) {$v$};
\node[draw, circle] (x) at (-1,-1) {$x$};
\node[draw, circle] (y) at (3,-1) {$y$};
\draw (u) -- (v);
\draw (u) -- (x);
\draw (v) -- (y);
\end{tikzpicture}
\end{center}
\textbf{Figure 3.1.}

Let $G_1, G_2, \ldots, G_n$ be connected component of a vertex decomposable graph $G$, by [10, Lemma 20] $G_i$ is vertex decomposable and so, it has a shedding vertex $u_i$ for $1 \leq i \leq n$. In the proof of Theorem 3.3 we showed that $u_i$ is shedding vertex of $G \cup \{u_iu_j\}$. Hence, By applying Theorem 3.3 $H = G_1 \cup G_2 \cup \ldots \cup G_n \cup \{u_iu_j | i \neq j\}$, which is a vertex decomposable graph. In addition, if $G$ is a well-covered graph, then $H$ will be a well-covered graph, too. As a special case, for a given graph $G$ with $V(G) = \{x_1, \ldots, x_n\}$ and the vertex decomposable graphs $G_1, \ldots, G_n$ with disjoint vertex sets, if $u_i$ is a shedding vertex of $G_i$, then $H = G_1 \cup G_2 \cup \ldots \cup G_n \cup \{u_iu_j | x_ix_j \in E(G)\}$ is a vertex decomposable graph. One can see that $H$ is isomorphic to the graph that obtained from $G$ by attaching $G_i$ to the vertices of $G$ in shedding vertices $u_i$. It means that attaching $G_i$ and $G_j$ to the vertices $x_i$ and $x_j$ of $G$, which $x_i, x_j \in E(G)$, in shedding vertices $u_i \in V(G_i)$ and $u_j \in V(G_j)$ is the same as saying that adding the edge $u_iu_j$ to the graph $\bigcup_{i=1}^n G_i$. As an example consider the following one.

\textbf{Example 3.5.} In the Figure 3.2, there are four vertex decomposable graphs $G_1, \ldots, G_4$ which contain the shedding vertices $u_1, \ldots, u_4$, respectively. These graphs have been connected according to edges of the graph $G$. To this end, we add the edge $u_iu_j$ if $x_ix_j \in E(G)$. Using this method, the vertex decomposable graph $H = G_1 \cup G_2 \cup G_3 \cup G_4 \cup \{u_1u_2, u_2u_3, u_3u_4, u_2u_4\}$ have been constructed. One can see, $H$, as well as $G_1$, is not well-covered. If $G_1$ is replaced with a
well-covered graph, then \( H \) will convert to a well-covered graph, too. One can see, \( H \) is isomorphic to the graph obtained by attaching vertex decomposable graphs \( G_1, \ldots, G_4 \) to each vertex of \( G \).

\[
G_1 \quad G_2 \\
\downarrow \quad \downarrow \\
G \quad G_3 \\
\downarrow \quad \downarrow \\
G_4 \quad G_3 \\
\downarrow \quad \downarrow \\
H
\]

**Figure 3.2.**

Let \( G \) be a graph and \( V(G) = \{x_1, \ldots, x_n\} \). If we set \( H = \bigcup_{i=1}^n \{u_i, v_i\} \cup \{u_i, u_j | x_i x_j \in E(G)\} \), then \( H \) is isomorphic to \( G^W \). Since \( K_2 \) is a well-covered vertex decomposable graph and its vertices are shedding vertices, so we have the following result.

**Corollary 3.6.** [3, Theorem 4.4] For every graph \( G \), \( G^W \) is well-covered and vertex decomposable.

Since the complete graphs are well-covered and vertex decomposable, and their vertices are shedding vertices, we have the following corollary:

**Corollary 3.7.** [5, Theorem 1.1] Let \( G \) be a finite simple graph on a vertex set \( V = \{x_1, \ldots, x_n\} \) and let \( k_1, \ldots, k_n \geq 2 \) be integers. Then the graph \( G' \) obtained from \( G \) by attaching the complete graph \( K_{k_i} \) to \( x_i \) for \( i = 1, \ldots, n \) is well-covered and vertex decomposable.

In the following theorem, a new construction is justified to generalize clique-whiskering of a graph.

**Theorem 3.8.** Let \( G \) be a graph and \( V(G) = \{W_1, \ldots, W_t\}, W_i = \{w_{i_1}, \ldots, w_{i_n}\} \) be a clique partition for its vertex set. Let \( G_1, G_2, \ldots, G_t \) be in class \( SQC \) such that \( G_i \) (\( 1 \leq i \leq t \)) contains a simplicial vertex \( y_i \) with the property that \( N_{G_i}(y_i) = \{u_{i_1}, \ldots, u_{i_n}\} \) which \( n_i \geq m_i \). Then \( H = G_1 \cup \ldots \cup G_t \cup \{u_i, u_k | w_{i_j} w_{k_l} \in E(G)\} \) is well-covered and vertex decomposable.

**Proof.** Let \( u \) belongs to the neighborhood of a simplicial vertex \( x_i \) of \( G_i \). It is not difficult to see that \( G_i \setminus \{u\} \) and \( G \setminus N_G[u] \) belong to \( SQC \) and for
any independent set of $G \setminus N_G[u]$, $S \cup \{x_i\}$ is independent of $G$. So, $u$ is a shedding vertex of $G_i$. On the other hand, if $v$ belongs to the neighborhood of a simplicial vertex of $G_j$ for $i \neq j$, then $v$ is shedding vertex of $G \setminus N_G[u]$. Hence by Theorem 3.3, we can add an edge between these two vertices and the resulting graph is well-covered and vertex decomposable. \hfill $\square$

The above theorem is illustrated using following example.

**Example 3.9.** In the Figure 3.3 (a), $\pi = \{\{w_1, w_1\}, \{w_2, \}, \{w_3, w_3\}\}$ is a partition for $V(G)$ and $y_1 \in G_1$, $y_2 \in G_2$, and $y_3 \in G_3$ are simplicial vertices. It is easy to see that $N_{G_1}(y_1) = \{u_1, u_1\}; N_{G_2}(y_2) = \{u_2, u_2\}; N_{G_3}(y_3) = \{u_3, u_3, u_3\}$. In the Figure 3.3 (b), as can be seen, $H = G_1 \cup G_2 \cup G_3 \cup \{u_1, u_1, u_1, u_1, u_1, u_1, u_1, u_1, u_1\}$. In fact, $H$ is isomorphic to the graph obtained by attaching vertex decomposable graphs $G_1, G_2, G_3$ to the cliques of $G$ in the neighborhoods of their simplicial vertices.

![Figure 3.3](image)

As a corollary of our result we have the following:
Corollary 3.10. \[ \text{[2, Theorem 3.3]} \] For every graph \( G \) and any partition \( \pi = \{W_1, \ldots, W_t\} \) of cliques for \( V(G) \), \( G^\pi \) is well-covered and vertex decomposable.

Proof. Let \( W_i = \{w_{i1}, \ldots, w_{im_i}\} \), corresponding to any \( W_i \) we consider a complete graph \( K_{|W_i| + 1} \) with vertex set \( \{u_{i1}, \ldots, u_{im_i}, w_i\} \) that is a well-covered and vertex decomposable, so, \( H = \bigcup_{i=1}^{\pi} K_{|W_i| + 1} \cup \{u_{ij}, w_{ij} \mid w_{ij}w_{kj} \in E(G)\} \cong G^\pi \) is well-covered and vertex decomposable.

Remark 3.11. (a) Theorem 3.8 remains true if the clique \( \{u_{i1}, \ldots, u_{in_i}\} \) is a subset of \( N_G(\{y^1_i, \ldots, y^{m_i}_i\}) \) where \( y^j_i \) is a simplicial vertex for \( 1 \leq j \leq n_i \).

(b) If \( W_i \) have simplicial vertex for some \( i \), then \( G_i \) can be the subgraph of \( G \) induced by \( W_i \). In fact, we attach vertex decomposable graphs just at the cliques that have not simplicial vertex and construct well-covered vertex decomposable graphs.

The above remark is illustrated using the following example:

Example 3.12. In the Figure 3.4, \( \pi = \{\{w_{11}, w_{12}\}, \{w_{21}, w_{22}\}, \{w_{31}, w_{32}\}\} \) is a partition of \( V(G) \). \( G_1 \) and \( G_2 \) are two vertex decomposable graphs which \( u_{21} \in N_{G_2}(y_{21}) \) and \( u_{22} \in N_{G_2}(y_{22}) \). Also, \( G_3 \) is the subgraph of \( G \) induced by \( W_3 \) (because \( w_{32} \in W_3 \) is a simplicial vertex). So, \( H = G_1 \cup G_2 \cup G_3 \cup \{u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}\} \) which it means that we attach vertex decomposable graphs just at \( W_1 \) and \( W_2 \).

Figure 3.4.

References


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