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TOTALLY UMBILICAL RADICAL TRANSVERSAL LIGHTLIKE HYPERSURFACES OF KÄHLER-NORDEN MANIFOLDS OF CONSTANT TOTALLY REAL SECTIONAL CURVATURES

G. NAKOVA

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Abstract. In this paper we study curvature properties of semi-symmetric type of totally umbilical radical transversal lightlike hypersurfaces (M, g) and $(M, \widetilde{g})$ of a Kähler-Norden manifold $(\overline{M}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\widetilde{\nu}}$ (g and $\widetilde{g}$ are the induced metrics on M by the Norden metrics $\overline{g}$ and $\overline{\widetilde{g}}$, respectively). We obtain a condition for $\overline{\widetilde{\nu}}$ (resp. $\overline{\nu}$) which is equivalent to each of the following conditions: (M, g) $(resp. (M, \widetilde{g}))$ is locally symmetric, semi-symmetric, Ricci semi-symmetric and almost Einstein. We construct an example of a totally umbilical radical transversal lightlike hypersurface, which is locally symmetric, semi-symmetric, Ricci semi-symmetric and almost Einstein.
Keywords: Kähler-Norden manifold, totally real sectional curvature, radical transversal lightlike hypersurface, totally umbilical lightlike hypersurface.
MSC(2010): Primary: 53C15; Secondary: 53C40, 53C50.

1. **Introduction**

Almost complex manifolds with Norden metric (or B-metric) were introduced by A. P. Norden [9] and their geometry has been investigated by G. Ganchev, K. Gribachev, A. Borisov, V. Mihova [4,5] and many others. There exists a difference between the geometry of an indefinite almost Hermitian manifold and the geometry of an almost complex manifold with Norden metric. It arises because in the first case the almost complex structure $\overline{J}$ is an isometry with respect to the semi-Riemannian metric $\overline{g}$ and in the second case $\overline{J}$ is an anti-isometry with respect to $\overline{g}$. Due to this property of the couple $(\overline{J}, \overline{g})$ of an almost complex manifold with Norden metric $\overline{M}$, there exists second Norden metric $\overline{\widetilde{g}}$ on $\overline{M}$ which is defined by $\overline{\widetilde{g}}(X, Y) = \overline{g}(\overline{J}X, Y)$. Both metrics $\overline{g}$ and $\overline{\widetilde{g}}$

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are indefinite of a neutral signature. Thus, we can consider two induced metrics g and $\widetilde{g}$ (by $\overline{g}$ and $\overline{\widetilde{g}}$, respectively) on a submanifold M of $\overline{M}$. In [7] we study submanifolds which are non-degenerate with respect to the one Norden metric and lightlike with respect to the other one. In [8] we introduced such class of hypersurfaces, namely, radical transversal lightlike hypersurfaces of almost complex manifolds with Norden metric. As it is well known, in case M is a lightlike submanifold of $\overline{M}$, a part of the normal bundle $TM^{\perp}$ lies in the tangent bundle TM. Therefore the geometry of lightlike submanifolds is very different from the Riemannian and the semi-Riemannian geometry. The general theory of lightlike submanifolds has been developed by K. Duggal and A. Bejancu in [2]. Many new classes of lightlike submanifolds of indefinite Kaehler, Sasakian, quaternion Kaehler manifolds are introduced by K. Duggal and B. Sahin in [3] and different applications of lightlike geometry in the mathematical physics are given.

An important class of lightlike hypersurfaces are the totally umbilical lightlike hypersurfaces. In [2] it is proved that there exist no totally umbilical lightlike real hypersurfaces of indefinite complex space forms $\overline{M}(c)$ with $c \neq 0$. In this paper we consider totally umbilical radical transversal lightlike hypersurfaces of Kähler-Norden manifolds of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\widetilde{\nu}}$. Our purpose is to study curvature conditions of semi-symmetric type for the considered hypersurfaces in case $(\overline{\nu}, \overline{\widetilde{\nu}}) \neq (0,0)$. Lightlike hypersurfaces which are semi-symmetric, Ricci semi-symmetric, parallel or semi-parallel in a semi-Euclidean space are investigated in [10]. The curvatures of lightlike hypersurfaces M of an indefinite Kenmotsu manifold $\overline{M}$ under the conditions that M is locally symmetric or M is semi-symmetric are studied in [6].

The main results in this paper are given in the following theorem:

Theorem 1.1. *Let (M, g) $(resp.\,(M, \widetilde{g}))$ be a totally umbilical radical transversal lightlike hypersurface of a Kähler-Norden manifold $(\overline{M}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ $(\dim \overline{M} = 2n \geq 4)$ of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\widetilde{\nu}}$ with respect to $\overline{g}$ such that $\overline{\widetilde{\nu}} \neq 0$ $(resp.\,\overline{\nu} \neq 0)$. Then the following assertions are equivalent*

- (i) (M, g) $(resp.\,(M, \widetilde{g}))$ *is semi-symmetric.*
- (ii) (M, g) $(resp.\,(M, \widetilde{g}))$ *is Ricci semi-symmetric.*
- (iii) $\overline{\widetilde{\nu}} = \dfrac{\rho^2}{b}$ $\left(resp.\ \overline{\nu} = \dfrac{\widetilde{\rho}^2}{\widetilde{b}}\right)$, *where $\rho\,(resp.\,\widetilde{\rho})$ and $b\,(resp.\,\widetilde{b})$ are the functions from* (2.13) *(resp.* (3.17)*) and* (2.16) *(resp.* (3.16)*).*
  Moreover, if $\overline{\widetilde{\nu}}$ (resp. $\overline{\nu}$) is a constant, the assertions (i), (ii), (iii) *and the assertions*
- (iv) (M, g) $(resp.\,(M, \widetilde{g}))$ *is locally symmetric.*
- (v) (M, g) $(resp.\,(M, \widetilde{g}))$ *is almost Einstein.*

are equivalent.

We note it is known that locally symmetric manifolds are semi-symmetric but the converse is not true. Moreover, the conditions the manifold to be semi-symmetric and Ricci semi-symmetric are not equivalent in general. An application of Theorem 1.1 is Example 5.2, where we construct a totally umbilical radical transversal lightlike hypersurface, which is locally symmetric, semi-symmetric, Ricci semi-symmetric and almost Einstein.

2. Preliminaries

2.1. **Almost complex manifolds with Norden metric.** A $2n$-dimensional semi-Riemannian manifold $\overline{M}$ is said to be *an almost complex manifold with Norden metric* (or *an almost complex manifold with B-metric*) [4] if it is equipped with an almost complex structure $\overline{J}$ and a semi-Riemannian metric $\overline{g}$ such that

$$\overline{J}^2 X = -X, \quad \overline{g}(\overline{J}X, \overline{J}Y) = -\overline{g}(X, Y), \quad X, Y \in \Gamma(T\overline{M}).$$

The tensor field $\overline{\widetilde{g}}$ on $\overline{M}$ defined by $\overline{\widetilde{g}}(X, Y) = \overline{g}(\overline{J}X, Y)$ is a Norden metric on $\overline{M}$, which is said to be *an associated metric* of $\overline{M}$. Both metrics $\overline{g}$ and $\overline{\widetilde{g}}$ are necessarily indefinite of signature (n, n). The Levi-Civita connection of $\overline{g}$ (resp. $\overline{\widetilde{g}}$) is denoted by $\overline{\nabla}$ (resp. $\overline{\widetilde{\nabla}}$). The tensor fields F and Φ are defined by $F(X, Y, Z) = \overline{g}((\overline{\nabla}_X \overline{J})Y, Z)$ and $\Phi(X, Y) = \overline{\widetilde{\nabla}}_X Y - \overline{\nabla}_X Y$. A classification of the almost complex manifolds with Norden metric with respect to the tensor F is given in [4] and eight classes W_i are obtained. These classes are characterized by conditions for the tensor Φ in [5]. The class W_0 of the Kähler manifolds with Norden metric is determined by the condition $\overline{J}$ to be parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$. The class W_0 is characterized by the following two equivalent conditions $F(X, Y, Z) = 0$ and $\Phi(X, Y) = 0$. We will call a manifold $(\overline{M}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ belonging to W_0 a Kähler-Norden manifold.

2.2. **Lightlike hypersurfaces of semi-Riemannian manifolds.** A hypersurface M of an $(m+2)$-dimensional $(m > 1)$ semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called *a lightlike hypersurface* [2,3] if at any $p \in M$ the tangent space T_pM and the normal space $T_pM^{\perp}$ have a non-empty intersection, which is denoted by $\mathrm{Rad}T_pM$. As $\dim\left(T_pM^{\perp}\right) = 1$ it follows that $\dim(\mathrm{Rad}T_pM) = 1$ and $\mathrm{Rad}T_pM = T_pM^{\perp}$. The mapping $\mathrm{Rad}TM : p \in M \longrightarrow \mathrm{Rad}T_pM$ defines a smooth distribution on M of rank 1 which is called *a radical distribution* on M. Thus the induced metric g by $\overline{g}$ on a lightlike hypersurface M has a constant rank m. There also exists a non-degenerate complementary vector bundle $S(TM)$ of the normal bundle $TM^{\perp}$ in the tangent bundle TM, called *a screen distribution* on M. We have the following decomposition of TM

$$TM = S(TM) \perp TM^{\perp}, \tag{2.1}$$

where $\perp$ denotes an orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$-module of smooth sections of a vector bundle E over M. It is well known [2, Theorem 1.1, p. 79] that there exists a unique *transversal vector bundle* $\mathrm{tr}(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $U \subset M$, there exists a unique section N of $\mathrm{tr}(TM)$ on U satisfying

$$\overline{g}(N,\xi) = 1, \qquad \overline{g}(N,N) = \overline{g}(N,W) = 0, \ \forall\, W \in \Gamma(S(TM)). \tag{2.2}$$

Hence for any screen distribution $S(TM)$ we have a unique $\mathrm{tr}(TM)$, which is a lightlike complementary vector bundle (but not orthogonal) to TM in $T\overline{M}$, such that

$$T\overline{M} = TM \oplus \mathrm{tr}(TM) = S(TM) \perp \left(TM^{\perp} \oplus \mathrm{tr}(TM)\right), \tag{2.3}$$

where $\oplus$ denotes a non-orthogonal direct sum.

Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{g}$ on $\overline{M}$ and P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.1). The local Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N; \tag{2.5}$$

and

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, respectively. The induced connections ∇ and ∇^* on TM and $S(TM)$, respectively, are linear connections. A_N and A^*_{ξ} are the shape operators on TM and $S(TM)$, respectively, τ is a 1-form on TM. Both local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_{\xi}X, Y), \quad \overline{g}(A^*_{\xi}X, N) = 0; \tag{2.8}$$

$$C(X,PY) = g(A_N X, PY), \quad \overline{g}(A_N X, N) = 0. \tag{2.9}$$

Since $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric on TM. From (2.4) we have $B(X,Y) = \overline{g}(\overline{\nabla}_X Y, \xi)$, $\forall X, Y \in \Gamma(TM)$ which implies

$$B(X,\xi) = 0, \quad \forall X \in \Gamma(TM). \tag{2.10}$$

From (2.8) and (2.10) it follows that the shape operator A^*_{ξ} is $S(TM)$-valued, self-adjoint with respect to g and $A^*_{\xi}\xi = 0$. In general, the induced connection ∇^* on $S(TM)$ is not torsion-free. This fact and (2.9) show that the shape operator A_N is not self-adjoint and it is $S(TM)$-valued. The linear connection ∇^* is a metric connection on $S(TM)$ but ∇ is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.11}$$

where η is a 1-form given by

$$\eta(X) = \overline{g}(X,N), \quad \forall X \in \Gamma(TM). \tag{2.12}$$

A lightlike hypersurface M is said to be *totally umbilical* [2] if on any coordinate neighborhood U there exists a smooth function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall X,Y \in \Gamma(TM_{|U}). \tag{2.13}$$

According to (2.8) and (2.10) the condition (2.13) is equivalent to

$$A^*_\xi(PX) = \rho PX, \quad X \in \Gamma(TM_{|U}). \tag{2.14}$$

Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively. By using (2.4) and (2.5) we get the Gauss equation of M

$$\begin{aligned}&\overline{R}(X,Y,Z) = R(X,Y,Z) + B(X,Z)A_NY - B(Y,Z)A_NX\\ &+\{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &+\tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}\, N,\end{aligned} \tag{2.15}$$

for any $X,Y,Z \in \Gamma(TM)$. The induced Ricci type tensor $R^{(0,2)}$ of M is defined by $R^{(0,2)}(X,Y) = \mathrm{trace}\{Z \longrightarrow R(X,Z)Y\}, \forall X,Y \in \Gamma(TM)$. In general, $R^{(0,2)}$ is not symmetric. According to [2, Theorem 3.2, p. 99], a necessary and sufficient condition for the induced Ricci tensor to be symmetric is each 1-form τ to be closed, i.e. $d\tau = 0$ on M. Therefore $R^{(0,2)}$ is denoted by Ric [3] only if the 1-form τ is closed.

2.3. **Radical transversal lightlike hypersurfaces of a Kähler-Norden manifold.** In [8] we introduced the class of radical transversal lightlike hypersurfaces of an almost complex manifold with Norden metric, which does not exist when the ambient manifold is an indefinite almost Hermitian manifold. Let $(M, g, S(TM))$ be a lightlike hypersurface of an almost complex manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g})$. We say that M is a *radical transversal lightlike hypersurface* of $\overline{M}$ if $\overline{J}(TM^\perp) = \mathrm{tr}(TM)$. For the considered hypersurfaces the following assertions are valid [8]: 1) M is a radical transversal lightlike hypersurface of $\overline{M}$ if and only if the screen distribution $S(TM)$ is holomorphic with respect to $\overline{J}$; 2) A radical transversal lightlike hypersurface M of $\overline{M}$ has a unique screen distribution up to a semi-orthogonal transformation and a unique transversal vector bundle.

There exist two Norden metrics $\overline{g}$ and $\widetilde{\overline{g}}$ on an almost complex manifold with Norden metric $\overline{M}$. Hence we can consider two induced metrics g and $\widetilde{g}$ on a hypersurface M of $\overline{M}$ by $\overline{g}$ and $\widetilde{\overline{g}}$, respectively. Denote by (M, g) a non-degenerate hypersurface of $(\overline{M}, \overline{J}, \overline{g}, \widetilde{\overline{g}})$, whose normal vector field $\overline{N}$ is a space-like unit ($\overline{g}(\overline{N}, \overline{N}) = 1$) or a time-like unit ($\overline{g}(\overline{N}, \overline{N}) = -1$) and $\overline{N}$ is orthogonal to $\overline{JN}$ with respect to $\overline{g}$. Then we proved [8] that (M, g) is a non-degenerate hypersurface of $\overline{M}$ if and only if $(M, \widetilde{g})$ is a radical transversal lightlike hypersurface.

Further, we consider a radical transversal lightlike hypersurface (M, g) of a Kähler-Norden manifold $(\overline{M}, \overline{J}, \overline{g}, \widetilde{\overline{g}})$. Let $\{\xi, N\}$ be the pair on (M, g) which satisfies condition (2.2). From the definition of M we have

$$\overline{J}\xi = bN, \tag{2.16}$$

where $b \in F(\overline{M})$. Taking into account (2.1) and (2.12), for an arbitrary $X \in \Gamma(TM)$ we have the following decomposition $X = PX + \eta(X)\xi$. From the last equality and (2.16) we obtain

$$\overline{J}X = \overline{J}(PX) + b\eta(X)N. \tag{2.17}$$

Since $S(TM)$ is holomorphic with respect to $\overline{J}$, it follows that $\overline{J}(PX)$ belongs to $S(TM)$. The shape operators A^*_ξ, A_N and the corresponding local second fundamental forms B, C are related as follows

$$A^*_\xi X = -b\overline{J}(A_N X), \ B(X,Y) = -bC(X, \overline{J}(PY)), \ X, Y \in \Gamma(TM). \tag{2.18}$$

The metric linear connection ∇^* of the considered hypersurfaces is such that the almost complex structure $\overline{J}$ is parallel with respect to ∇^*, i.e.

$$(\nabla^*_X \overline{J})PY = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2.19}$$

The 1-form τ is expressed by the function b in the following way

$$\tau(X) = -\frac{1}{2b}X(b), \tag{2.20}$$

which means that τ is closed and consequently the induced Ricci tensor on (M, g) is symmetric.

3. Totally umbilical radical transversal lightlike hypersurfaces of a Kähler-Norden manifold of constant totally real sectional curvatures

The characteristic condition $\Phi = 0$ of a $2n$-dimensional Kähler-Norden manifold $(\overline{M}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ implies that the Levi-Civita connections $\overline{\nabla}$ and $\overline{\widetilde{\nabla}}$ of $\overline{M}$ coincide. Hence the curvature tensors of type (1,3) $\overline{R}$ and $\overline{\widetilde{R}}$ of $\overline{\nabla}$ and $\overline{\widetilde{\nabla}}$ coincide, too. Further, $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ (resp. $\overline{x}, \overline{y}, \overline{z}, \overline{w}$) will stand for arbitrary differentiable vector fields on $\overline{M}$ (resp. vectors in $T_p\overline{M}, p \in \overline{M}$). The curvature tensors $\overline{R}$ and $\overline{\widetilde{R}}$ of type (0,4) are given by $\overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \overline{g}(\overline{R}(\overline{X}, \overline{Y}, \overline{Z}), \overline{W})$ and $\overline{\widetilde{R}}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \overline{\widetilde{g}}(\overline{\widetilde{R}}(\overline{X}, \overline{Y}, \overline{Z}), \overline{W})$, respectively.
Moreover, we have

$$\overline{\widetilde{R}}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{JW}). \tag{3.1}$$

From the condition $\overline{\nabla J} = \overline{J\nabla}$ it follows $\overline{R}(\overline{X}, \overline{Y}, \overline{JZ}) = \overline{JR}(\overline{X}, \overline{Y}, \overline{Z})$. Then using the fact that $\overline{J}$ is an anti-isometry with respect to $\overline{g}$ and $\overline{\widetilde{g}}$ we obtain

$$\begin{aligned} &\overline{R}(\overline{X}, \overline{Y}, \overline{JZ}, \overline{JW}) = -\overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}), \\ &\overline{\widetilde{R}}(\overline{X}, \overline{Y}, \overline{JZ}, \overline{JW}) = -\overline{\widetilde{R}}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}), \end{aligned} \tag{3.2}$$

i.e. $\overline{R}$ and $\overline{\overline{R}}$ are Kähler tensors. The following tensors are essential in the geometry of the Kähler-Norden manifolds

$$\begin{aligned} \overline{\pi}_1(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \overline{g}(\overline{Y},\overline{Z})\overline{g}(\overline{X},\overline{W}) - \overline{g}(\overline{X},\overline{Z})\overline{g}(\overline{Y},\overline{W}), \\ \overline{\pi}_2(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= \overline{g}(\overline{Y},\overline{JZ})\overline{g}(\overline{X},\overline{JW}) - \overline{g}(\overline{X},\overline{JZ})\overline{g}(\overline{Y},\overline{JW}), \\ \overline{\pi}_3(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= -\overline{g}(\overline{Y},\overline{Z})\overline{g}(\overline{X},\overline{JW}) + \overline{g}(\overline{X},\overline{Z})\overline{g}(\overline{Y},\overline{JW}) \\ &\quad -\overline{g}(\overline{X},\overline{W})\overline{g}(\overline{Y},\overline{JZ}) + \overline{g}(\overline{Y},\overline{W})\overline{g}(\overline{X},\overline{JZ}). \end{aligned} \tag{3.3}$$

By $\overline{\overline{\pi}}_i\,(i=1,2,3)$ are denoted the corresponding tensors with respect to $\overline{\overline{g}}$. They are related with $\overline{\pi}_i\,(i=1,2,3)$ as follows

$$\overline{\overline{\pi}}_1 = \overline{\pi}_2, \quad \overline{\overline{\pi}}_2 = \overline{\pi}_1, \quad \overline{\overline{\pi}}_3 = -\overline{\pi}_3. \tag{3.4}$$

For every non-degenerate section $\alpha = span\{\overline{x},\overline{y}\}$ with respect to $\overline{g}$ in $T_p\overline{M}$ the following two sectional curvatures $\overline{\nu}$ and $\overline{\overline{\nu}}$ are defined in [1]

$$\overline{\nu}(\alpha;p) = \frac{\overline{R}(\overline{x},\overline{y},\overline{y},\overline{x})}{\overline{\pi}_1(\overline{x},\overline{y},\overline{y},\overline{x})}, \quad \overline{\overline{\nu}}(\alpha;p) = \frac{\overline{R}(\overline{x},\overline{y},\overline{y},\overline{J}\overline{x})}{\overline{\pi}_1(\overline{x},\overline{y},\overline{y},\overline{x})}.$$

Analogously, if $\alpha = span\{\overline{x},\overline{y}\}$ is a non-degenerate section with respect to $\overline{\overline{g}}$, we can define two sectional curvatures $\overline{\nu}'$ and $\overline{\overline{\nu}}'$ given by

$$\overline{\nu}'(\alpha;p) = \frac{\overline{\overline{R}}(\overline{x},\overline{y},\overline{y},\overline{x})}{\overline{\overline{\pi}}_1(\overline{x},\overline{y},\overline{y},\overline{x})}, \quad \overline{\overline{\nu}}'(\alpha;p) = \frac{\overline{\overline{R}}(\overline{x},\overline{y},\overline{y},\overline{J}\overline{x})}{\overline{\overline{\pi}}_1(\overline{x},\overline{y},\overline{y},\overline{x})}.$$

A section α is said to be *holomorphic* if $\overline{J}\alpha = \alpha$ and its sectional curvature is called *a holomorphic sectional curvature*. A section α is said to be *totally real* with respect to $\overline{g}$ (resp. $\overline{\overline{g}}$) if $\overline{J}\alpha$ is orthogonal to α with respect to $\overline{g}$ (resp. $\overline{\overline{g}}$). We will call the sectional curvatures $\overline{\nu}$ and $\overline{\overline{\nu}}$ (resp. $\overline{\nu}'$ and $\overline{\overline{\nu}}'$) of a non-degenerate totally real section with respect to $\overline{g}$ (resp. $\overline{\overline{g}}$) *totally real sectional curvatures* with respect to $\overline{g}$ (resp. $\overline{\overline{g}}$). It is well known that *an indefinite complex space form* is a connected indefinite Kähler manifold $\overline{M}$ of constant holomorphic sectional curvature c and it is denoted by $\overline{M}(c)$. When $\overline{M}$ is a Kähler-Norden manifold the property (3.2) gives that all of the holomorphic sectional curvatures are zero. Therefore the sectional curvatures of the totally real sections are important in our considerations.

Theorem 3.1. *[1] Let $(\overline{M},\overline{J},\overline{g},\overline{\overline{g}})$ be a $2n$-dimensional ($2n \geq 4$) Kähler-Norden manifold. $\overline{M}$ is of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\overline{\nu}}$ with respect to $\overline{g}$, i.e. $\overline{\nu}(\alpha;p) = \overline{\nu}(p)$, $\overline{\overline{\nu}}(\alpha;p) = \overline{\overline{\nu}}(p)$ $\quad (p \in \overline{M})$, if and only if*

$$\overline{R} = \overline{\nu}[\overline{\pi}_1 - \overline{\pi}_2] + \overline{\overline{\nu}}\overline{\pi}_3. \tag{3.5}$$

Both functions $\overline{\nu}$ and $\overline{\overline{\nu}}$ are constants if $\overline{M}$ is connected and $2n \geq 6$.

Remark 3.2. Suppose $\overline{M}$ is of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\overline{\nu}}$ with respect to $\overline{g}$. Then from (3.5) by using (3.1), (3.3) and (3.4) we obtain

$\overline{\widetilde{R}} = -\overline{\widetilde{\nu}}[\overline{\widetilde{\pi}}_1 - \overline{\widetilde{\pi}}_2] + \overline{\nu}\overline{\widetilde{\pi}}_3$. From Theorem 3.1 it follows that $\overline{M}$ is of constant totally real sectional curvatures $\overline{\nu}' = -\overline{\widetilde{\nu}}$ and $\overline{\widetilde{\nu}}' = \overline{\nu}$ with respect to $\overline{\widetilde{g}}$.

Theorem 3.3. *Let (M, g) be a totally umbilical radical transversal lightlike hypersurface of a Kähler-Norden manifold $(\overline{M}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ $(\dim \overline{M} = 2n \geq 4)$ of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\widetilde{\nu}}$ with respect to $\overline{g}$. Then $\overline{\nu} = 0$ and ρ from (2.13) satisfies the partial differential equations*

$$b\overline{\widetilde{\nu}} - \rho^2 + \xi(\rho) + \rho\tau(\xi) = 0, \tag{3.6}$$

$$PX(\rho) + \rho\tau(PX) = 0, \quad \forall X \in \Gamma(TM), \tag{3.7}$$

where b is the function from (2.16). Moreover, the curvature tensor R and the Ricci tensor Ric of (M, g) are given by

$$\begin{aligned} R(X,Y,Z) = &\left(\overline{\widetilde{\nu}} - \frac{\rho^2}{b}\right)\left[g(X,Z)\overline{J}(PY) - g(Y,Z)\overline{J}(PX)\right] \\ &+\overline{\widetilde{\nu}}\left[\overline{g}(X,\overline{J}Z)Y - \overline{g}(Y,\overline{J}Z)X\right], \end{aligned} \tag{3.8}$$

$$\mathrm{Ric}(X,Y) = -2(n-1)\overline{\widetilde{\nu}}\widetilde{g}(X,Y) + \left(\overline{\widetilde{\nu}} - \frac{\rho^2}{b}\right)\widetilde{g}(PX,PY), \tag{3.9}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. By using (2.15), (3.5), (2.16) and (2.13) we obtain

$$\begin{aligned} &b\overline{\nu}\left[-\overline{g}(Y,\overline{J}Z)\eta(X) + \overline{g}(X,\overline{J}Z)\eta(Y)\right] + b\overline{\widetilde{\nu}}\left[-g(Y,Z)\eta(X) + g(X,Z)\eta(Y)\right] \\ &= X(\rho)g(Y,Z) - Y(\rho)g(X,Z) \\ &+\rho\left[(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + \tau(X)g(Y,Z) - \tau(Y)g(X,Z)\right], \end{aligned}$$

where $X, Y, Z \in \Gamma(TM)$. Taking into account (2.11) and (2.13), the above equality becomes

$$\begin{aligned} &b\overline{\nu}\left[-\overline{g}(Y,\overline{J}Z)\eta(X) + \overline{g}(X,\overline{J}Z)\eta(Y)\right] \\ &+b\overline{\widetilde{\nu}}\left[-g(Y,Z)\eta(X) + g(X,Z)\eta(Y)\right] \\ &= X(\rho)g(Y,Z) - Y(\rho)g(X,Z) \\ &+\rho\left[\rho g(X,Z)\eta(Y) - \rho g(Y,Z)\eta(X) \right. \\ &\left. +\tau(X)g(Y,Z) - \tau(Y)g(X,Z)\right]. \end{aligned} \tag{3.10}$$

Replacing X, Y and Z in (3.10) by PX, ξ and PZ, respectively, we get

$$b\overline{\nu}g(PX, \overline{J}(PZ)) + b\overline{\widetilde{\nu}}g(PX,PZ) = \left[\rho^2 - \xi(\rho) - \rho\tau(\xi)\right]g(PX,PZ).$$

Because $S(TM)$ is non-degenerate we have,

$$\left[b\overline{\widetilde{\nu}} - \rho^2 + \xi(\rho) + \rho\tau(\xi)\right]PZ + b\overline{\nu}\overline{J}(PZ) = 0.$$

Since PZ and $\overline{J}(PZ)$ are linearly independent and $b \neq 0$, we obtain (3.6) and $\overline{\nu} = 0$. Now if we take $X = PX, Y = PY, Z = PZ$ in (3.10) and by using $S(TM)$ is non-degenerate we have,

$$(3.11) \qquad [PX(\rho) + \rho\tau(PX)]\, PY = [PY(\rho) + \rho\tau(PY)]\, PX.$$

Suppose there exists a vector field $X_0 \in \Gamma(TM_{|U})$ such that $PX_0(\rho)+\rho\tau(PX_0) \neq 0$ at a point $p \in M$. Then from (3.11) it follows that all vectors from $S(TM)$ are collinear with $(PX_0)_p$. This is a contradiction because $\dim S(TM) = 2n-2 \geq 2$. Hence (3.7) is true at any point $p \in M$. Next, by using (3.5) and taking into account that $\overline{\nu} = 0$ and (2.17), we obtain

$$(3.12) \qquad \begin{aligned} \overline{R}(X,Y,Z) = \overline{\overline{\nu}}\,\big[&-g(Y,Z)\overline{J}(PX) + g(X,Z)\overline{J}(PY) \\ &-\overline{g}(Y,\overline{J}Z)X + \overline{g}(X,\overline{J}Z)Y\big]. \end{aligned}$$

The equalities (2.14) and (2.18) imply

$$(3.13) \qquad A_N PX = \frac{\rho}{b}\overline{J}(PX), \quad \forall X \in \Gamma(TM).$$

So, (3.8) follows from (2.15), (3.12), (3.13) and (2.13). Finally, we obtain (3.9) by using the expression of the Ricci tensor $R^{(0,2)}$ of a lightlike hypersurface given in [3] by

$$(3.14) \qquad \begin{aligned} &R^{(0,2)}(X,Y) = \overline{\mathrm{Ric}}(X,Y) \\ &+B(X,Y)\mathrm{tr}A_N - g(A_N X, A^*_\xi Y) - \overline{g}(R(\xi,Y,X),N), \end{aligned}$$

where $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\overline{M}$. From (3.13) it follows that $\mathrm{tr}A_N = 0$. By using (3.5) and $\overline{\nu} = 0$ we find

$$(3.15) \qquad \overline{\mathrm{Ric}}(X,Y) = -2(n-1)\overline{\overline{\nu}}\,\overline{g}(X,\overline{J}Y), \quad \forall X,Y \in \Gamma(TM).$$

Substituting (3.15) in (3.14) and taking into account (2.14), (3.8) and (3.13), we get (3.9). □

Now, let $(M,\widetilde{g})$ be a radical transversal lightlike hypersurface of a Kähler-Norden manifold $(\overline{M},\overline{J},\overline{g},\overline{\overline{g}})$. Denote by $\widetilde{TM}, \widetilde{TM}^{\perp}, S(\widetilde{TM})$ and $\mathrm{tr}(\widetilde{TM})$ the tangent bundle, the normal bundle, the screen distribution and the transversal vector bundle of $(M,\widetilde{g})$, respectively. Let $\{\xi, N\}$ be the pair of vector fields $\xi \in \Gamma(\widetilde{TM}^{\perp})$, $N \in \Gamma(\mathrm{tr}(\widetilde{TM}))$ satisfying

$$(3.16) \qquad \overline{J}\xi = \widetilde{b}N,\ \overline{\overline{g}}(N,\xi) = 1,\ \overline{\overline{g}}(N,N) = \overline{\overline{g}}(N,W) = 0,\ \forall W \in \Gamma(S(\widetilde{TM})).$$

The local second fundamental form $\widetilde{B}$ and the 1-form $\widetilde{\tau}$ are given by

$$(3.17) \qquad \widetilde{B}(X,Y) = \widetilde{\rho}\widetilde{g}(X,Y), \quad \widetilde{\tau}(X) = -\frac{1}{2\widetilde{b}}X(\widetilde{b}), \quad \forall X,Y \in \Gamma(\widetilde{TM}).$$

Taking into account Remark 3.2, the following theorem follows in a similar way as Theorem 3.3

Theorem 3.4. *Let $(M,\widetilde{g})$ be a totally umbilical radical transversal lightlike hypersurface of a Kähler-Norden manifold $(\overline{M},\overline{J},\overline{g},\overline{\overline{g}})$ $(\dim\overline{M}=2n\geq 4)$ of constant totally real sectional curvatures $\overline{\nu}$ and $\overline{\overline{\nu}}$ with respect to $\overline{g}$. Then $\overline{\overline{\nu}}=0$ and the function $\widetilde{\rho}$ from (3.17) satisfies the partial differential equations*

$$\widetilde{b}\overline{\nu}-\widetilde{\rho}^2+\xi(\widetilde{\rho})+\widetilde{\rho}\widetilde{\tau}(\xi)=0, \tag{3.18}$$

$$PX(\widetilde{\rho})+\widetilde{\rho}\widetilde{\tau}(PX)=0,\quad \forall X\in\Gamma(\widetilde{TM}), \tag{3.19}$$

where $\widetilde{b}$ is the function from (3.16). Moreover, the curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{\mathrm{Ric}}$ of $(M,\widetilde{g})$ are given by

$$\begin{aligned}\widetilde{R}(X,Y,Z) &= \left(\overline{\nu}-\frac{\widetilde{\rho}^2}{\widetilde{b}}\right)\left[\widetilde{g}(X,Z)\overline{J}(PY)-\widetilde{g}(Y,Z)\overline{J}(PX)\right]\\ &\quad+\overline{\nu}\left[\widetilde{\overline{g}}(X,\overline{J}Z)Y-\widetilde{\overline{g}}(Y,\overline{J}Z)X\right],\end{aligned}\tag{3.20}$$

$$\widetilde{\mathrm{Ric}}(X,Y) = 2(n-1)\overline{\nu}g(X,Y)-\left(\overline{\nu}-\frac{\widetilde{\rho}^2}{\widetilde{b}}\right)g(PX,PY), \tag{3.21}$$

for any $X,Y,Z\in\Gamma(\widetilde{TM})$.

4. **Proof of Theorem 1.1**

Proof. Let (M,g) be a totally umbilical radical transversal lightlike hypersurface of $\overline{M}$. From Theorem 3.3 it follows that $\overline{\nu}=0$. Denote $a=\overline{\overline{\nu}}-\dfrac{\rho^2}{b}$.

(i)$\Longleftrightarrow$(iii) A lightlike hypersurface M is said to be *semi-symmetric* [3] if $(\mathcal{R}(X,Y)\cdot R)(U,V,W)=0$, where

$$\begin{aligned}&(\mathcal{R}(X,Y)\cdot R)(U,V,W)=R(X,Y,R(U,V,W))\\ &-R(U,V,R(X,Y,W))-R(R(X,Y,U),V,W)\\ &-R(U,R(X,Y,V),W),\end{aligned}\tag{4.1}$$

$X,Y,U,V,W\in\Gamma(TM)$. By using (3.8) and (4.1) we obtain

$$\begin{aligned}(\mathcal{R}R(X,Y)\cdot R)(U,V,W) &= a\Big\{g(V,W)\left[ag(Y,\overline{J}(PU))-\overline{\overline{\nu}}\overline{g}(Y,\overline{J}U)\right]\\ &\quad-g(U,W)\left[ag(Y,\overline{J}(PV))-\overline{\overline{\nu}}\overline{g}(Y,\overline{J}V)\right]\Big\}\overline{J}(PX)\\ &\quad-a\Big\{g(V,W)\left[ag(X,\overline{J}(PU))-\overline{\overline{\nu}}\overline{g}(X,\overline{J}U)\right]\\ &\quad-g(U,W)\left[ag(X,\overline{J}(PV))-\overline{\overline{\nu}}\overline{g}(X,\overline{J}V)\right]\Big\}\overline{J}(PY)\\ &\quad+a\left[g(U,W)g(V,Y)-g(V,W)g(U,Y)\right]\left[\overline{\overline{\nu}}X-aPX\right]\end{aligned}$$

$$
\begin{aligned}
&-a\left[g(U,W)g(V,X)-g(V,W)g(U,X)\right]\left[\overline{\overline{\nu}}Y-aPY\right]\\
&+a\Big\{g(Y,U)\left[ag(W,\overline{J}(PX))-\overline{\overline{\nu}}\overline{g}(W,\overline{J}X)\right]\\
&-g(X,U)\left[ag(W,\overline{J}(PY))-\overline{\overline{\nu}}\overline{g}(W,\overline{J}Y)\right]\\
&+g(Y,W)\left[ag(U,\overline{J}(PX))-\overline{\overline{\nu}}\overline{g}(U,\overline{J}X)\right]\\
&-g(X,W)\left[ag(U,\overline{J}(PY))-\overline{\overline{\nu}}\overline{g}(U,\overline{J}Y)\right]\Big\}\overline{J}(PV)\\
&-a\Big\{g(Y,V)\left[ag(W,\overline{J}(PX))-\overline{\overline{\nu}}\overline{g}(W,\overline{J}X)\right]\\
&-g(X,V)\left[ag(W,\overline{J}(PY))-\overline{\overline{\nu}}\overline{g}(W,\overline{J}Y)\right]\\
&+g(Y,W)\left[ag(V,\overline{J}(PX))-\overline{\overline{\nu}}\overline{g}(V,\overline{J}X)\right]\\
&-g(X,W)\left[ag(V,\overline{J}(PY))-\overline{\overline{\nu}}\overline{g}(V,\overline{J}Y)\right]\Big\}\overline{J}(PU).
\end{aligned}
$$

Assume that (M,g) is semi-symmetric. Then $(\mathcal{R}(X,Y)\cdot R)(U,V,W)=0$ for any $X,Y,U,V,W\in\Gamma(TM)$. If we take $X=PX$, $Y=\xi$, $U=\xi$, $V=PV$, $W=PW$, by using (2.16) we get $ab\overline{\overline{\nu}}\left[-g(PV,PW)\overline{J}(PX)+g(PX,PW)\overline{J}(PV)\right]=0$ for any $X,V,W\in\Gamma(TM)$. Suppose $a\neq 0$. Since $b\neq 0$ and $\overline{\overline{\nu}}\neq 0$ it follows that $g(PV,PW)\overline{J}(PX)=g(PX,PW)\overline{J}(PV)$ for any $X,V,W\in\Gamma(TM)$. Applying $\overline{J}$ to the last equality and replacing W by V we have

$$
g(PV,PV)PX=g(PX,PV)PV. \tag{4.2}
$$

Suppose there exists a vector field $V_0\in\Gamma(TM)$ such that $g(PV_0,PV_0)\neq 0$ at some point $p\in M$. Then from (4.2) it follows that all vectors from $S(TM)$ are collinear with PV_0. This is a contradiction because $\dim S(TM)=2n-2\geq 2$. Hence $a=0$, i.e. $\overline{\overline{\nu}}=\dfrac{\rho^2}{b}$ at any point $p\in M$. Conversely, substituting $a=0$ in $(\mathcal{R}(X,Y)\cdot R)(U,V,W)$, we obtain (i).

(ii)$\Longleftrightarrow$(iii) A lightlike hypersurface M is said to be *Ricci semi-symmetric* [3] if $(\mathcal{R}(X,Y)\cdot \mathrm{Ric})(X_1,X_2)=0$, where

$$
\begin{aligned}
&(\mathcal{R}(X,Y)\cdot \mathrm{Ric})(X_1,X_2)\\
&=-\mathrm{Ric}(R(X,Y,X_1),X_2)-\mathrm{Ric}(X_1,R(X,Y,X_2)),
\end{aligned} \tag{4.3}
$$

$X,Y,X_1,X_2\in\Gamma(TM)$. By using (3.8), (3.9), (4.3) and (2.17) we have

$$
\begin{aligned}
&(\mathcal{R}(X,Y)\cdot \mathrm{Ric})(X_1,X_2)\\
&=-ab\overline{\overline{\nu}}\eta(X_1)\left[\eta(X)g(Y,\overline{J}(PX_2))-\eta(Y)g(X,\overline{J}(PX_2))\right]\\
&-ab\overline{\overline{\nu}}\eta(X_2)\left[\eta(X)g(Y,\overline{J}(PX_1))-\eta(Y)g(X,\overline{J}(PX_1))\right].
\end{aligned} \tag{4.4}
$$

Assume that (M,g) is Ricci semi-symmetric. Then $(\mathcal{R}(X,Y)\cdot\mathrm{Ric})(X_1,X_2)=0$ for any $X,Y,X_1,X_2\in\Gamma(TM)$. Taking $X=\xi$, $Y=PY$, $X_1=PX_1$, $X_2=\xi$ we obtain $ab\overline{\overline{\nu}}g(PY,\overline{J}(PX_1))=0$. From the last equality, taking into account that $b \neq 0$, $\overline{\overline{\nu}} \neq 0$ and $S(TM)$ is non-degenerate, it follows that $a = 0$. The implication (iii)$\Longrightarrow$(ii) follows from (4.4).

Now, let $\overline{\overline{\nu}}$ be a constant. We note that according to Theorem 3.1 $\overline{\nu}$ and $\overline{\overline{\nu}}$ are always constants if $\overline{M}$ is connected and $\dim\overline{M} = 2n \geq 6$.

(iv)$\Longleftrightarrow$(iii) A lightlike hypersurface M is said to be *locally symmetric* [3] if $(\nabla_U R)(X,Y,Z) = 0$, where

$$(\nabla_U R)(X,Y,Z) = \nabla_U R(X,Y,Z) - R(\nabla_U X,Y,Z) - R(X,\nabla_U Y,Z) - R(X,Y,\nabla_U Z), \tag{4.5}$$

$X,Y,Z,U \in \Gamma(TM)$. By using (3.8), (4.5) we obtain

$$\begin{aligned}(\nabla_U R)(X,Y,Z) \quad = \quad & \\ & -\left[a(\nabla_U g)(Y,Z) + g(Y,Z)U(a)\right]\overline{J}(PX) \\ & +\left[a(\nabla_U g)(X,Z) + g(X,Z)U(a)\right]\overline{J}(PY) \\ & -\overline{\overline{\nu}}\left[U(\overline{g}(Y,\overline{J}Z)) - \overline{g}(\nabla_U Y,\overline{J}Z) - \overline{g}(Y,\overline{J}(\nabla_U Z))\right]X \\ & +\overline{\overline{\nu}}\left[U(\overline{g}(X,\overline{J}Z)) - \overline{g}(\nabla_U X,\overline{J}Z) - \overline{g}(X,\overline{J}(\nabla_U Z))\right]Y \\ & -ag(Y,Z)\left[\nabla_U\overline{J}(PX) - \overline{J}(P(\nabla_U X))\right] \\ & +ag(X,Z)\left[\nabla_U\overline{J}(PY) - \overline{J}(P(\nabla_U Y))\right].\end{aligned} \tag{4.6}$$

Consider the following expressions from (4.6)

$$\begin{aligned}(A) &= a(\nabla_U g)(X,Z) + g(X,Z)U(a),\\ (B) &= U(\overline{g}(X,\overline{J}Z)) - \overline{g}(\nabla_U X,\overline{J}Z) - \overline{g}(X,\overline{J}(\nabla_U Z)),\\ (C) &= \nabla_U\overline{J}(PX) - \overline{J}(P(\nabla_U X)).\end{aligned}$$

For the expression (A) by using (2.11) and (2.13) we get

$$(A) = a\rho\left[g(U,X)\eta(Z) + g(U,Z)\eta(X)\right] + g(X,Z)U(a). \tag{4.7}$$

Taking into account (2.4) and the fact that $\overline{\nabla}$ is a metric connection, the expression (B) becomes

$$(B) = \overline{g}((\overline{\nabla}_U\overline{J})Z,X) - \overline{g}((\overline{\nabla}_U\overline{J})X,Z) + B(U,X)\overline{g}(Z,\overline{J}N) + B(U,Z)\overline{g}(X,\overline{J}N).$$

Then (2.16) and $\overline{\nabla J} = 0$ imply that (B) vanishes for any $X,Z,U \in \Gamma(TM)$. By using $\overline{J}(PX) \in S(TM)$, (2.6), (2.13) and (2.18), for the first term of (C) we have

$$\nabla_U\overline{J}(PX) = \nabla^*_U\overline{J}(PX) - \frac{\rho}{b}g(U,X)\xi. \tag{4.8}$$

According to (2.6), (2.7) and (2.14), the second term of (C) becomes

$$\overline{J}(P(\nabla_U X)) = \overline{J}(\nabla^*_U PX) - \rho\eta(X)\overline{J}(PU). \tag{4.9}$$

From (4.8), (4.9) and (2.19) we obtain

$$(C) = -\frac{\rho}{b}g(U,X)\xi + \rho\eta(X)\overline{J}(PU). \tag{4.10}$$

Now, (4.7), $(B) = 0$ and (4.10) imply

$$
\begin{aligned}
(\nabla_U R)(X,Y,Z) = &- \{a\rho\,[g(U,Y)\eta(Z) + g(U,Z)\eta(Y)] \\
&+ g(Y,Z)U(a)\}\,\overline{J}(PX) \\
&+ \{a\rho\,[g(U,X)\eta(Z) + g(U,Z)\eta(X)] \\
&+ g(X,Z)U(a)\}\,\overline{J}(PY) \\
&- ag(Y,Z)\left[-\frac{\rho}{b}g(U,X)\xi + \rho\eta(X)\overline{J}(PU)\right] \\
&+ ag(X,Z)\left[-\frac{\rho}{b}g(U,Y)\xi + \rho\eta(Y)\overline{J}(PU)\right].
\end{aligned} \tag{4.11}
$$

Assume that (M, g) is locally symmetric. Then for any $X, Y, Z, U \in \Gamma(TM)$ we have $(\nabla_U R)(X, Y, Z) = 0$. Taking $Z = \xi$ in (4.11) we obtain

$$
a\rho\left[-g(U,Y)\overline{J}(PX) + g(U,X)\overline{J}(PY)\right] = 0, \quad \forall X, Y, U \in \Gamma(TM).
$$

Taking into account that $\rho \neq 0$ we conclude that $a = 0$ in the same way as in the implication (i)$\Longrightarrow$(iii). The implication (iii)$\Longrightarrow$(iv) follows from (4.11).

We say that a lightlike hypersurface M of an almost complex manifold with Norden metric $(\overline{M}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ is *almost Einstein* if $\mathrm{Ric} = kg + c\widetilde{g}$, where k and c are constants and g, $\widetilde{g}$ are the induced metrics on M by $\overline{g}$, $\overline{\widetilde{g}}$, respectively.

(v)$\Longleftrightarrow$(iii) Let (M, g) be almost Einstein. Then by using (3.9) we have

$$
kg(X,Y) + c\widetilde{g}(X,Y) = -2(n-1)\overline{\widetilde{\nu}}\widetilde{g}(X,Y) + a\widetilde{g}(PX,PY), \quad X, Y \in \Gamma(TM).
$$

If we replace X and Y from the above equality by PX and PY, respectively, we obtain

$$
kg(PX,PY) + \left[c + 2(n-1)\overline{\widetilde{\nu}} - a\right] g(PX, \overline{J}(PY)) = 0. \tag{4.12}
$$

Since $S(TM)$ is non-degenerate, from (4.12) it follows that

$$
kPX + \left[c + (2n-3)\overline{\widetilde{\nu}} + \frac{\rho^2}{b}\right] \overline{J}(PX) = 0. \tag{4.13}
$$

Because PX and $\overline{J}(PX)$ are linearly independent, (4.13) implies that $k = 0$ and $c + (2n-3)\overline{\widetilde{\nu}} + \dfrac{\rho^2}{b} = 0$. The last equality is equivalent to $b = -\dfrac{\rho^2}{c + (2n-3)\overline{\widetilde{\nu}}}$. Since $\overline{\widetilde{\nu}}$ is a constant and by using (2.20), we find $\tau(\xi) = -\dfrac{1}{\rho}\xi(\rho)$. Substituting $\tau(\xi)$ in (3.6) we obtain $\overline{\widetilde{\nu}} = \dfrac{\rho^2}{b}$. The implication (iii) $\Longrightarrow$(v) follows from (3.9). □

Remark 4.1. Analogously, by using Theorem 3.4, we can prove Theorem 1.1 in the case when $(M, \widetilde{g})$ is a totally umbilical radical transversal lightlike hypersurface of $\overline{M}$.

5. An example of a totally umbilical radical transversal lightlike hypersurface, which is locally symmetric, semi-symmetric, Ricci semi-symmetric and almost Einstein

Proposition 5.1. *Let (M, g) (resp. $(M, \widetilde{g})$) be the hypersurface from Theorem 3.3 (resp. Theorem 3.4). If b and ρ (resp. $\widetilde{b}$ and $\widetilde{\rho}$) are constants, then (M, g) (resp. $(M, \widetilde{g})$) is locally symmetric, semi-symmetric, Ricci semi-symmetric and almost Einstein.*

Proof. By using (2.20) (resp. (3.17)), from (3.6) (resp. (3.18)) we obtain $\overline{\overline{\nu}} = \dfrac{\rho^2}{b}$ $\left(resp.\ \overline{\nu} = \dfrac{\widetilde{\rho}^2}{\widetilde{b}}\right)$. Then our assertion follows from Theorem 1.1. □

Further, we construct a Kähler-Norden manifold of constant totally real sectional curvatures and its hypersurface, which satisfies the conditions of Proposition 5.1.

Example 5.2. Consider the Lie group $\overline{G}$ of all (2×2) complex upper triangular matrices with a determinant equal to 1, i.e.

$$\overline{G} = \left\{ \begin{pmatrix} z_1 & z_2 \\ 0 & z_1^{-1} \end{pmatrix} : z_1 \in \mathbb{C}\backslash\{0\}, z_2 \in \mathbb{C} \right\}.$$

The real Lie algebra $\overline{\mathfrak{g}}$ of $\overline{G}$ consists of all (2×2) complex upper triangular traceless matrices, i.e.
$\overline{\mathfrak{g}} = \left\{ \begin{pmatrix} w_1 & w_2 \\ 0 & -w_1 \end{pmatrix} : w_1, w_2 \in \mathbb{C} \right\}$. The Lie algebra $\overline{\mathfrak{g}}$ is spanned by the left invariant vector fields $\{X_1, X_2, X_3, X_4\}$, where

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ X_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix},\ X_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The non-zero Lie brackets of the basic vector fields are given by

$$[X_1, X_2] = -[X_3, X_4] = -2X_4\,; \qquad [X_1, X_4] = -[X_2, X_3] = 2X_2\,. \tag{5.1}$$

Since $\overline{\mathfrak{g}}$ is a complex Lie algebra, we define a complex structure $\overline{J}$ on $\overline{\mathfrak{g}}$ by $\overline{J}\overline{X} = -i\overline{X}$ for any left invariant vector field $\overline{X}$ belonging to $\overline{\mathfrak{g}}$. Hence we have $[\overline{J}\overline{X}, \overline{Y}] = \overline{J}[\overline{X}, \overline{Y}]$ for any $\overline{X}, \overline{Y} \in \overline{\mathfrak{g}}$, i.e. $\overline{J}$ is a bi-invariant complex structure. Now, we define a left invariant metric $\overline{g}$ on $\overline{\mathfrak{g}}$ by

$$\begin{aligned} &\overline{g}(X_i, X_i) = -\overline{g}(X_j, X_j) = 1; \quad i = 1, 2; \quad j = 3, 4; \\ &\overline{g}(X_i, X_j) = 0; \quad i \neq j; \quad i, j = 1, 2, 3, 4. \end{aligned} \tag{5.2}$$

From $\overline{J}X_1 = X_3$, $\overline{J}X_2 = X_4$ it follows that the introduced metric is a Norden metric on $\overline{G}$. Thus $(\overline{G}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ is a 4-dimensional complex manifold with Norden metric. Since the metric $\overline{g}$ is left invariant, for the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ we have

$$2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) = \overline{g}([\overline{X}, \overline{Y}], \overline{Z}) + \overline{g}([\overline{Z}, \overline{X}], \overline{Y}) + \overline{g}([\overline{Z}, \overline{Y}], \overline{X}) \tag{5.3}$$

for any $\overline{X}, \overline{Y}, \overline{Z} \in \overline{\mathfrak{g}}$. By using the fact that $\overline{J}$ is bi-invariant and (5.3), we get $F(\overline{X}, \overline{Y}, \overline{Z}) = 0$. Hence $(\overline{G}, \overline{J}, \overline{g}, \overline{\widetilde{g}})$ is a Kähler-Norden manifold. Next, by using (5.1), (5.2) and (5.3) we obtain the components of $\overline{\nabla}$. For the non-zero of them we have

$$\begin{array}{ll} \overline{\nabla}_{X_2} X_1 = -\overline{\nabla}_{X_4} X_3 = 2X_4\,; & \overline{\nabla}_{X_2} X_2 = -\overline{\nabla}_{X_4} X_4 = -2X_3\,; \\ \overline{\nabla}_{X_2} X_3 = \overline{\nabla}_{X_4} X_1 = -2X_2\,; & \overline{\nabla}_{X_2} X_4 = \overline{\nabla}_{X_4} X_2 = 2X_1\,. \end{array} \tag{5.4}$$

We denote $\overline{R}_{ijkl} = \overline{R}(X_i, X_j, X_k, X_l)$, $(i, j, k, l = 1, 2, 3, 4)$, where $\overline{R}$ is the curvature tensor on $\overline{G}$. Taking into account (5.4) we get the non-zero components of $\overline{R}$

$$\overline{R}_{1441} = \overline{R}_{2332} = \overline{R}_{1423} = -\overline{R}_{1221} = -\overline{R}_{3443} = -\overline{R}_{1234} = -4. \tag{5.5}$$

By using (3.3), (5.2) and (5.5) we find

$$\overline{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = 4\left\{(\overline{\pi}_1 - \overline{\pi}_2)(\overline{X}, \overline{Y}, \overline{Z}, \overline{W})\right\}$$

for any $\overline{X}, \overline{Y}, \overline{Z}, \overline{W} \in \overline{\mathfrak{g}}$. Then from Theorem 3.1 it follows that $\overline{G}$ is of constant totally real sectional curvatures $\overline{\nu} = 4$ and $\overline{\widetilde{\nu}} = 0$ with respect to $\overline{g}$. Further we consider the Lie subalgebra $\mathfrak{g}$ of $\overline{\mathfrak{g}}$ which is spanned by $\{X_2, X_3, X_4\}$. The corresponding to $\mathfrak{g}$ Lie subgroup G of $\overline{G}$ is given by

$$G = \left\{ \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}\backslash\{0\}, z \in \mathbb{C} \right\}.$$

Denote by g and $\widetilde{g}$ the induced metrics on G of $\overline{g}$ and $\overline{\widetilde{g}}$, respectively. Hence (G, g) and $(G, \widetilde{g})$ are hypersurfaces of $\overline{G}$. By using $\overline{J}X_1 = X_3$, $\overline{J}X_2 = X_4$, $\overline{J}X_3 = -X_1$, $\overline{J}X_4 = -X_2$ and (5.2) we get

$$\begin{array}{l} \widetilde{g}(X_2, X_2) = \widetilde{g}(X_3, X_3) = \widetilde{g}(X_4, X_4) = \widetilde{g}(X_2, X_3) = \widetilde{g}(X_3, X_4) = 0; \\ \widetilde{g}(X_2, X_4) = -1 \end{array} \tag{5.6}$$

and

$$\overline{\widetilde{g}}(X_1, X_1) = \overline{\widetilde{g}}(X_1, X_2) = \overline{\widetilde{g}}(X_1, X_4) = 0; \quad \overline{\widetilde{g}}(X_1, X_3) = -1. \tag{5.7}$$

The equalities (5.6) imply that $(G, \widetilde{g})$ is a lightlike hypersurface of $\overline{G}$. Taking into account (5.7) it is easy to check that the normal space $\mathfrak{g}^{\widetilde{\perp}}$ of $\mathfrak{g}$ with respect to $\overline{\widetilde{g}}$ is spanned by $\{\xi = -X_3\}$ and the screen distribution $S(\mathfrak{g})$ is spanned by $\{X_2, X_4\}$. The pair $\{N = X_1, \xi\}$ satisfies the conditions (3.16). Since $\overline{J}\xi = N$, we have $\widetilde{b} = 1$. Hence $(G, \widetilde{g})$ is a radical transversal lightlike hypersurface of $\overline{G}$. As the Levi-Civita connections $\overline{\nabla}$ and $\overline{\widetilde{\nabla}}$ on $\overline{G}$ coincide, from (5.4) we obtain

$$\overline{\widetilde{\nabla}}_{\xi}\xi = 0, \quad \overline{\widetilde{\nabla}}_{X_2}\xi = 2X_2, \quad \overline{\widetilde{\nabla}}_{X_4}\xi = 2X_4. \tag{5.8}$$

Now, (2.7) and (5.8) imply that $A^*_\xi X = A^*_\xi PX = -2PX$ for any $X \in \mathfrak{g}$, i.e. $(G, \widetilde{g})$ is totally umbilical and $\widetilde{\rho} = -2$. Then, according to Proposition 5.1, the totally umbilical radical transversal lightlike hypersurface $(G, \widetilde{g})$ of $\overline{G}$ is locally symmetric, semi-symmetric, Ricci semi-symmetric and almost Einstein.

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