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A NOTE ON INEQUALITIES FOR TSALLIS RELATIVE OPERATOR ENTROPY

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Abstract. In this short note, we present some inequalities for relative operator entropy which are generalizations of some results obtained by Zou [Operator inequalities associated with Tsallis relative operator entropy, *Math. Inequal. Appl.* 18 (2015), no. 2, 401–406]. Meanwhile, we also show some new lower and upper bounds for relative operator entropy and Tsallis relative operator entropy.
Keywords: Relative operator entropy, Tsallis relative operator entropy, operator inequalities.
MSC(2010): Primary: 47A63; Secondary: 47B65.

1. Introduction

In this note we mainly adopt the notation and terminology in [7]. For convenience, recall that. For two invertible positive operators A and B and $\lambda \in (0,\, 1]$, the Tsallis relative operator entropy $T_\lambda(A|B)$ and the relative operator entropy $S(A|B)$ are defined by

$$T_\lambda(A|B) = \frac{A\#_\lambda B - A}{\lambda},$$

$$S(A|B) = A^{1/2} \log\left(A^{-1/2} B A^{-1/2}\right) A^{1/2},$$

where

$$A\#_\lambda B = A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^\lambda A^{1/2}$$

is the wieghted geometric mean.

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Recently, Zou [7] proved that if $a>0$ and $\lambda\in(0,\ 1]$, then for any invertible positive operators A and B,

(1.1)
$$\begin{aligned}(1-\log a)A-\frac{1}{a}AB^{-1}A &\le -\left(\frac{1-a^{\lambda}}{\lambda a^{\lambda}}+\log a\right)A+a^{-\lambda}T_{-\lambda}(A|B)\\ &\le S(A|B)\\ &\le (\log a)A+T_{\lambda}(A|B)+\frac{1-a^{\lambda}}{\lambda a^{\lambda}}A\#_{\lambda}B\\ &\le (\log a-1)A+\frac{1}{a}B.\end{aligned}$$

This is a refinement of the following inequality:

$$(1-\log a)A-\frac{1}{a}AB^{-1}A\le S(A|B)\le(\log a-1)A+\frac{1}{a}B,$$

which is due to Furuta [2]. For more information on the Tsallis relative entropy the reader is referred to [1–6] and the references therein.

In this short note, we will present some generalizations of (1.1). Meanwhile, we also show some new lower and upper bounds for relative operator entropy $S(A|B)$ and Tsallis relative operator entropy $T_{\lambda}(A|B)$.

2. Main results

We begin this section with the following result.

Theorem 2.1. *Let $a>0$, $\lambda\in(0,1]$ and $v\in[0,1]$. For any invertible positive operators A and B, we have*

$$\begin{aligned}S(A|B) &\le \left(v+(1-v)a^{\lambda}\right)T_{\lambda}(A|B)-v\frac{1-a^{\lambda}}{\lambda}A\#_{\lambda}B\\ &+ \left((1-v)\frac{a^{\lambda}-1}{\lambda}-\log a\right)A.\end{aligned}$$

Proof. Note that

$$\frac{(ax)^{\lambda}-1}{\lambda}=\frac{x^{\lambda}-1}{\lambda}+x^{\lambda}\frac{a^{\lambda}-1}{\lambda}, \tag{2.1}$$

$$\frac{(ax)^{\lambda}-1}{\lambda}=a^{\lambda}\frac{x^{\lambda}-1}{\lambda}+\frac{a^{\lambda}-1}{\lambda}. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}\frac{(ax)^{\lambda}-1}{\lambda} &= v\left(\frac{x^{\lambda}-1}{\lambda}+x^{\lambda}\frac{a^{\lambda}-1}{\lambda}\right)\\ &+ (1-v)\left(a^{\lambda}\frac{x^{\lambda}-1}{\lambda}+\frac{a^{\lambda}-1}{\lambda}\right)\\ &= \left(v+(1-v)a^{\lambda}\right)\frac{x^{\lambda}-1}{\lambda}\\ &+ \left(v\left(x^{\lambda}-1\right)+1\right)\frac{a^{\lambda}-1}{\lambda}.\end{aligned} \tag{2.3}$$

It is known that for positive real number x, we have

$$\log ax \leq \frac{(ax)^{\lambda}-1}{\lambda}. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\log x \leq \left(v+(1-v)\,a^{\lambda}\right)\frac{x^{\lambda}-1}{\lambda}+v\frac{a^{\lambda}-1}{\lambda}x^{\lambda}+(1-v)\,\frac{a^{\lambda}-1}{\lambda}-\log a. \tag{2.5}$$

The result

$$\begin{aligned} S\left(A|B\right) &\leq \left(v+(1-v)\,a^{\lambda}\right)T_{\lambda}\left(A|B\right)-v\frac{1-a^{\lambda}}{\lambda}A\#_{\lambda}B \\ &+ \left((1-v)\,\frac{a^{\lambda}-1}{\lambda}-\log a\right)A. \end{aligned}$$

follows from (2.5) by applying the functional calculus $x=A^{-1/2}BA^{-1/2}$ and then multiplying both sides by $A^{1/2}$. □

Remark 2.2. Putting $v=1$ in (2.1), we get the third part of (1.1). Putting $v=0$ in (2.1), we have

$$S\left(A|B\right) \leq a^{\lambda}T_{\lambda}\left(A|B\right)+\left(\frac{a^{\lambda}-1}{\lambda}-\log a\right)A, \tag{2.6}$$

which is an upper bound for $S\left(A|B\right)$.

Theorem 2.3. *Let $a>0$, $\lambda\in(0,1]$ and $v\in[0,1]$. For any invertible positive operators A and B, we have*

$$\begin{aligned} \left(v+(1-v)\,a^{-\lambda}\right)T_{-\lambda}\left(A|B\right) &- v\frac{a^{-\lambda}-1}{\lambda}A\#_{-\lambda}B \\ &- \left((1-v)\,\frac{a^{-\lambda}-1}{\lambda}+\log a\right)A \\ &\leq S\left(A|B\right). \end{aligned} \tag{2.7}$$

Proof. Substituting x by x^{-1} and a by a^{-1} in (2.5), respectively, we have

$$\left(v+(1-v)\,a^{-\lambda}\right)\frac{x^{-\lambda}-1}{-\lambda}-v\frac{a^{-\lambda}-1}{\lambda}x^{-\lambda}-(1-v)\,\frac{a^{-\lambda}-1}{\lambda}-\log a \leq \log x$$

for all $x>0$. The result

$$\begin{aligned} \left(v+(1-v)\,a^{-\lambda}\right)T_{-\lambda}\left(A|B\right) &- v\frac{a^{-\lambda}-1}{\lambda}A\#_{-\lambda}B \\ &- \left((1-v)\,\frac{a^{-\lambda}-1}{\lambda}+\log a\right)A \\ &\leq S\left(A|B\right). \end{aligned}$$

follows from this last inequality by applying the functional calculus $x=A^{-1/2}BA^{-1/2}$ and then multiplying both sides by $A^{1/2}$. □

Remark 2.4. Putting $v = 0$ in (2.7), we get the second part of (1.1). Putting $v = 1$ in (2.7), we have

$$T_{-\lambda}(A|B) - \frac{a^{-\lambda}-1}{\lambda} A\#_{-\lambda}B - (\log a)\, A \le S(A|B)\,, \tag{2.8}$$

which is a lower bound for $S(A|B)$.

Remark 2.5. It follows from (2.6) and (2.8) that

$$\begin{aligned} T_{-\lambda}(A|B) \quad &- \quad \frac{a^{-\lambda}-1}{\lambda} A\#_{-\lambda}B - (\log a)\, A \\ &\le \quad S(A|B) \\ &\le \quad a^{\lambda} T_{\lambda}(A|B) + \left(\frac{a^{\lambda}-1}{\lambda} - \log a\right) A. \end{aligned} \tag{2.9}$$

Putting $a = 1$ in (2.9), we have

$$T_{-\lambda}(A|B) \le S(A|B) \le T_{\lambda}(A|B)\,,$$

which is due to Furuichi, Yanagi and Kuriyama [1].

Theorem 2.6. *Let $a > 0$, $\lambda \in (0,1]$ and $v \in [0,1]$. For any invertible positive operators A and B, we have*

$$\begin{aligned} A - \frac{1}{a} AB^{-1}A \quad &+ \quad v\frac{a^{-\lambda}-1}{\lambda} A\#_{-\lambda}B + (1-v)\,\frac{a^{-\lambda}-1}{\lambda} A \\ &\le \quad \left(v + (1-v)\, a^{-\lambda}\right) T_{-\lambda}(A|B)\,. \end{aligned} \tag{2.10}$$

Proof. Substituting λ by $-\lambda$ in (2.3), we have

$$\begin{aligned} \frac{(ax)^{-\lambda}-1}{-\lambda} \quad &= \quad \left(v + (1-v)\, a^{-\lambda}\right) \frac{x^{-\lambda}-1}{-\lambda} \\ &+ \quad \left(v\left(x^{-\lambda}-1\right)+1\right) \frac{a^{-\lambda}-1}{-\lambda}. \end{aligned} \tag{2.11}$$

For $a > 0$ and $\lambda \in [0,\ 1]$, we have

$$1 - \frac{1}{ax} \le \frac{(ax)^{-\lambda}-1}{-\lambda}, \quad x > 0. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$1 - \frac{1}{ax} + v\frac{a^{-\lambda}-1}{\lambda} x^{-\lambda} + (1-v)\,\frac{a^{-\lambda}-1}{\lambda} \le \left(v + (1-v)\, a^{-\lambda}\right) \frac{x^{-\lambda}-1}{-\lambda}.$$

The result

$$\begin{aligned} A - \frac{1}{a} AB^{-1}A \quad &+ \quad v\frac{a^{-\lambda}-1}{\lambda} A\#_{-\lambda}B + (1-v)\,\frac{a^{-\lambda}-1}{\lambda} A \\ &\le \quad \left(v + (1-v)\, a^{-\lambda}\right) T_{-\lambda}(A|B)\,. \end{aligned}$$

follows from this last inequality by applying the functional calculus $x = A^{-1/2}BA^{-1/2}$ and then multiplying both sides by $A^{1/2}$. □

Remark 2.7. Putting $v = 0$ in (2.10), we get the first part of (1.1). Putting $v = 1$ in (2.10), we have

$$A - \frac{1}{a}AB^{-1}A + \frac{a^{-\lambda} - 1}{\lambda}A\#_{-\lambda}B \leq T_{-\lambda}(A|B),$$

which is a lower bound for $T_{-\lambda}(A|B)$. Putting $a = 1$ in this last inequality, we have

$$A - AB^{-1}A \leq T_{-\lambda}(A|B),$$

which is due to Zou [7].

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