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CENTER–LIKE SUBSETS IN RINGS WITH DERIVATIONS OR EPIMORPHISMS

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ABSTRACT. We introduce center–like subsets $Z(R, f), Z^{**}(R, f)$ and $Z_{1}(R, f)$, where $R$ is a ring and $f$ is a map from $R$ to $R$. For $f$ a derivation or a non-identity epimorphism and $R$ a suitably–chosen prime or semiprime ring, we prove that these sets coincide with the center of $R$.

Keywords: Prime ring, semiprime ring, derivation, epimorphism, center–like subset.


1. Introduction

Let $R$ denote a ring with center $Z = Z(R)$. For each $x, y \in R$, let $[x, y]$ denote the commutator $xy - yx$; and recall that $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$.

Several results in the literature assert that certain subsets of a ring $R$, defined by some sort of commutativity condition, must coincide with $Z(R)$. We call such subsets center–like subsets. A classical result of Herstein [11] states that the hypercenter $S(R)$, defined as $\{a \in R \mid ax^n = x^n a, n = n(x, a) \geq 1, \text{ for all } x \in R\}$, coincides with $Z(R)$ if $R$ has no nonzero nil ideals. Following Herstein, Chacron [7] introduced the cohypercenter $G(R)$ as follows: $a \in G(R)$ if and only if for each $x \in R$ there exists a polynomial $p(X) \in Z[X]$, depending on $a$ and $x$, such that $[a, x - x^2p(x)] = 0$; and he established equality of $Z(R)$ and $G(R)$ for semiprime $R$. Similar results of this kind are to be found in [3, 4, 6, 8, 9] and [10].

Our purpose is to study center–like subsets the definition of which involves a map $f : R \to R$, which have not been extensively studied. Apparently the first example of such a set was $H(R, d) = \{a \in R \mid ad(x) = d(x)a \text{ for all } x \in R\}$, where $d$ is a derivation. Herstein introduced this set in [13], and he proved that
if $R$ is prime with $\text{char}(R) \neq 2$ and $d$ is a nonzero derivation, then $H(R, d) = Z(R)$.

In [1] it is proved that a semiprime ring must be commutative if there exists a derivation $d$ on $R$ such that $[x, y] = [d(x), d(y)]$ for all $x, y \in R$; and in [2] it is shown that a prime ring is commutative if for some nonzero derivation $d$, $[d(x), d(y)] = [d(x), y] + [x, d(y)]$ for all $x, y \in R$. Motivated by these results, we define the following subsets of a ring $R$ equipped with a map $f : R \to R$:

$$Z^*(R, f) = \{y \in R \mid [x, y] = [f(y), f(x)] \text{ for all } x \in R\};$$

$$Z^{**}(R, f) = \{y \in R \mid [x, y] = [f(x), f(y)] \text{ for all } x \in R\};$$

$$Z_1(R, f) = \{y \in R \mid [f(x), f(y)] = [f(y), x] + [y, f(x)] \text{ for all } x \in R\}.$$

We shall be concerned with these sets when $f$ is a derivation or an epimorphism.

2. Results on derivations

**Theorem 2.1.** Let $R$ be a semiprime ring and $d$ a derivation on $R$. Then $Z^*(R, d) = Z$.

**Proof.** Since $d(Z) \subseteq Z$, $Z \subseteq Z^*(R, d)$; thus we only need to show that $Z^*(R, d) \subseteq Z$. Let $y \in Z^*(R, d)$, i.e.,

$$[x, y] = [d(y), d(x)] \text{ for all } x \in R.$$  \hspace{1cm} (2.1)

Substituting $xy$ for $x$ in (2.1) and then using (2.1), we obtain

$$d(x)[y, d(y)] + [x, d(y)]d(y) = 0 \text{ for all } x \in R.$$  \hspace{1cm} (2.2)

Replacing $x$ by $xy$ in (2.2) and simplifying using (2.2), we get

$$d(x)w[y, d(y)] + [x, d(y)]wd(y) = 0 \text{ for all } x, w \in R;$$

and taking $x = d(y)$ now gives $d^2(y)R[y, d(y)] = \{0\}$. It follows that

$$d^2(y)R[y, d(y)] = \{0\}.$$  \hspace{1cm} (2.4)

But by (2.1) with $x = d(y)$, $[d(y), y] = [d(y), d^2(y)]$, so (2.4) yields $[y, d(y)]R[y, d(y)] = \{0\}$; and semiprimeness gives $[y, d(y)] = 0$. By (2.3) we now get $[x, d(y)]R[x, d(y)] = \{0\}$ for all $x \in R$, so that $d(y) \in Z$ and by (2.1) $y \in Z$.

A similar proof yields

**Theorem 2.2.** If $R$ is a semiprime ring and $d$ is a derivation on $R$, then $Z^{**}(R, d) = Z$.

**Corollary 2.3.** Let $R$ be a semiprime ring and $U$ a nonzero left ideal of $R$. If $R$ admits a derivation $d$ such that $d(U) \neq \{0\}$ and $[x, [y, d(y)]] = [[y, d^2(y)], d(x)]$ for all $x \in R$ and $y \in U$, then $R$ contains a nonzero central ideal.

**Proof.** By Theorem 2.1, $d$ is centralizing on $U$; hence by Theorem 3 of [5], $R$ contains a nonzero central ideal. \hfill $\square$
Corollary 2.4. Let $R$ be a semiprime ring, $U$ a nonzero ideal of $R$ and $d$ a derivation on $R$ such that $d(U) \neq \{0\}$ and $d(U) \subseteq U$. If $[x, [y, d(y)]] = [[y, d^2(y)], d(x)]$ for all $x, y \in U$, then $R$ contains a nonzero central ideal.

Proof. Since $U$ is an ideal of a semiprime ring, $U$ is a semiprime ring; and $d$ restricts to a derivation on $U$. Hence, by Theorem 2.1, $[y, d(y)] \in Z(U)$ for all $y \in U$. But, as is easily can be shown, $Z(U) \subseteq Z(R)$; therefore $d$ is centralizing on $R$. Again invoking Theorem 3 of [5], we see that $R$ must contain a nonzero central ideal. □

We now proceed to study $Z_1(R, d)$.

Theorem 2.5. Let $R$ be a prime ring with $\text{char}(R) \neq 2$ and $d$ a nonzero derivation on $R$. Then $Z_1(R, d) = Z$.

Proof. We only need to prove that $Z_1(R, d) \subseteq Z$, since the other inclusion is immediate. Let $y \in Z_1(R, d)$, so that

\[ [d(x), d(y)] = [d(y), x] + [y, d(x)] \quad \text{for all} \quad x \in R. \tag{2.5} \]

Substituting $xy$ for $x$ in (2.5), we obtain

\[ d(x)[y, d(y)] + [x, d(y)]d(y) = [y, x]d(y) \quad \text{for all} \quad x \in R; \tag{2.6} \]

and substituting $xw$ for $x$ in (2.6), we get

\[ d(x)w[y, d(y)] + [x, d(y)]wd(y) = [y, x]wd(y) \quad \text{for all} \quad x, w \in R. \tag{2.7} \]

Replacing $x$ by $d(y)$ in (2.7) gives

\[ d^2(y)w[y, d(y)] = [y, d(y)]wd(y) \quad \text{for all} \quad w \in R; \tag{2.8} \]

and taking $x = y$ in (2.7) gives

\[ d(y)w[y, d(y)] + [y, d(y)]wd(y) = 0 \quad \text{for all} \quad w \in R. \tag{2.9} \]

From (2.8) and (2.9) we conclude that

\[ (d(y) + d^2(y))w[y, d(y)] = 0 \quad \text{for all} \quad w \in R; \tag{2.10} \]

and since $R$ is prime, either $d(y) + d^2(y) = 0$ or $[y, d(y)] = 0$. Suppose that $d(y) + d^2(y) = 0$. Then $[d^2(y), d(y)] = 0$; and by taking $x = d(y)$ in (2.5), we get $[y, d^2(y)] = 0$, so that $[y, d(y)] = 0$. Thus, in either case $[y, d(y)] = 0$.

It now follows from (2.7) that $[x, y + d(y)]wd(y) = 0$ for all $x, w \in R$, so by primeness of $R$, either $d(y) = 0$ or $y + d(y) \in Z$. If $y + d(y) \in Z$, then $[d(x), d(y)] = [d(x), y]$ for all $x \in R$. This fact, together with (2.5), gives $d(y) \in Z$; and (2.5) now gives $[y, d(R)] = \{0\}$. If $d(y) = 0$, we also have $[y, d(R)] = \{0\}$, so $Z_1(R, d) \subseteq H(R, d)$; hence by Herstein’s result mentioned earlier, $Z_1(R, d) \subseteq Z$. □
If we assume $R$ is semiprime instead of prime and consider a family $\{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals for which $\bigcap P_\alpha = \{0\}$, the argument used to prove Theorem 2.5 can be modified to yield $[y, d(R)] \subseteq P_\alpha$ for all $\alpha \in \Lambda$. Hence $Z_1(R, d) \subseteq H(R, d)$. But we cannot prove that $Z_1(R, d) \subseteq Z$, as the following example shows.

**Example 2.6.** Let $R = R_1 \oplus R_2$, where $R_1$ is a commutative domain with nonzero derivation $d_1$ and $R_2$ is a noncommutative prime ring. Then $R$ is semiprime with nonzero derivation $d$ given by $d((r_1, r_2)) = (d_1(r_1), 0)$. For $y = (0, r_2)$, we have $d(y) = 0$ and $[y, d(x)] = 0$ for all $x \in R$; hence $y \in Z_1(R, d)$. Thus $S = \{(0, r_2) \mid r_2 \in R_2\} \subseteq Z_1(R, d)$, but $S \not\subseteq Z$.

**3. Results on epimorphisms**

We turn now to the results involving epimorphisms.

**Lemma 3.1** ([12, Lemma 1.1.9]). Let $R$ be a 2-torsion-free semiprime ring. If $y \in R$ and $[[x, y], y] = 0$ for all $x \in R$, then $y \in Z$.

**Lemma 3.2.** Let $R$ be a prime ring and $T$ an endomorphism which is not the identity map. If $u \in R$ and $u(x - T(x)) = 0$ for all $x \in R$, then $u = 0$.

**Proof.** Assume $ux = uT(x)$ for all $x \in R$. Then for all $r, x \in R$, $uxr = uT(xr) = uT(x)T(r) = uT(xT(r))$, so that $ux(r - T(r)) = 0$ for all $x \in R$; and by primeness of $R$ we get $u = 0$. \hfill \Box

**Lemma 3.3.** Let $R$ be an arbitrary ring and $T$ an epimorphism of $R$. Then

(i) $Z^*(R, T)$ is an additive subgroup of $R$;

(ii) if $T(u) = u$ and $y \in Z^*(R, T)$, then $[u, y - T(y)] = 0$;

(iii) if $y \in Z^*(R, T)$, then $T(y) \in Z^*(R, T)$;

(iv) if $y \in Z^*(R, T)$, then $y - T(y) \in Z^*(R, T)$.

**Proof.** (i) and (ii) are immediate from the definition of $Z^*(R, T)$ and (iv) follows from (i) and (iii). Therefore we only need to prove (iii). Let $y \in Z^*(R, T)$, so that $[x, y] = [T(x), T(y)]$ for all $x \in R$. Applying $T$ to both sides of this equation yields $[T(x), T(y)] = [T(T(x)), T(T(y))]$; and since $T$ is an epimorphism, $[w, T(y)] = [T(w), T(T(y))]$ for all $w \in R$. Therefore, $T(y) \in Z^*(R, T)$. \hfill \Box

**Theorem 3.4.** Let $R$ be a prime ring with $\text{char}(R) \neq 2$, and let $T$ be an epimorphism of $R$ which is not the identity map. Then $Z^*(R, T) = Z$.

**Proof.** Clearly, $Z \subseteq Z^*(R, T)$, since $T(Z) \subseteq Z$; therefore we only need to show that $Z^*(R, T) \subseteq Z$. Let $y \in Z^*(R, T)$ and note that for any $x \in R, u = [x, y]$ satisfies $T(u) = u$. Therefore, by Lemma 3.3 (ii) and (iv),

\[(3.1) \quad [x, y - T(y)], y - T(y) = 0 \text{ for all } x \in R,\]
hence by Lemma 3.1 \( y - T(y) \in Z \). Thus, for all \( x \in R \), \([x, y] = [T(x), T(y)] = [T(x), y] \), so that
\[
(3.2) \quad [x - T(x), y] = 0 \text{ for all } x \in R.
\]
Substituting \( rx \) for \( x \) in (3.2), we have \([rx, y] - [T(r)T(x), y] = 0 \), so
\[
r[x, y] + [r, y]x - T(r)[T(x), y] - [T(r), y]T(x) = 0,
\]
which by (3.2) may be rewritten as
\[
(3.3) \quad (r - T(r))[x, y] + [r, y](x - T(x)) = 0 \text{ for all } r, x \in R.
\]
Recalling that \( T([r, y]) = [r, y] \) and replacing \( r \) by \([r, y]\) in (3.3), we get \([r, y](x - T(x)) = 0 \) for all \( r, x \in R \). Using Lemma 3.2 we conclude that \([r, y] = 0 \) for all \( r \in R \), hence by Lemma 3.1, \( y \in Z \). □

We note that Theorem 3.4 cannot be extended to semiprime rings, as the following example shows.

**Example 3.5.** Let \( R = R_1 \oplus R_2 \), where \( R_1 \) is a commutative domain with epimorphism \( T_1 \) which is not the identity map on \( R_1 \), and \( R_2 \) is a noncommutative prime ring; and define \( T : R \to R \) by \( T((r_1, r_2)) = (T_1(r_1), r_2) \). Then \( R \) is semiprime, \( T \) is a non-identity epimorphism, and \( \{(0, r_2) \mid r_2 \in R_2\} \) is a noncentral subset of \( Z^*(R, T) \).

Our final theorem involves \( Z^*(R, T) \) when \( T \) is an epimorphism.

**Theorem 3.6.** Let \( R \) be a prime ring with \( \text{char}(R) \neq 2 \), and let \( T : R \to R \) be an epimorphism such that \( T^2 \) is not the identity map. Then \( Z^*(R, T) = Z \).

**Proof.** By an argument similar to the one in the proof of Theorem 3.4, we get
\[
(3.4) \quad [x + T(x), y] = 0 \text{ for all } x \in R, y \in Z^*(R, T).
\]
Replacing \( x \) by \( rx \) in (3.4), we have for all \( r, x \in R, y \in Z^*(R, T) \)
\[
r[x, y] + [r, y]x + T(r)[T(x), y] + [T(r), y]T(x) = 0;
\]
and by (3.4) we can rewrite this equation as
\[
(3.5) \quad (r - T(r))[x, y] + [r, y](x - T(x)) = 0 \text{ for all } x, r \in R.
\]
We now substitute \( w + T(w) \) for \( r \) in (3.5), and since \([w + T(w), y] = 0 \), we obtain
\[
(3.6) \quad (w - T^2(w))[x, y] = 0 \text{ for all } w, x \in R.
\]
This easily yields \((w - T^2(w))R[x, y] = \{0\} \), so by primeness of \( R, y \in Z \). □
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