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CENTER–LIKE SUBSETS IN RINGS WITH DERIVATIONS OR EPIMORPHISMS

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Abstract. We introduce center–like subsets $Z^*(R,f), Z^{**}(R,f)$ and $Z_1(R,f)$, where R is a ring and f is a map from R to R. For f a derivation or a non-identity epimorphism and R a suitably–chosen prime or semiprime ring, we prove that these sets coincide with the center of R.
Keywords: Prime ring, semiprime ring, derivation, epimorphism, center–like subset.
MSC(2010): Primary: 16W20; Secondary: 16W25, 16U70, 16U80.

1. Introduction

Let R denote a ring with center $Z = Z(R)$. For each $x, y \in R$, let $[x, y]$ denote the commutator $xy - yx$; and recall that $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$.

Several results in the literature assert that certain subsets of a ring R, defined by some sort of commutativity condition, must coincide with $Z(R)$. We call such subsets *center–like subsets*. A classical result of Herstein [11] states that the *hypercenter* $S(R)$, defined as $\{a \in R \mid ax^n = x^n a, n = n(x, a) \geq 1, \text{ for all } x \in R\}$, coincides with $Z(R)$ if R has no nonzero nil ideals. Following Herstein, Chacron [7] introduced the *cohypercenter* $G(R)$ as follows: $a \in G(R)$ if and only if for each $x \in R$ there exists a polynomial $p(X) \in \mathbb{Z}[X]$, depending on a and x, such that $[a, x - x^2 p(x)] = 0$; and he established equality of $Z(R)$ and $G(R)$ for semiprime R. Similar results of this kind are to be found in [3, 4, 6, 8, 9] and [10].

Our purpose is to study center–like subsets the definition of which involves a map $f : R \to R$, which have not been extensively studied. Apparently the first example of such a set was $H(R, d) = \{a \in R \mid ad(x) = d(x)a \text{ for all } x \in R\}$, where d is a derivation. Herstein introduced this set in [13], and he proved that

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if R is prime with $char(R) \neq 2$ and d is a nonzero derivation, then $H(R,d) = Z(R)$.

In [1] it is proved that a semiprime ring must be commutative if there exists a derivation d on R such that $[x,y] = [d(x), d(y)]$ for all $x, y \in R$; and in [2] it is shown that a prime ring is commutative if for some nonzero derivation d, $[d(x), d(y)] = [d(x), y] + [x, d(y)]$ for all $x, y \in R$. Motivated by these results, we define the following subsets of a ring R equipped with a map $f : R \to R$:

$$Z^*(R,f) = \{y \in R \mid [x,y] = [f(y), f(x)] \text{ for all } x \in R\};$$
$$Z^{**}(R,f) = \{y \in R \mid [x,y] = [f(x), f(y)] \text{ for all } x \in R\};$$
$$Z_1(R,f) = \{y \in R \mid [f(x), f(y)] = [f(y), x] + [y, f(x)] \text{ for all } x \in R\}.$$

We shall be concerned with these sets when f is a derivation or an epimorphism.

2. **Results on derivations**

Theorem 2.1. *Let R be a semiprime ring and d a derivation on R. Then $Z^*(R,d) = Z$.*

Proof. Since $d(Z) \subseteq Z$, $Z \subseteq Z^*(R,d)$; thus we only need to show that $Z^*(R,d) \subseteq Z$. Let $y \in Z^*(R,d)$, i.e.,

$$[x,y] = [d(y), d(x)] \text{ for all } x \in R. \tag{2.1}$$

Substituting xy for x in (2.1) and then using (2.1), we obtain

$$d(x)[y, d(y)] + [x, d(y)]d(y) = 0 \text{ for all } x \in R. \tag{2.2}$$

Replacing x by xw in (2.2) and simplifying using (2.2), we get

$$d(x)w[y, d(y)] + [x, d(y)]wd(y) = 0 \text{ for all } x, w \in R; \tag{2.3}$$

and taking $x = d(y)$ now gives $d^2(y)R[y, d(y)] = \{0\}$. It follows that

$$[d^2(y), d(y)]R[y, d(y)] = \{0\}. \tag{2.4}$$

But by (2.1) with $x = d(y)$, $[d(y), y] = [d(y), d^2(y)]$, so (2.4) yields $[y, d(y)]R[y, d(y)] = \{0\}$; and semiprimeness gives $[y, d(y)] = 0$. By (2.3) we now get $[x, d(y)]R[x, d(y)] = \{0\}$ for all $x \in R$, so that $d(y) \in Z$ and by (2.1) $y \in Z$. □

A similar proof yields

Theorem 2.2. *If R is a semiprime ring and d is a derivation on R, then $Z^{**}(R,d) = Z$.*

Corollary 2.3. *Let R be a semiprime ring and U a nonzero left ideal of R. If R admits a derivation d such that $d(U) \neq \{0\}$ and $[x, [y, d(y)]] = [[y, d^2(y)], d(x)]$ for all $x \in R$ and $y \in U$, then R contains a nonzero central ideal.*

Proof. By Theorem 2.1, d is centralizing on U; hence by Theorem 3 of [5], R contains a nonzero central ideal. □

Corollary 2.4. *Let R be a semiprime ring, U a nonzero ideal of R and d a derivation on R such that $d(U) \neq \{0\}$ and $d(U) \subseteq U$. If $[x,[y,d(y)]] = [[y,d^2(y)],d(x)]$ for all $x, y \in U$, then R contains a nonzero central ideal.*

Proof. Since U is an ideal of a semiprime ring, U is a semiprime ring; and d restricts to a derivation on U. Hence, by Theorem 2.1, $[y,d(y)] \in Z(U)$ for all $y \in U$. But, as is easily can be shown, $Z(U) \subseteq Z(R)$; therefore d is centralizing on R. Again invoking Theorem 3 of [5], we see that R must contain a nonzero central ideal. □

We now proceed to study $Z_1(R,d)$.

Theorem 2.5. *Let R be a prime ring with char$(R) \neq 2$ and d a nonzero derivation on R. Then $Z_1(R,d) = Z$.*

Proof. We only need to prove that $Z_1(R,d) \subseteq Z$, since the other inclusion is immediate. Let $y \in Z_1(R,d)$, so that

$$[d(x),d(y)] = [d(y),x] + [y,d(x)] \text{ for all } x \in R. \tag{2.5}$$

Substituting xy for x in (2.5), we obtain

$$d(x)[y,d(y)] + [x,d(y)]d(y) = [y,x]d(y) \text{ for all } x \in R; \tag{2.6}$$

and substituting xw for x in (2.6), we get

$$d(x)w[y,d(y)] + [x,d(y)]wd(y) = [y,x]wd(y) \text{ for all } x,w \in R. \tag{2.7}$$

Replacing x by $d(y)$ in (2.7) gives

$$d^2(y)w[y,d(y)] = [y,d(y)]wd(y) \text{ for all } w \in R; \tag{2.8}$$

and taking $x = y$ in (2.7) gives

$$d(y)w[y,d(y)] + [y,d(y)]wd(y) = 0 \text{ for all } w \in R. \tag{2.9}$$

From (2.8) and (2.9) we conclude that

$$(d(y) + d^2(y))w[y,d(y)] = 0 \text{ for all } w \in R; \tag{2.10}$$

and since R is prime, either $d(y) + d^2(y) = 0$ or $[y,d(y)] = 0$. Suppose that $d(y) + d^2(y) = 0$. Then $[d^2(y),d(y)] = 0$; and by taking $x = d(y)$ in (2.5), we get $[y,d^2(y)] = 0$, so that $[y,d(y)] = 0$. Thus, in either case $[y,d(y)] = 0$.

It now follows from (2.7) that $[x,y+d(y)]wd(y) = 0$ for all $x,w \in R$, so by primeness of R, either $d(y) = 0$ or $y + d(y) \in Z$. If $y + d(y) \in Z$, then $[d(x),d(y)] = -[d(x),y]$ for all $x \in R$. This fact, together with (2.5), gives $d(y) \in Z$; and (2.5) now gives $[y,d(R)] = \{0\}$. If $d(y) = 0$, we also have $[y,d(R)] = \{0\}$, so $Z_1(R,d) \subseteq H(R,d)$; hence by Herstein's result mentioned earlier, $Z_1(R,d) \subseteq Z$. □

If we assume R is semiprime instead of prime and consider a family $\{P_\alpha \mid \alpha \in \Lambda\}$ of prime ideals for which $\bigcap P_\alpha = \{0\}$, the argument used to prove Theorem 2.5 can be modified to yield $[y, d(R)] \subseteq P_\alpha$ for all $\alpha \in \Lambda$. Hence $Z_1(R, d) \subseteq H(R, d)$. But we cannot prove that $Z_1(R, d) \subseteq Z$, as the following example shows.

Example 2.6. Let $R = R_1 \oplus R_2$, where R_1 is a commutative domain with nonzero derivation d_1 and R_2 is a noncommutative prime ring. Then R is semiprime with nonzero derivation d given by $d((r_1, r_2)) = (d_1(r_1), 0)$. For $y = (0, r_2)$, we have $d(y) = 0$ and $[y, d(x)] = 0$ for all $x \in R$; hence $y \in Z_1(R, d)$. Thus $S = \{(0, r_2) \mid r_2 \in R_2\} \subseteq Z_1(R, d)$, but $S \nsubseteq Z$.

3. Results on epimorphisms

We turn now to the results involving epimorphisms.

Lemma 3.1 ([12, Lemma 1.1.9])**.** *Let R be a 2–torsion–free semiprime ring. If $y \in R$ and $[[x, y], y] = 0$ for all $x \in R$, then $y \in Z$.*

Lemma 3.2. *Let R be a prime ring and T an endomorphism which is not the identity map. If $u \in R$ and $u(x - T(x)) = 0$ for all $x \in R$, then $u = 0$.*

Proof. Assume $ux = uT(x)$ for all $x \in R$. Then for all $r, x \in R$, $uxr = uT(xr) = uT(x)T(r) = uxT(r)$, so that $ux(r - T(r)) = 0$ for all $x \in R$; and by primeness of R we get $u = 0$. □

Lemma 3.3. *Let R be an arbitrary ring and T an epimorphism of R. Then*

(i) $Z^{**}(R, T)$ is an additive subgroup of R;
(ii) if $T(u) = u$ and $y \in Z^{**}(R, T)$, then $[u, y - T(y)] = 0$;
(iii) if $y \in Z^{**}(R, T)$, then $T(y) \in Z^{**}(R, T)$;
(iv) if $y \in Z^{**}(R, T)$, then $y - T(y) \in Z^{**}(R, T)$.

Proof. (i) and (ii) are immediate from the definition of $Z^{**}(R, T)$ and (iv) follows from (i) and (iii). Therefore we only need to prove (iii). Let $y \in Z^{**}(R, T)$, so that $[x, y] = [T(x), T(y)]$ for all $x \in R$. Applying T to both sides of this equation yields $[T(x), T(y)] = [T(T(x)), T(T(y))]$; and since T is an epimorphism, $[w, T(y)] = [T(w), T(T(y))]$ for all $w \in R$. Therefore, $T(y) \in Z^{**}(R, T)$. □

Theorem 3.4. *Let R be a prime ring with $char(R) \neq 2$, and let T be an epimorphism of R which is not the identity map. Then $Z^{**}(R, T) = Z$.*

Proof. Clearly, $Z \subseteq Z^{**}(R, T)$, since $T(Z) \subseteq Z$; therefore we only need to show that $Z^{**}(R, T) \subseteq Z$. Let $y \in Z^{**}(R, T)$ and note that for any $x \in R, u = [x, y]$ satisfies $T(u) = u$. Therefore, by Lemma 3.3 (ii) and (iv),

$$[[x, y - T(y)], y - T(y)] = 0 \text{ for all } x \in R, \tag{3.1}$$

hence by Lemma 3.1 $y - T(y) \in Z$. Thus, for all $x \in R, [x, y] = [T(x), T(y)] = [T(x), y]$, so that

$$[x - T(x), y] = 0 \text{ for all } x \in R. \tag{3.2}$$

Substituting rx for x in (3.2), we have $[rx, y] - [T(r)T(x), y] = 0$, so

$$r[x, y] + [r, y]x - T(r)[T(x), y] - [T(r), y]T(x) = 0,$$

which by (3.2) may be rewritten as

$$(r - T(r))[x, y] + [r, y](x - T(x)) = 0 \text{ for all } r, x \in R. \tag{3.3}$$

Recalling that $T([r, y]) = [r, y]$ and replacing r by $[r, y]$ in (3.3), we get $[[r, y], y](x - T(x)) = 0$ for all $r, x \in R$. Using Lemma 3.2 we conclude that $[[r, y], y] = 0$ for all $r \in R$, hence by Lemma 3.1, $y \in Z$. □

We note that Theorem 3.4 cannot be extended to semiprime rings, as the following example shows.

Example 3.5. Let $R = R_1 \oplus R_2$, where R_1 is a commutative domain with epimorphism T_1 which is not the identity map on R_1, and R_2 is a noncommutative prime ring; and define $T : R \to R$ by $T((r_1, r_2)) = (T_1(r_1), r_2)$. Then R is semiprime, T is a non-identity epimorphism, and $\{(0, r_2) \mid r_2 \in R_2\}$ is a noncentral subset of $Z^{**}(R, T)$.

Our final theorem involves $Z^*(R, T)$ when T is an epimorphism.

Theorem 3.6. *Let R be a prime ring with char$(R) \neq 2$, and let $T : R \to R$ be an epimorphism such that T^2 is not the identity map. Then $Z^*(R, T) = Z$.*

Proof. By an argument similar to the one in the proof of Theorem 3.4, we get

$$[x + T(x), y] = 0 \text{ for all } x \in R, y \in Z^*(R, T). \tag{3.4}$$

Replacing x by rx in (3.4), we have for all $r, x \in R, y \in Z^*(R, T)$

$$r[x, y] + [r, y]x + T(r)[T(x), y] + [T(r), y]T(x) = 0;$$

and by (3.4) we can rewrite this equation as

$$(r - T(r))[x, y] + [r, y](x - T(x)) = 0 \text{ for all } x, r \in R. \tag{3.5}$$

We now substitute $w + T(w)$ for r in (3.5), and since $[w + T(w), y] = 0$, we obtain

$$(w - T^2(w))[x, y] = 0 \text{ for all } w, x \in R. \tag{3.6}$$

This easily yields $(w - T^2(w))R[x, y] = \{0\}$, so by primeness of R, $y \in Z$. □

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