Title:
Coefficient estimates for a subclass of analytic and bi-univalent functions

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COEFFICIENT ESTIMATES FOR A SUBCLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and investigate a subclass \( H_{h,p}^{h,p}(\beta) \) of analytic and bi-univalent functions in the open unit disk \( \mathbb{U} \). Upper bounds for the second and third coefficients of functions in this subclass are founded. Our results generalize and improve over the existing results in the literature.

Keywords: Analytic functions, bi-univalent functions, coefficient estimates, starlike functions, Koebe one-quarter theorem.


1. Introduction

Let \( \mathcal{A} \) be a class of analytic functions in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \), of the form

\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

We also denote by \( \mathcal{S} \) the class of functions \( f \in \mathcal{A} \) which are univalent in \( \mathbb{U} \). Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk \( \mathbb{U} \). The Koebe one-quarter theorem \cite{3} ensures that the image of \( \mathbb{U} \) under every univalent function \( f \in \mathcal{S} \) contains a disk of radius \( \frac{1}{4} \). Hence every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), which is defined by

\[
    f^{-1}(f(z)) = z \ (z \in \mathbb{U}),
\]

and

\[
    f(f^{-1}(w)) = w \ \left( |w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),
\]

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where
\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). The class consisting of bi-univalent functions are denoted by \( \Sigma \).

Determination of the bounds for the coefficients \( a_n \) is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient \( a_2 \) of functions \( f \in \mathcal{S} \) gives the growth and distortion bounds as well as covering theorems.

Lewin [9] investigated the class \( \Sigma \) of bi-univalent functions and showed that \( |a_2| < 1.51 \) for the functions belonging to \( \Sigma \). Subsequently, Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \). Kedzierawski [8] proved this conjecture for a special case when the function \( f \) and \( f^{-1} \) are starlike functions. Tan [12] obtained the bound for \( |a_n| \) namely \( |a_n| < 1 \) which is the best known estimate for functions in the class \( \Sigma \). Recently there interest to study the bi-univalent functions class \( \Sigma \) (see [4,5,13,14]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \).

Recently, Frasin [6] introduced two subclasses of class \( \Sigma \) and obtained estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in these subclasses as follow.

**Definition 1.1** ([6]). A function \( f(z) \) given by (1.1) is said to be in the class \( H_\Sigma(\alpha, \beta) \) if the following conditions are satisfied:

\[ f \in \Sigma \quad \text{and} \quad |\arg(f'(z) + \beta zf''(z))| < \frac{\alpha \pi}{2} \quad (z \in U), \]

and

\[ |\arg(g'(w) + \beta wg''(w))| < \frac{\alpha \pi}{2} \quad (w \in U), \]

where, \( \beta > 0, \ 0 < \alpha < 1, \ 2(1 - \alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta^{m+1}} - 1 \), and the function \( g \) is the extension of \( f^{-1} \) to \( U \), which

(1.2) \( g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \)

**Theorem 1.2** ([6]). Let \( f(z) \) given by (1.1) be in the class \( H_\Sigma(\alpha, \beta) \) where \( \beta > 0, \ 0 < \alpha < 1, \ 2(1 - \alpha) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta^{m+1}} - 1 \). Then

\[ |a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha + 2)} + 4\beta(\alpha + \beta + 2 - \alpha\beta)}. \]
and

$$|a_3| \leq \frac{\alpha^2}{(1 + \beta)^2} + \frac{2\alpha}{3(1 + 2\beta)}.$$ 

**Definition 1.3** ([6]). A function \( f(z) \) given by (1.1) is said to be in the class \( H_{\Sigma}(\gamma, \beta) \) if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re(f'(z) + \beta zf''(z)) > \gamma \ (z \in \mathbb{U}),$$

and

$$\Re(g'(w) + \beta wg''(w)) > \gamma \ (w \in \mathbb{U}),$$

where, \( \beta > 0, \ 0 \leq \gamma < 1, \ 2(1 - \gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1, \) and \( g \) is given by (1.2).

**Theorem 1.4** ([6]). Let \( f(z) \) given by (1.1) be in the class \( H_{\Sigma}(\gamma, \beta) \) where \( \beta > 0, \ 0 \leq \gamma < 1, \ 2(1 - \gamma) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m+1} \leq 1. \) Then

$$|a_2| \leq \sqrt{\frac{2(1 - \gamma)}{3(1 + 2\beta)}},$$

and

$$|a_3| \leq \frac{(1 - \gamma)^2}{(1 + \beta)^2} + \frac{2(1 - \gamma)}{3(1 + 2\beta)}.$$

The purpose of our study is to obtain estimates of coefficients \( |a_2| \) and \( |a_3| \) for functions in subclasses \( H_{\Sigma}^{h;p}(\beta) \) which improve Theorem 1.2 and Theorem 1.4.

2. **Coefficient Estimates**

In this section, we introduce and investigate the general subclass \( H_{\Sigma}^{h;p}(\beta) \) where \( \beta \geq 0. \)

**Definition 2.1.** Let \( h, p : \mathbb{U} \to \mathbb{C} \) be analytic functions and

$$\min\{|\Re(h(z))|, |\Re(p(z))|\} > 0 \ (z \in \mathbb{U}) \text{ and } h(0) = p(0) = 1.$$

A univalent function \( f \in \mathcal{S} \) given by (1.1) is said to be in the class \( H_{\Sigma}^{h;p}(\beta) \) if the following conditions are satisfied:

\[
(2.1) \quad f \in \Sigma \text{ and } f'(z) + \beta zf''(z) \in h(\mathbb{U}) \ (z \in \mathbb{U}),
\]

and

\[
(2.2) \quad g'(w) + \beta wg''(w) \in p(\mathbb{U}) \ (w \in \mathbb{U}),
\]

where, \( \beta \geq 0 \) and the function \( g \) is given by (1.2).

**Remark 2.2.** There are many choices of \( h, p \) and \( \beta \) which would provide interesting subclasses of class \( H_{\Sigma}^{h;p}(\beta) \). For example,
(1) For $\beta > 0$ and $h(z) = p(z) = \left( \frac{1-z}{1-z} \right)^{\alpha}$ where $0 < \alpha \leq 1$, it can be directly verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Now if $f \in H^{h,p}_{\Sigma}(\beta)$ then

$$|\arg(f'(z) + \beta zf''(z))| < \frac{\alpha \pi}{2} \quad \text{and} \quad |\arg(g'(w) + \beta wg''(w))| < \frac{\alpha \pi}{2}. $$

Therefore in this case, the class $H^{h,p}_{\Sigma}(\beta)$ reduces to class $H_{\Sigma}(\alpha, \beta)$ in Definition 1.1.

(2) For $\beta > 0$ and $h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z}$, where $0 \leq \gamma < 1$, the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Now if $f \in H^{h,p}_{\Sigma}(\beta)$, then

$$\Re(f'(z) + zf''(z)) > \gamma \quad \text{and} \quad \Re(g'(w) + wg''(w)) > \gamma. $$

This means that the class $H^{h,p}_{\Sigma}(\beta)$ reduces to class $H_{\Sigma}(\gamma, \beta)$ in Definition 1.3.

(3) For $\beta = 0$ and $h(z) = p(z) = \left( \frac{1+z}{1-z} \right)^{\alpha}$ where $0 < \alpha \leq 1$, the class $H^{h,p}_{\Sigma}(0)$ reduces to the class $H^\gamma_{\Sigma}$ which is defined by Srivastava et al. [10, Definition 1].

(4) For $\beta = 0$ and $h(z) = p(z) = \frac{1+(1-2\gamma)z}{1-z}$, where $0 \leq \gamma < 1$, the class $H^{h,p}_{\Sigma}(0)$ reduces to the class $H_{\Sigma}(\gamma)$ which is defined by Srivastava et al. [10, Definition 2].

Now, we derive the estimates of the coefficients $|a_2|$ and $|a_3|$ for class $H^{h,p}_{\Sigma}(\beta)$.

**Theorem 2.3.** If $f \in H^{h,p}_{\Sigma}(\beta)$ where $\beta \geq 0$, then

$$|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{8(1+\beta)^2}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{12(1+2\beta)}} \right\}, $$

and

$$|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{8(1+\beta)^2} + \frac{|h''(0)| + |p''(0)|}{12(1+2\beta)}, \frac{|h'''(0)|}{6(1+2\beta)} \right\}. $$

**Proof.** Since $f \in H^{h,p}_{\Sigma}(\beta)$ and $g = f^{-1}$, from relations (2.1) and (2.2) we have,

$$f'(z) + \beta zf''(z) = h(z) \quad (z \in \mathbb{U}), $$

and

$$g'(w) + \beta wg''(w) = p(w) \quad (w \in \mathbb{U}), $$

respectively, where functions $h$ and $p$ satisfy the conditions of Definition 2.1. Also, functions $h$ and $p$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1z + h_2z^2 + h_3z^3 + \cdots, $$

respectively.
and

\[ p(w) = 1 + p_1w + p_2w^2 + p_3w^2 + \cdots. \tag{2.8} \]

Now, by substituting (2.7) and (2.8) into (2.5) and (2.6), respectively, and equating the coefficients, we get

\[ 2(1 + \beta)a_2 = h_1, \tag{2.9} \]
\[ 3(1 + 2\beta)a_3 = h_2, \tag{2.10} \]
\[ -2(1 + \beta)a_2 = p_1, \tag{2.11} \]

and

\[ 6(1 + 2\beta)a_2^2 - 3(1 + 2\beta)a_3 = p_2. \tag{2.12} \]

From (2.9) and (2.11), it yields

\[ h_1 = -p_1, \tag{2.13} \]

and

\[ 8(1 + \beta)^2a_2^2 = h_1^2 + p_1^2. \tag{2.14} \]

Adding (2.10) and (2.12), gives

\[ 6(1 + 2\beta)a_2^2 = p_2 + h_2. \tag{2.15} \]

Consequently, from (2.14) and (2.15), we have that

\[ a_2^2 = \frac{h_1^2 + p_1^2}{8(1 + \beta)^2}, \tag{2.16} \]

and

\[ a_2^2 = \frac{p_2 + h_2}{6(1 + 2\beta)}, \tag{2.17} \]

respectively. Therefore, we find from the equations (2.16) and (2.17), that

\[ |a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{8(1 + \beta)^2}, \]

and

\[ |a_2| \leq \frac{|h''(0)| + |p''(0)|}{12(1 + 2\beta)}. \]

Thus, the desired estimate on the coefficient \( |a_2| \) as asserted in (2.3).

Next, in order to find the bound of the coefficient \( |a_3| \), by subtracting (2.12) from (2.10), we get

\[ 6(1 + 2\beta)a_3 - 6(1 + 2\beta)a_2^2 = h_2 - p_2. \tag{2.18} \]

Upon substituting the value of \( a_2^2 \) from (2.16) into (2.18), it follows that

\[ a_3 = \frac{h_1^2 + p_1^2}{8(1 + \beta)^2} + \frac{h_2 - p_2}{6(1 + 2\beta)}. \]
Therefore, we obtain

\[(2.19) \quad |a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{8(1 + \beta)^2} + \frac{|h''(0)| + |p''(0)|}{12(1 + 2\beta)}.\]

On the other hand, by substituting the value of \(a_2^2\) from \((2.17)\) into \((2.18)\), it follows that

\[a_3 = \frac{p_2 + h_2}{6(1 + 2\beta)} + \frac{h_2 - p_2}{6(1 + 2\beta)} = \frac{h_2}{3(1 + 2\beta)}.\]

Hence

\[(2.20) \quad |a_3| \leq \frac{|h''(0)|}{6(1 + 2\beta)}.\]

The desired estimate of the coefficient \(|a_3|\) as asserted in \((2.4)\) will be obtained from \((2.19)\) and \((2.20)\).

By choosing

\[h(z) = p(z) = \left(\frac{1 + z}{1 - z}\right)\alpha \quad (0 < \alpha \leq 1, \ z \in \mathbb{U}),\]

in Theorem 2.3, we have the following result.

**Corollary 2.4.** Let the function \(f\) be given by \((1.1)\) in the class \(H_{\Sigma}^{h,p}(\beta)\) where \(\beta \geq 0\). Then

\[|a_2| \leq \min \left\{ \frac{\alpha}{1 + \beta}, \frac{\sqrt{2}\alpha}{3(1 + 2\beta)} \right\},\]

and

\[|a_3| \leq \frac{2\alpha^2}{3(1 + 2\beta)}.\]

**Remark 2.5.** Corollary 2.4 is an improvement of estimates obtained by Frasin [6] in Theorem 1.2. To see this, for the coefficient \(|a_2|\), we have that

(i) If \(\beta \geq \frac{1}{2} + \frac{\sqrt{3}}{2} \), then \(\min \left\{ \frac{\alpha}{1 + \beta}, \frac{\sqrt{2}\alpha}{\sqrt{3(1 + 2\beta)}} \right\} = \frac{\alpha}{1 + \beta}, \) and

\[\frac{\alpha}{1 + \beta} \leq \frac{2\alpha}{\sqrt{2(\alpha + 2) + 4\beta(\alpha + \beta + 2 - \alpha\beta)}}.\]

(ii) If \(\beta \leq \frac{1}{2} + \frac{\sqrt{3}}{2} \), then \(\min \left\{ \frac{\alpha}{1 + \beta}, \frac{\sqrt{2}\alpha}{\sqrt{3(1 + 2\beta)}} \right\} = \frac{\sqrt{2}\alpha}{\sqrt{3(1 + 2\beta)}}, \) and

\[\frac{\sqrt{2}\alpha}{\sqrt{3(1 + 2\beta)}} \leq \frac{2\alpha}{\sqrt{2(\alpha + 2) + 4\beta(\alpha + \beta + 2 - \alpha\beta)}}.\]
Also for the coefficient $|a_3|$, it can be concluded that
\[
\frac{2\alpha^2}{3(1 + 2\beta)} \leq \frac{\alpha^2}{(1 + \beta)^2} + \frac{2\alpha}{3(1 + 2\beta)}.
\]
Therefore the bounds obtained in Corollary 2.4 is a refinement of the estimates obtained in Theorem 1.2.

If we take $\beta = 1$ in Theorem 2.3, then we have the following result.

**Corollary 2.6.** Let the function $f$ be given by (1.1) in the class $H_{\alpha}^{h,p}(1)$. Then
\[
|a_2| \leq \min \left\{ \sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{32}}, \sqrt{\frac{|h''(0)| + |p''(0)|}{36}} \right\},
\]
and
\[
|a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{32} + \frac{|h''(0)| + |p''(0)|}{36}, \frac{|h''(0)|}{18} \right\}.
\]

By taking
\[
h(z) = p(z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha} \quad (0 < \alpha \leq 1, \ z \in \mathbb{U}),
\]
in Corollary 2.6, the following result will be concluded.

**Corollary 2.7.** Let the function $f$ be given by (1.1) in the class $H_{\alpha}^{h,p}(1)$. Then
\[
|a_2| \leq \min \left\{ \frac{\alpha}{2}, \frac{\sqrt{2\alpha}}{3} \right\} = \frac{\sqrt{2\alpha}}{3},
\]
and
\[
|a_3| \leq \frac{2\alpha^2}{9}.
\]

By setting $\beta = 0$ in Corollary 2.4, we obtain the following consequence which is an improvement of the estimates obtained by Srivastava et al. in [10, Theorem 1].

**Corollary 2.8.** Let the function $f$ be given by (1.1) in the class $H_{\alpha}^{1}(1)$. Then
\[
|a_2| \leq \sqrt{\frac{2}{3}} \alpha,
\]
and
\[
|a_3| \leq \frac{2\alpha^2}{3}.
\]

By letting
\[
h(z) = p(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} \quad (0 \leq \gamma < 1, \ z \in \mathbb{U}),
\]
in Theorem 2.3, we deduce the following corollary.
Corollary 2.9. Let the function $f$ be given by (1.1) in the class $H_{\Sigma}^{h,p}(\beta)$ where $\beta \geq 0$. Then,

$$|a_2| \leq \min \left\{ \frac{1 - \gamma}{1 + \beta}, \sqrt[3]{\frac{2(1 - \gamma)}{3(1 + 2\beta)}} \right\},$$

and

$$|a_3| \leq \frac{2(1 - \gamma)}{3(1 + 2\beta)}.$$

Remark 2.10. Corollary 2.9 is an improvement of the estimates obtained by Frasin [6] in Theorem 1.4. To see this, for the coefficient $|a_2|$, if $\beta > \frac{3\delta - 2 + \sqrt{3\delta(3\delta - 2)}}{2}$ and $\delta > \frac{2}{3}$, then

$$\min \left\{ \frac{1 - \gamma}{1 + \beta}, \sqrt[3]{\frac{2(1 - \gamma)}{3(1 + 2\beta)}} \right\} < \sqrt[3]{\frac{2(1 - \gamma)}{3(1 + 2\beta)}}.$$

Also for the coefficient $|a_3|$, we have

$$\frac{2(1 - \gamma)}{3(1 + 2\beta)} < \frac{(1 - \gamma)^2}{(1 + \beta)^2} + \frac{2(1 - \gamma)}{3(1 + 2\beta)}.$$

Therefore the bounds obtained in Corollary 2.9 is a refinement of the estimates obtained in Theorem 1.4.

If we take $\beta = 1$ and

$$h(z) = p(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha \quad (0 < \alpha \leq 1, \ z \in \mathbb{U}),$$

in Theorem 2.3, we have the following result.

Corollary 2.11. Let the function $f$ be given by (1.1) in the class $H_{\Sigma}^{h,p}(1)$. Then,

$$|a_2| \leq \min \left\{ \frac{1 - \gamma}{2}, \sqrt[3]{\frac{2(1 - \gamma)}{3}} \right\},$$

and

$$|a_3| \leq \frac{2(1 - \gamma)}{9}.$$

By setting $\beta = 0$ in Corollary 2.9, we obtain the following result which is an improvement of the estimates obtained by Srivastava et al. [10, Theorem 2].

Corollary 2.12. Let the function $f$ be given by (1.1) in the class $H_{\Sigma}(\gamma)$. Then

$$|a_2| \leq \min \left\{ 1 - \gamma, \sqrt[3]{\frac{2(1 - \gamma)}{3}} \right\},$$

and

$$|a_3| \leq \frac{2(1 - \gamma)}{3}.$$
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