ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 4, pp. 891–902

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 4, pp. 891–902
Online ISSN: 1735-8515

RELATIVE (CO)HOMOLOGY OF F-GORENSTEIN MODULES

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(Communicated by Mohammad-Taghi Dibaei)

Abstract. We investigate the relative cohomology and relative homology theories of F-Gorenstein modules, consider the relations between classical and F-Gorenstein (co)homology theories.

Keywords: F-Gorenstein projective module, F-Gorenstein injective module, relative cohomology, relative homology.
MSC(2010): Primary: 13D05; Secondary: 16E30, 18G25.

1. Introduction

It is well known that among commutative Noetherian local rings, the finiteness of G-dimensions can be used to measure the Gorensteinness (see [1, Thm. 4.20]).

On the other hand, in their series of papers [2–4] Auslander and Solberg provided certain foundational aspects of relative homological algebra theory in terms of subbifunctors of the functor $\mathrm{Ext}^1_\Lambda(-,-)$. Later, they further developed the theory and generalized the notion of Gorenstein algebras to F-Gorenstein algebras for the purpose of constructing Gorenstein algebras and algebras with dominant dimension at least 2 in [5].

Pursuing the themes described above, over an Artin algebra Λ, for an additive subbifunctor F of $\mathrm{Ext}^1_\Lambda(-,-)$ with enough projectives and injectives, Tang in [13] characterized F-Gorenstein algebras and F-self-injective algebras by finite F-Gorenstein dimensions.

The goal of this paper is to continue studying F-Gorenstein dimensions, investigate the relative cohomology and relative homology theories of F-Gorenstein modules, consider the relations between classical and F-Gorenstein (co)homology theories.

Article electronically published on August 20, 2016.
Received: 5 March 2015, Accepted: 27 May 2015.
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Throughout this paper, Λ denotes an Artin algebra over a commutative Artinian ring k, and modΛ denotes the category of finitely generated left Λ-modules. We denote by $(\text{mod}\Lambda)^{op}$ the opposite category of modΛ. Moreover, we assume that all the subcategories are full subcategories.

2. F-Gorenstein projective (injective) modules

In this section, we recall basic definitions and notions used in this paper.

Suppose F is an additive subbifunctor of the additive bifunctor $\text{Ext}^1_\Lambda(-,-) : (\text{mod}\Lambda)^{op} \times \text{mod}\Lambda \to Ab$ (some examples of additive subbifunctors see [13, Example 2.1]).

Definition 2.1. [2] A short exact sequence $\eta : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in modΛ is said to be F-exact if η is in $F(C, A)$. Moreover, f is called an F-monomorphism and g is called an F-epimorphism.

An exact sequence $X = \cdots \to X_{l+1} \xrightarrow{f_{l+1}} X_l \xrightarrow{f_l} X_{l-1} \to \cdots$ in modΛ is called an F-exact sequence provided that $0 \to \text{Im} f_{l+1} \to X_l \to \text{Im} f_l \to 0$ is F-exact for all l.

Each additive subbifunctor F corresponds to a class of short exact sequences that is closed under the operations of pushout, pullback, and direct sums (and hence Baer sums).

Definition 2.2. A Λ-module P (respectively, I) in modΛ is said to be F-projective (respectively, F-injective) if for each F-exact sequence $0 \to A \to B \to C \to 0$, the sequence $0 \to \text{Hom}_\Lambda(P, A) \to \text{Hom}_\Lambda(P, B) \to \text{Hom}_\Lambda(P, C) \to 0$ (respectively, $0 \to \text{Hom}_\Lambda(C, I) \to \text{Hom}_\Lambda(B, I) \to \text{Hom}_\Lambda(A, I) \to 0$) is exact.

The full subcategory of modΛ consisting of all F-projective (respectively, F-injective) modules is denoted by $\mathcal{P}(F)$ (respectively, $\mathcal{I}(F)$).

Definition 2.3. F is said to have enough projectives (respectively, injectives) if for any $A \in \text{mod}\Lambda$ there is an F-exact sequence $0 \to B \to P \to A \to 0$ (respectively, $0 \to A \to I \to C \to 0$) with P in $\mathcal{P}(F)$ (respectively, I in $\mathcal{I}(F)$).

Definition 2.4. For any $M \in \text{mod}\Lambda$, a left F-projective resolution of M is an F-exact sequence $\mathbf{P} = \cdots \to P_1 \to P_0 \to M \to 0$ with $P_l \in \mathcal{P}(F)$ for all l. Furthermore, we say $\mathbf{P}$ is proper if the sequence $\text{Hom}_\Lambda(Q, \mathbf{P})$ is exact for all $Q \in \mathcal{P}(F)$. A right F-projective resolution of M is an F-exact sequence $\mathbf{P} = 0 \to M \to P^0 \to P^1 \to \cdots$ with $P^i \in \mathcal{P}(F)$ for all i. Furthermore, we say $\mathbf{P}$ is co-proper if the sequence $\text{Hom}_\Lambda(\mathbf{P}, Q)$ is exact for all $Q \in \mathcal{P}(F)$.

F-injective resolutions are defined dually.

If F has enough projectives and injectives, then for any A and C in modΛ the right derived functors of $\text{Hom}_\Lambda(C, -)$ and $\text{Hom}_\Lambda(-, A)$ using right F-injective and left F-projective resolutions, respectively, coincide. We denote by $\text{Ext}^i_F(C, -)$ (respectively, $\text{Ext}^i_F(-, A)$) the right derived functors of $\text{Hom}_\Lambda(C, -)$ (respectively, $\text{Hom}_\Lambda(-, A)$).

Definition 2.5. If F has enough projectives, we can define the relative projective dimension of a given M in modΛ, that is, $\mathrm{pd}_F M = \inf\{n \mid 0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ is a left F-projective resolution of $M\}$.

If F has enough injectives, we define the relative injective dimension dually.

Remark 2.6. It is easy to check that $\mathrm{pd}_F M = \inf\{n \mid \mathrm{Ext}_F^{n+1}(M,B) = 0$ for any $B \in \mathrm{mod}\Lambda\}$ and $\mathrm{id}_F M = \inf\{n \mid \mathrm{Ext}_F^{n+1}(B,M) = 0$ for any $B \in \mathrm{mod}\Lambda\}$. We use $\mathcal{P}^\infty(F)$ (respectively, $\mathcal{I}^\infty(F)$) to denote the subcategory of modΛ consisting of the modules with finite F-projective (respectively, F-injective) dimensions.

From now on, we always assume that F has enough projectives and injectives.

Definition 2.7. *(* [13]*)* A Λ-module M in modΛ is called F-Gorenstein projective if there exists a $\mathrm{Hom}_\Lambda(-,\mathcal{P}(F))$-exact F-exact sequence

$$T = \cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots$$

of F-projective Λ-modules with $M = \mathrm{Im}(P_{n+1} \to P_n)$, and in this case T is called a complete F-projective resolution of M. Denote the class of F-Gorenstein projective Λ-modules by $\mathcal{GP}(F)$.

A Λ-module N in modΛ is called F-Gorenstein injective if there exists a $\mathrm{Hom}_\Lambda(\mathcal{I}(F), -)$-exact F-exact sequence

$$S = \cdots \to I_{n+1} \to I_n \to I_{n-1} \to \cdots$$

of F-injective Λ-modules with $N = \mathrm{Ker}(I_n \to I_{n-1})$, and in this case S is called a complete F-injective resolution of N. Denote the class of F-Gorenstein injective Λ-modules by $\mathcal{GI}(F)$.

Remark 2.8. $\mathcal{GP}(F)$ is also called Gorenstein subcategory of mod Λ (see [12, Def. 4.1]). Obviously, if we take subfunctor $F = \mathrm{Ext}_\Lambda^1(-,-)$, all modules in $\mathcal{GP}(F)$ are precisely finitely generated Gorenstein projective modules or modules with G-dimension zero (see [1, 7, 11]).

Definition 2.9. Let $M \in \mathrm{mod}\Lambda$. The F-Gorenstein projective dimension of M $\mathrm{Gpd}_F M = \inf\{n \mid$ there is an F-exact sequence $\mathbf{G} = 0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ with $G_i \in \mathcal{GP}(F)$ for $0 \le i \le n\}$. If $M = 0$, we set $\mathrm{Gpd}_F M = -1$. If there is no such n, we set $\mathrm{Gpd}_F M = \infty$.

Left $\mathcal{GP}(F)$-resolution $\mathbf{G}$ is called proper if it is $\mathrm{Hom}_\Lambda(\mathcal{GP}(F), \mathbf{G})$-exact.

We define (co-proper) right $\mathcal{GI}(F)$-resolutions and F-Gorenstein injective dimensions dually.

Notation 2.10. In particular, we abbreviate as follows:

$\mathcal{GP}^\infty(F) =$ the subcategory of modΛ with $\mathrm{Gpd}_F M < \infty$.

$\mathcal{GI}^\infty(F) =$ the subcategory of modΛ with $\mathrm{Gid}_F M < \infty$.

$\widetilde{\mathcal{GP}}(F) =$ the subcategory of modΛ admitting a proper left $\mathcal{GP}(F)$-resolution.

$\widetilde{\mathcal{GI}}(F) =$ the subcategory of modΛ admitting a co-proper right $\mathcal{GI}(F)$-resolution.

Definition 2.11. Let $\mathcal{X}$ be any subclass of modΛ and $M \in$ modΛ. An $\mathcal{X}$-*precover* of M is a Λ-homomorphism $\varphi : X \to M$, where $X \in \mathcal{X}$ and such that the sequence

$$\mathrm{Hom}_\Lambda(X', X) \xrightarrow{\mathrm{Hom}_\Lambda(X',\varphi)} \mathrm{Hom}_\Lambda(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. If, moreover, $\varphi f = \varphi$ for $f \in \mathrm{Hom}_\Lambda(X, X)$ implies f is an automorphism of X, then φ is called an $\mathcal{X}$-cover of M. Also, an $\mathcal{X}$-preenvelope and $\mathcal{X}$-envelope of M are defined "dually".

3. Relative cohomology

It is well known that if F has enough projectives and injectives, then it is standard to derive $\mathrm{Hom}_\Lambda(-,-)$ using left F-projective resolutions in the first variable, or right F-injective resolutions in the second variable, and doing this, one obtains $\mathrm{Ext}^i_F(-,-)$ in both cases. In this section, we examine the situation where F-projective and F-injective modules are replaced by F-Gorenstein projective and F-Gorenstein injective ones, respectively.

At first, we will need the following:

Lemma 3.1. *(1) If M is an F-Gorenstein projective Λ-module, then $\mathrm{Ext}^{\geq 1}_F(M, W) = 0$ for all $W \in \mathcal{P}^\infty(F)$ or $W \in \mathcal{I}^\infty(F)$.*

(2) If N is an F-Gorenstein injective Λ-module, then $\mathrm{Ext}^{\geq 1}_F(U, N) = 0$ for all $U \in \mathcal{P}^\infty(F)$ or $U \in \mathcal{I}^\infty(F)$.

Proof. (1) For $W \in \mathcal{P}^\infty(F)$, $\mathrm{Ext}^{\geq 1}_F(M, W) = 0$ is an immediate consequence of [13, Thm. 2.13].

Assume that $\mathrm{id}_F W = m < \infty$. If $m = 0$, there is nothing to prove. Now we suppose that $m > 0$. Since M is F-Gorenstein projective, we have an F-exact sequence

$$0 \to M \to P^0 \to P^1 \to \cdots \to P^{m-1} \to C \to 0,$$

where all P^i are F-projective. Breaking this sequence into short F-exact ones, we see that $\mathrm{Ext}^i_F(M, W) \cong \mathrm{Ext}^{i+m}_F(C, W)$ for $i > 0$, so the Exts vanish as desired since $\mathrm{Ext}^{i+m}_F(C, W) = 0$ for $i > 0$.

(2) The proof is dual to (1). □

By [13, Prop. 2.11] and the duality, we have the following:

Lemma 3.2. *(1) If $M \in$ modΛ with $\mathrm{Gpd}_F M < \infty$, then there exists an F-exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is a $\mathcal{GP}(F)$-precover of M, and $\mathrm{pd}_F K = \mathrm{Gpd}_F M - 1$. Consequently, M has a proper left $\mathcal{GP}(F)$-resolution.*

(2) If $N \in$ modΛ with $\mathrm{Gid}_F N < \infty$, then there exists an F-exact sequence $0 \to N \to E \to C \to 0$, where $N \to E$ is a $\mathcal{GI}(F)$-preenvelope of N, and $\mathrm{id}_F C = \mathrm{Gid}_F N - 1$. Consequently, N has a co-proper right $\mathcal{GI}(F)$-resolution.

Following the previous lemma, one can get the inclusions $\mathcal{GP}^{\infty}(F) \subseteq \widetilde{\mathcal{GP}}(F)$ and $\mathcal{GI}^{\infty}(F) \subseteq \widetilde{\mathcal{GI}}(F)$.

Lemma 3.3. *(1) Let $M \in \mathcal{GP}^{\infty}(F)$. Then any proper left $\mathcal{GP}(F)$-resolution $\mathbf{G} \to M$ is $\mathrm{Hom}_{\Lambda}(-, \mathcal{GI}(F))$-exact.*

(2) Let $N \in \mathcal{GI}^{\infty}(F)$. Then any co-proper right $\mathcal{GI}(F)$-resolution $N \to \mathbf{E}$ is $\mathrm{Hom}_{\Lambda}\ (\mathcal{GP}(F), -)$-exact.

Proof. (1) Let $\mathbf{G} = \cdots \to G_1 \to G_0 \to M \to 0$ be a proper left $\mathcal{GP}(F)$-resolution of M. We split the resolution $\mathbf{G}$ into short F-exact sequences. Hence it suffices to show exactness of $\mathrm{Hom}_{\Lambda}(\mathbf{X}, H)$ for all F-Gorenstein injective Λ-modules H and all short F-exact sequences

$$\mathbf{X} = 0 \to L \to G \to M \to 0,$$

where $G \to M$ is a $\mathcal{GP}(F)$-precover of M. By Lemma 3.2 (1), there is a short F-exact sequence

$$\mathbf{X}' = 0 \longrightarrow L' \xrightarrow{\ \iota\ } G' \xrightarrow{\ \pi\ } M \longrightarrow 0,$$

where $\pi : G' \to M$ is a $\mathcal{GP}(F)$-precover of M and $\mathrm{pd}_F L' < \infty$.

By [10, Prop. 2.2], the complexes $\mathbf{X}$ and $\mathbf{X}'$ are homotopically equivalent, and thus so are the complexes $\mathrm{Hom}_{\Lambda}(\mathbf{X}, H)$ and $\mathrm{Hom}_{\Lambda}(\mathbf{X}', H)$ for every (F-Gorenstein injective) Λ-module H. Hence it suffices to show the exactness of $\mathrm{Hom}_{\Lambda}(\mathbf{X}', H)$ whenever H is F-Gorenstein injective.

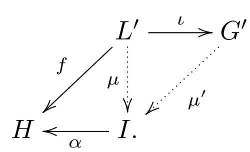
Now let H be any F-Gorenstein injective Λ-module. We need to prove the exactness of

$$\mathrm{Hom}_{\Lambda}(G', H) \xrightarrow{\mathrm{Hom}_{\Lambda}(\iota, \mathrm{H})} \mathrm{Hom}_{\Lambda}(L', H) \longrightarrow 0.$$

To show this, let $f : L' \to H$ be any homomorphism. We wish to find $g : G' \to H$ such that $g\iota = f$. Now pick an F-exact sequence

$$0 \longrightarrow H' \longrightarrow I \xrightarrow{\ \alpha\ } H \longrightarrow 0,$$

where I is F-injective, and H' is F-Gorenstein injective. Since H' is F-Gorenstein injective and $\mathrm{pd}_F L' < \infty$, we get $\mathrm{Ext}^1_F(L', H') = 0$ by Lemma 3.1 (2), and thus a lifting $\mu : L' \to I$ with $\alpha\mu = f$:



Next, F-injectivity of I gives $\mu' : G' \to I$ with $\mu'\iota = \mu$. Now $g = \alpha\mu' : G' \to H$ is the desired map.

(2) The proof of (2) is dual to that of (1). $\square$

Definition 3.4. Let $M \in \widetilde{\mathcal{GP}}(F)$ and consider a proper left $\mathcal{GP}(F)$-resolution $\mathbf{G} \to M$. For every $n \in \mathbb{Z}$ and every $N \in \mathrm{mod}\Lambda$, define a relative cohomology group

$$\mathrm{Ext}^n_{\mathcal{GP}(F)}(M, N) = \mathrm{H}^n(\mathrm{Hom}_\Lambda(\mathbf{G}, N)).$$

Similarly, choosing for each $N \in \widetilde{\mathcal{GI}}(F)$ a co-proper right $\mathcal{GI}(F)$-resolution $N \to \mathbf{E}$, we define for each $n \in \mathbb{Z}$ and each $M \in \mathrm{mod}\Lambda$ a relative cohomology group

$$\mathrm{Ext}^n_{\mathcal{GI}(F)}(M, N) = \mathrm{H}^n(\mathrm{Hom}_\Lambda(M, \mathbf{E})).$$

Lemma 3.3 can be stated in the language of Enochs and Jenda [8, Thm. 8.2.13] as follows: the bifunctor $\mathrm{Hom}_\Lambda(-,-)$ is right balanced by $\mathcal{GP}^\infty(F) \times \mathcal{GI}^\infty(F)$. So we have the following result.

Theorem 3.5. *Let $M \in \mathcal{GP}^\infty(F)$ and $N \in \mathcal{GI}^\infty(F)$. Then for all $i \in \mathbb{Z}$, we have isomorphisms*

$$\mathrm{Ext}^i_{\mathcal{GP}(F)}(M, N) \cong \mathrm{Ext}^i_{\mathcal{GI}(F)}(M, N).$$

Proof. Use [10, Thm. 2.6]. □

Definition 3.6. (Definition of Gext) Let $M \in \mathcal{GP}^\infty(F)$ and $N \in \mathcal{GI}^\infty(F)$. Then we write

$$\mathrm{Gext}^n_F(M, N) := \mathrm{Ext}^n_{\mathcal{GP}(F)}(M, N) \cong \mathrm{Ext}^n_{\mathcal{GI}(F)}(M, N)$$

for the isomorphic abelian groups in Theorem 3.5.

The following proposition extends [6, Thm. 4.2(2.b), Prop. (4.4) and Prop.(4.6)] to the more general settings.

Proposition 3.7. *(1)* $\mathrm{Gext}^0_F(-,-) = \mathrm{Hom}_\Lambda(-,-)$.

(2) If the F-exact sequence $0 \to M' \to M \to M'' \to 0$ of modΛ in $\widetilde{\mathcal{GP}}(F)$ is $\mathrm{Hom}_\Lambda(\mathcal{GP}(F), -)$*-exact, then for each $N \in$ modΛ there is a long exact sequence*

$$\cdots \to \mathrm{Ext}^i_{\mathcal{GP}(F)}(M, N) \to \mathrm{Ext}^i_{\mathcal{GP}(F)}(M', N) \to \mathrm{Ext}^{i+1}_{\mathcal{GP}(F)}(M'', N) \to \cdots.$$

(3) If the F-exact sequence $0 \to N' \to N \to N'' \to 0$ of modΛ is $\mathrm{Hom}_\Lambda(\mathcal{GP}(F), -)$*-exact, then for each $M \in \widetilde{\mathcal{GP}}(F)$ there is a long exact sequence*

$$\cdots \to \mathrm{Ext}^i_{\mathcal{GP}(F)}(M, N) \to \mathrm{Ext}^i_{\mathcal{GP}(F)}(M, N'') \to \mathrm{Ext}^{i+1}_{\mathcal{GP}(F)}(M, N') \to \cdots.$$

(4) There are natural transformations

$$\mathrm{Gext}^i_F(-,-) \longrightarrow \mathrm{Ext}^i_F(-,-)$$

which are also natural in the long exact sequences as in (2) *and* (3) *above.*

In the following, we want to compare Gext with Ext.

Theorem 3.8. *Let M, $N \in \mathrm{mod}\Lambda$. Then the following conclusions hold:*

(1) There are natural isomorphisms $\mathrm{Ext}^n_{\mathcal{GP}(F)}(M,N) \cong \mathrm{Ext}^n_F(M,N)$ under each of the conditions

$$(a)\quad M \in \mathcal{P}^\infty(F) \quad or \quad (b)\quad M \in \widetilde{\mathcal{GP}}(F) \quad and \quad N \in \mathcal{I}^\infty(F).$$

(2) There are natural isomorphisms $\mathrm{Ext}^n_{\mathcal{GI}(F)}(M,N) \cong \mathrm{Ext}^n_F(M,N)$ under each of the conditions

$$(a)\quad N \in \mathcal{I}^\infty(F) \quad or \quad (b)\quad N \in \widetilde{\mathcal{GI}}(F) \quad and \quad M \in \mathcal{P}^\infty(F).$$

(3) Assume that $M \in \mathcal{GP}^\infty(F)$ and $N \in \mathcal{GI}^\infty(F)$. If either $M \in \mathcal{P}^\infty(F)$ or $N \in \mathcal{I}^\infty(F)$, then

$$\mathrm{Gext}^n_F(M,N) \cong \mathrm{Ext}^n_F(M,N)$$

is functorial in M and N.

Proof. (1) (a) Assume that $M \in \mathcal{P}^\infty(F)$, and pick any F-projective resolution $\mathbf{P}$ of M. Since $\mathbf{P}$ is also a proper left $\mathcal{GP}(F)$-resolution of M, and thus

$$\mathrm{Ext}^n_{\mathcal{GP}(F)}(M,N) = \mathrm{H}^n(\mathrm{Hom}_\Lambda(\mathbf{P},N)) = \mathrm{Ext}^n_F(M,N).$$

(b) Assume that $M \in \widetilde{\mathcal{GP}}(F)$ and $\mathrm{id}_F N = m < \infty$. By Lemma 3.1 (1), we see that $\mathrm{Ext}^i_F(G,N) = 0$ for every F-Gorenstein projective Λ-module G and all $i > 0$. Therefore [9, Chapter III, Prop. 1.2 A] implies that $\mathrm{Ext}^i_F(-,N)$ can be computed using (proper) left $\mathcal{GP}(F)$-resolutions of the argument in the first variable, as desired.

The proof of (2) is similar.

The claim (3) is a direct consequence of (1) and (2), together with the Definition 3.6 of $\mathrm{Gext}^i_F(-,-)$. □

Proposition 3.9. *Assume that $M \in \mathcal{GP}^\infty(F)$. Then the following statements are equivalent for any non-negative integer m.*

(1) $\mathrm{Gpd}_F M \leq m$.
(2) For any F-exact sequence $0 \to K_m \to P_{m-1} \to \cdots \to P_0 \to M \to 0$ with each $P_i \in \mathcal{P}(F)$, $K_m \in \mathcal{GP}(F)$.
(3) $Ext^{m+i}_F(M,P) = 0$ for all $P \in \mathcal{P}(F)$ and all $i \geq 1$.
(4) $Ext^{m+i}_F(M,L) = 0$ for all $L \in \mathcal{P}^\infty(F)$ and all $i \geq 1$.
(5) $Ext^{m+i}_{\mathcal{GP}(F)}(M,N) = 0$ for all $N \in \mathrm{mod}\Lambda$ and all $i \geq 1$.
(6) $Ext^{m+1}_{\mathcal{GP}(F)}(M,N) = 0$ for all $N \in \mathrm{mod}\Lambda$.

Proof. (1)⇔(2)⇔(3)⇔(4) follows from [13, Thm. 2.13].

(2)⇒(5). Let $\cdots \to G_1 \to G_0 \to M \to 0$ be a proper left $\mathcal{GP}(F)$-resolution of M and $C_m = \mathrm{Ker}(G_{m-1} \to G_{m-2})$. Then $C_m \in \mathcal{GP}(F)$ by [13, Cor. 2.10]. So $0 \to C_m \to G_{m-1} \cdots \to G_1 \to G_0 \to M \to 0$ is a proper left $\mathcal{GP}(F)$-resolution of M and thus $\mathrm{Ext}^{m+i}_{\mathcal{GP}(F)}(M,N) = 0$ for all $N \in \mathrm{mod}\Lambda$ and all $i \geq 1$.

(5)⇒(6) is obvious.

(6)⇒(1). Let $\cdots \to G_1 \to G_0 \to M \to 0$ be a proper left $\mathcal{GP}(F)$-resolution of M and $C_{m+1} = \mathrm{Ker}(G_m \to G_{m-1})$. Then (6) implies that $\mathrm{Hom}_\Lambda(G_m, N) \to \mathrm{Hom}_\Lambda(C_{m+1}, N) \to 0$ is exact for all $N \in \mathrm{mod}\Lambda$. So by setting $N = C_{m+1}$, we see that $0 \to C_{m+1} \to G_m \to C_m \to 0$ is split F-exact, where $C_m = \mathrm{Ker}(G_{m-1} \to G_{m-2})$, and so C_{m+1}, $C_m \in \mathcal{GP}(F)$ by [13, Cor. 2.9], as desired. □

With a dual proof we get the following.

Proposition 3.10. *Assume that $N \in \mathcal{GI}^\infty(F)$. Then the following statements are equivalent for any non-negative integer m.*

(1) $\mathrm{Gid}_F N \leq m$.
(2) For any F-exact sequence $0 \to N \to I^0 \to \cdots \to I^{m-1} \to K^m \to 0$ with each $I^i \in \mathcal{I}(F)$, $K^m \in \mathcal{GI}(F)$.
(3) $\mathrm{Ext}_F^{m+i}(E, N) = 0$ for all $E \in \mathcal{I}(F)$ and all $i \geq 1$.
(4) $\mathrm{Ext}_F^{m+i}(L, N) = 0$ for all $L \in \mathcal{I}^\infty(F)$ and all $i \geq 1$.
(5) $\mathrm{Ext}_{\mathcal{GI}(F)}^{m+i}(M, N) = 0$ for all $M \in \mathrm{mod}\Lambda$ and all $i \geq 1$.
(6) $\mathrm{Ext}_{\mathcal{GI}(F)}^{m+1}(M, N) = 0$ for all $M \in \mathrm{mod}\Lambda$.

It was proved by Tang that every $M \in \mathrm{mod}\Lambda$ has finite F-Gorenstein projective (respectively, F-Gorenstein injective) dimension whenever Λ is F-Gorenstein [13, Thm. 3.4]. Now we get the following result.

Theorem 3.11. *Assume that Λ is F-Gorenstein. Then the following statements are equivalent for some non-negative integer m.*

(1) fin.dim$\Lambda \leq m$.
(2) fin.Gdim$\Lambda \leq m$.
(3) $\mathrm{Gext}_F^{m+i}(M, N) = 0$ for all M, $N \in \mathrm{mod}\Lambda$ and all $i \geq 1$.
(4) $\mathrm{Gext}_F^{m+1}(M, N) = 0$ for all M, $N \in \mathrm{mod}\Lambda$.

Proof. This easily follows from Propositions 3.9, 3.10 and [13, Thms. 3.2, 3.4]. □

Corollary 3.12. *Assume that Λ is F-Gorenstein. Then any submodule of an F-Gorenstein projective module is F-Gorenstein projective if and only if any quotient of an F-Gorenstein injective module is F-Gorenstein injective.*

Proof. This is an immediate consequence of Theorem 3.11 by setting $m = 1$. □

If we consider the elements of $\mathrm{Ext}_\Lambda^1(M, N)$ as classes of short exact sequences $0 \to N \to L \to M \to 0$ in $\mathrm{mod}\Lambda$, then we get the following result.

Corollary 3.13. *Let Λ be F-Gorenstein. Then the following are equivalent for an F-exact sequence $0 \to N \to L \to M \to 0$ in $\mathrm{mod}\Lambda$.*

(1) *The sequence $0 \to N \to L \to M \to 0$ corresponds to an element of*

$Gext_F^1(M,N) \subset Ext_\Lambda^1(M,N)$.

(2) *The sequence $Hom_\Lambda(L,C) \to Hom_\Lambda(N,C) \to 0$ is exact for all $C \in \mathcal{GI}(F)$.*

(3) *The sequence $Hom_\Lambda(C,L) \to Hom_\Lambda(C,M) \to 0$ is exact for all $C \in \mathcal{GP}(F)$.*

Proof. (1)⇒(2). If $0 \to N \to L \to M \to 0$ corresponds to an element of $\mathrm{Gext}_F^1(M,N)$, then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

with F-exact rows such that $N \to E$ is a $\mathcal{GI}(F)$-preenvelope. But if $N \to C$ is a map with $C \in \mathcal{GI}(F)$, then $N \to C$ can be extended to E. But then $N \to C$ can be extended to L and so the result follows.

(2)⇒(1). Exactness of the sequence $\mathrm{Hom}_\Lambda(L,C) \to \mathrm{Hom}_\Lambda(N,C) \to 0$ for all $C \in \mathcal{GI}(F)$ implies that we have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & C & \longrightarrow & K & \longrightarrow & 0 \end{array}$$

with F-exact rows, where $N \to C$ is a $\mathcal{GI}(F)$-preenvelope of N. This shows that the sequence $0 \to N \to L \to M \to 0$ corresponds to an element of $\mathrm{Gext}_F^1(M,N)$.

(1)⇔(3). It follows by a dual argument. □

4. **Relative homology**

Now we consider the functor $-\otimes_\Lambda -$: $(\mathrm{mod}\Lambda)^{op} \times \mathrm{mod}\Lambda \to \mathcal{A}b$. It was proved by Auslander and Solberg that $- \otimes_\Lambda -$ is left balanced by $\mathcal{P}(F^{op}) \times \mathcal{P}(F)$.

Lemma 4.1. *Assume that Λ is F-Gorenstein. Then the following statements are equivalent for $M \in \mathrm{mod}\Lambda$.*

(1) M is F-Gorenstein projective.
(2) $Ext_F^i(M,L) = 0$ for all $L \in \mathcal{I}^\infty(F)$ and all $i \geq 1$.
(3) $Ext_F^1(M,L) = 0$ for all $L \in \mathcal{I}^\infty(F)$.
(4) $Tor_i^F(L,M) = 0$ for all $L \in \mathcal{P}^\infty(F^{op})$ and all $i \geq 1$.
(5) $Tor_1^F(L,M) = 0$ for all $L \in \mathcal{P}^\infty(F^{op})$.
(6) DM is F-Gorenstein injective Λ^{op}-module.

Proof. (1)⇒(2) by Lemma 3.1 (1) and (2)⇒(3) is trivial.

(3)⇒(1). By [13, Thm. 3.4].

(2)⇒(4). Following from [2, Lem. 2.1], we have $D\mathrm{Tor}_i^F(L,M) \cong \mathrm{Ext}_F^i(M,DL)$. If $\mathrm{pd}_{F^{op}}L < \infty$ then $\mathrm{id}_F DL < \infty$, so $\mathrm{Ext}_F^i(M,DL) = 0$ for all $i \geq 1$ by (2). Hence $\mathrm{Tor}_i^F(L,M) = 0$ for all $i \geq 1$ and all $L \in \mathcal{P}^\infty(F^{op})$.

(4)⇒(5) is obvious.

(5)⇒(6) and (6)⇒(3). Note that $\mathrm{Tor}_1^F(L,M) = 0$ if and only if $\mathrm{Ext}_F^1(L,DM) = D\mathrm{Tor}_1^F(L,M) = 0$ for all $L \in \mathcal{P}^\infty(F^{op})$. Now using [13, Thm. 3.4] we get our claims. □

Lemma 4.2. *Assume that Λ is F-Gorenstein and $M \in \mathrm{mod}\Lambda$. Then any proper left $\mathcal{GP}(F)$-resolution $\mathbf{G} \to M$ is $\mathcal{GP}(F^{op}) \otimes_\Lambda -$-exact.*

Proof. Let $\mathbf{G} = \cdots \to G_1 \to G_0 \to M \to 0$ be a proper left $\mathcal{GP}(F)$-resolution of M. We note that $\cdots \to H \otimes_\Lambda G_1 \to H \otimes_\Lambda G_0 \to H \otimes_\Lambda M \to 0$ is exact for all $H \in \mathcal{GP}(F^{op})$ if and only if $0 \to \mathrm{Hom}_\Lambda(M,DH) \to \mathrm{Hom}_\Lambda(G_0,DH) \to \mathrm{Hom}_\Lambda(G_1,DH) \to \cdots$ is exact for all $H \in \mathcal{GP}(F^{op})$ by the adjointness isomorphism. Hence our desired result follows from Lemmas 3.3 (1) and 4.1. □

Definition 4.3. Let $M \in \widetilde{\mathcal{GP}}(F^{op})$ and consider a proper left $\mathcal{GP}(F^{op})$-resolution $\mathbf{G} \to M$. For every $n \in \mathbb{Z}$ and every $N \in \mathrm{mod}\Lambda$, define a relative homology group

$$\mathrm{Tor}_n^{\mathcal{GP}(F^{op})}(M,N) = \mathrm{H}_n(\mathbf{G} \otimes_\Lambda N).$$

Similarly, choosing for each $N \in \widetilde{\mathcal{GP}}(F)$ a proper left $\mathcal{GP}(F)$-resolution $\mathbf{D} \to N$, we define for each $n \in \mathbb{Z}$ and each $M \in \mathrm{mod}\Lambda$ a relative homology group

$$\mathrm{Tor}_n^{\mathcal{GP}(F)}(M,N) = \mathrm{H}_n(M \otimes_\Lambda \mathbf{D}).$$

Lemma 4.2 can be stated as follows: the bifunctor $-\otimes_\Lambda -$: is left balanced by $\mathcal{GP}(F^{op}) \times \mathcal{GP}(F)$ over F-Gorenstein Artin agebras. So we have the following result.

Theorem 4.4. *Assume that Λ is F-Gorenstein. For any M, $N \in \mathrm{mod}\Lambda$ and $i \in \mathbb{Z}$, we have isomorphisms*

$$\mathrm{Tor}_n^{\mathcal{GP}(F^{op})}(M,N) \cong \mathrm{Tor}_n^{\mathcal{GP}(F)}(M,N).$$

Definition 4.5. (Definition of Gtor) Assume that Λ is F-Gorenstein. For any M, $N \in \mathrm{mod}\Lambda$ and $i \in \mathbb{Z}$, we write

$$\mathrm{Gtor}_n^F(M,N) := \mathrm{Tor}_n^{\mathcal{GP}(F^{op})}(M,N) \cong \mathrm{Tor}_n^{\mathcal{GP}(F)}(M,N)$$

for the isomorphic abelian groups in Theorem 4.4.

Then it is easy to check the following properties of Gtor_n^F:

Proposition 4.6. *(1) $\mathrm{Gtor}_0^F(-,-) = - \otimes_\Lambda -$.*

(2) $\mathrm{Gtor}_i^F(M,-) = 0$ for all $i \geq 1$ and all $M \in \mathcal{GP}(F^{op})$.

(3) $\mathrm{Gtor}_i^F(-,N) = 0$ for all $i \geq 1$ and all $N \in \mathcal{GP}(F)$.

(4) *If the exact sequence* $0 \to M' \to M \to M'' \to 0$ *of* $mod\Lambda^{op}$ *is* $-\otimes_\Lambda N$*-exact for all* $N \in \mathcal{GP}(F)$*, then by part (1) of [8, Theorem 8.2.3] there is a long exact sequence*

$$\cdots \to Gtor_{i+1}^F(M'', N) \to Gtor_i^F(M', N) \to Gtor_i^F(M, N) \to \cdots .$$

(5) *Same as* (4) *for an exact sequence* $0 \to N' \to N \to N'' \to 0$ *of* $mod\Lambda$*.*

(6) *There are natural transformations*

$$Tor_i^F(-,-) \longrightarrow Gtor_i^F(-,-)$$

which are also natural in the long exact sequences as in (4) *and* (5) *above.*

In the next proposition, we give the relationships between functor Gtor and the classical one Tor.

Proposition 4.7. *Let* Λ *be* F*-Gorenstein and* $L \in \mathrm{mod}\Lambda$*. If* $\mathrm{pd}_F L < \infty$ *then the natural transformation* $Tor_i^F(-, L) \longrightarrow Gtor_i^F(-, L)$ *is a natural isomorphism for all* $i \geq 0$*.*

Proof. We consider the F-exact sequence $0 \to H \to G \to N \to 0$ in $\mathrm{mod}\Lambda$, where $G \to N$ is a $\mathcal{GP}(F)$-precover of N. If $\mathrm{pd}_F L < \infty$, then we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}_{i+1}^F(N, L) & \longrightarrow & \mathrm{Tor}_i^F(H, L) & \longrightarrow & \mathrm{Tor}_i^F(G, L) & \longrightarrow & \mathrm{Tor}_i^F(N, L) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Gtor}_{i+1}^F(N, L) & \longrightarrow & \mathrm{Gtor}_i^F(H, L) & \longrightarrow & \mathrm{Gtor}_i^F(G, L) & \longrightarrow & \mathrm{Gtor}_i^F(N, L)
\end{array}$$

by Lemma 4.1.

If $i = 0$, then $\mathrm{Tor}_0^F(-,-) = \mathrm{Gtor}_0^F(-,-) = -\otimes_\Lambda -$ and so the last three maps are isomorphisms. Hence $\mathrm{Tor}_1^F(N, L) \longrightarrow \mathrm{Gtor}_1^F(N, L)$ is an isomorphism. The result now follows by induction on i. □

We also have the following result.

Proposition 4.8. *Let* Λ *be* F*-Gorenstein and* $N \in \mathrm{mod}\Lambda$*. Then*

(1) *If* $M \in \mathrm{mod}\Lambda$*, then* $Gext_F^1(M, N) \longrightarrow Ext_F^1(M, N)$ *is an injection.*

(2) *If* $M \in \mathrm{mod}\Lambda^{op}$*, then* $Tor_1^F(M, N) \longrightarrow Gtor_1^F(M, N)$ *is a surjection.*

Proof. (1) Let $N \to C$ be a $\mathcal{GI}(F)$-preenvelope of N and $K = C/N$. Then we have the following diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_\Lambda(M, C) & \longrightarrow & \mathrm{Hom}_\Lambda(M, K) & \longrightarrow & \mathrm{Gext}_F^1(M, N) & \longrightarrow & \mathrm{Gext}_F^1(M, C) = 0 \\
\| & & \| & & \downarrow & & \\
\mathrm{Hom}_\Lambda(M, C) & \longrightarrow & \mathrm{Hom}_\Lambda(M, K) & \longrightarrow & \mathrm{Ext}_F^1(M, N) & &
\end{array}$$

with exact rows. It is easy to see that $\mathrm{Gext}^1_F(M,N) \longrightarrow \mathrm{Ext}^1_F(M,N)$ is an injection.

(2) The proof is similar to that of (1). □

Acknowledgments

Research supported by National Natural Science Foundation of China (11261050, 11301240, 11401475).

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