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ON BLACK-SCHOLES EQUATION; METHOD OF HEIR-EQUATIONS, NONLINEAR SELF-ADJOINTNESS AND CONSERVATION LAWS

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(Communicated by Asadollah Aghajani)

Abstract. In this paper, Heir-equations method is applied to investigate nonclassical symmetries and new solutions of the Black-Scholes equation. Nonlinear self-adjointness is proved and infinite number of conservation laws are computed by a new conservation law theorem.
Keywords: Black-Scholes equation, Heir-equation, nonclassical symmetry, nonlinear self-adjointness, conservation law.
MSC(2010): Primary: 58J70, 35A30; Secondary: 70s10.

1. Introduction

Mathematical finance is a field of applied mathematics, concerned with financial markets. Generally, mathematical finance derive and extend the mathematical or numerical models suggested by financial economics.
In the recent years stock option was one of the most popular financial derivative. Indeed, an option is a financial contract which gives its owner the right to buy or sell a specified amount of a particular asset at a fixed price, called the exercise price, on or before a specified date, called the maturity date. Black and Scholes have shown in [2] that option prices satisfy a second-order partial differential equation (PDE) with respect to the time t and asset price x. This equation is known as the *Black-Scholes* equation and is given by

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Bu = 0, \tag{1.1}$$

where $A,\ B$ are arbitrary constants.

Gazizov and Ibragimov in [14] have shown that the one-dimensional Black-Scholes equation is included in Sophous Lie's classification of linear second-order PDEs with two independent variables. Then, they obtained the invariant

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solutions of the Black-Scholes equation. Pooe et al. in [30] utilized two sets of transformations, introduced in [14], which reduce the Black-Scholes equation to the one-dimensional heat equation. Then they exploited an optimal system of one-dimensional subalgebras for the heat equation to obtain two classes of optimal systems of one-dimensional subalgebras for Eq. (1.1). Complete Lie symmetry group of the one dimensional Black-Scholes equation is derived in [24] and infinite dimensional Lie algebras of Eq. (1.1) and related invariant solution are obtained. However, invariant solutions of Eq. (1.1) presented in [17, 24] are different from our obtained solutions, by nonclassical symmetries, in Section 3. More recently, classical symmetries of the Black-Scholes equation with time dependent coefficient is considered by O'Hara et al. in [17] and Group classification of a generalized Black-Scholes-Merton equation is considered in [5].

The nonclassical symmetry method due to Bluman and Cole [4] is one of the most well known generalizations of Lie's classical method for finding group-invariant solutions of a PDE which these solutions are not deducible from Lie group analysis. It consists in adding the invariant surface condition to the given equation, and then apply the Lie group analysis. The main difficulty of this approach is that the determining equations are no longer linear. M.C. Nucci in [26], has found that iterations of the nonclassical symmetries method give rise to new nonlinear equations, which inherit the Lie point symmetry algebra of the given equation.

Concept of conservation laws, which are mathematical formulations of the fundamental physical principles, such as conservation of momentum, mass, charge, energy and so on, arises in a wide variety of applications and contexts. Conservation laws are widely applied in analysis of PDEs, particularly, investigation of existence, uniqueness and stability of solutions of nonlinear PDEs. Moreover, conservation laws play a vital role in studying the integrability of nonlinear PDEs. The existence of infinite conservation laws is an important indicator of integrability of the system. Edelstein and Govinder in [7] have found the conservation laws of Black-Scholes equation by the method of Kara and Mahomed [23], which utilizes the point symmetries. Also, we found that our obtained conservation laws are different from the reported results in [7]. However, as mentioned by Edlestein, et al. comparison of Ibragimov method with method of Kara and Mahomed is not possible. They obtained only six conserved vectors for Eq. (1.1), whereas in this paper we present infinite number of conservation laws. In [20] a general theorem on conservation laws for arbitrary differential equations which do not require the existence of Lagrangians has been proved. This new theorem is based on the concept of adjoint equations for both linear and non-linear equations. There are many equations with physical significance which are not self-adjoint and several generalizations of this concept have been introduced. In [21] Ibragimov has introduced the definition of quasi self-adjoint equations. Then, after introducing the concept of weakly self-adjoint equations by Gandarias in [12], this concept is utilized to construct the conservation laws of the Hamilton-Jacobi-Bellman equations which arises from financial mathematics [13]. Finally, Ibragimov in [22] introduced the concept of nonlinear self-adjointness where substitution $v = h(u)$ can be replaced with a more general substitution $v = h(x, t, u, u_t, u_x, \ldots)$. Some recent papers in this field are [6, 8–11, 18, 31].

In this paper we look for nonclassical symmetries of Eq. (1.1) with the purpose of deriving nonclassical symmetry solutions and then finding the conservation laws [20] using the concept of nonlinear self-adjointness of Black-Scholes equation and classical symmetries.

2. Heir-equations and nonclassical symmetries

Let us consider an evolution equation in two independent variables and one dependent variable of second order:

$$u_t = H(t, x, u, u_x, u_{xx}). \tag{2.1}$$

If

$$\Gamma = V_1(t, x, u)\partial_t + V_2(t, x, u)\partial_x - F(t, x, u)\partial_u, \tag{2.2}$$

is a generator of a Lie point symmetry[1] of equation (2.1) then the invariant surface condition (more details about invariant surface condition can be found in [3]) is given by:

$$V_1(t, x, u)u_t + V_2(t, x, u)u_x = F(t, x, u). \tag{2.3}$$

Let us take the case with $V_1 = 0$ and $V_2 = 1$, so that (2.3) becomes[2]:

$$u_x = G(t, x, u) \tag{2.4}$$

Then, an equation for G, namely, G-equation is easily obtained [28]. Its invariant surface condition is given by:

$$\xi_1(t, x, u, G)G_t + \xi_2(t, x, u, G)G_x + \xi_3(t, x, u, G)G_u = \eta(t, x, u, G). \tag{2.5}$$

Let us consider the case $\xi_1 = 0$, $\xi_2 = 1$, and $\xi_3 = G$, so that (2.5) becomes:

$$G_x + GG_u = \eta(t, x, u, G). \tag{2.6}$$

Then, an equation for η is derived. We call this equation η-equation. Clearly

$$G_x + GG_u \equiv u_{xx} \equiv \eta. \tag{2.7}$$

We could keep iterating to obtain the Ω-equation, which corresponds to:

$$\eta_x + G\eta_u + \eta\eta_G \equiv u_{xxx} \equiv \Omega(t, x, u, G, \eta) \tag{2.8}$$

[1] The minus sign in front of $F(t, x, u)$ was put there for the sake of simplicity: it could be replaced with a plus sign without affecting the following results.

[2] We have replaced $F(t, x, u)$ with $G(t, x, u)$ in order to avoid any ambiguity in the following discussion.

and so on. Each of these equations inherits the symmetry algebra of the original equation, with the right prolongation: first prolongation for the G-equation, second prolongation for the η-equation, and so on. Therefore, these equations were named Heir-equations in [26]. This implies that even in the case of few Lie point symmetries many more Lie symmetry reductions can be performed by using the invariant symmetry solution of any of the possible Heir-equations, as it was shown in [1, 19, 25, 26]. However, we recall that the Heir-equations are just some of the many possible n-extended equations as defined by Guthrie in [16].

In [26] it was shown that this iterating method yields both partial symmetries as given by Vorob'ev in [32], and differential constraints as given by Olver [29].

In [15] Goard has shown that Nucci's method of constructing Heir equations by iterating the nonclassical symmetries method is equivalent to the generalized conditional symmetries method.

The difficulty in applying the method of nonclassical symmetries consists in solving nonlinear determining equations in contrast with the linearity of the determining equations in the case of classical symmetries. In [27] it was shown that one can find the nonclassical symmetries of any evolution equations of any order by using a suitable heir-equation and searching for a given particular solution among all its solutions, thus avoiding any complicated calculations. We recall the method as applicable to equation (2.1).

We derive u_t from (2.1) and replace it into (2.3), with the condition $V_1 = 1$, i.e.:

$$H(t, x, u, u_x, u_{xx}) + V_2(t, x, u)u_x = F(t, x, u). \tag{2.9}$$

Then, we generate the η-equation with $\eta = \eta(x, t, u, G)$, and replace $u_x = G$, $u_{xx} = \eta$ into (2.9), i.e.:

$$H(t, x, u, G, \eta) = F(t, x, u) - V_2(t, x, u)G. \tag{2.10}$$

For Dini's theorem, we can isolate η in (2.10), e.g.:

$$\eta = [h_1(t, x, u, G) + F(t, x, u) - V_2(t, x, u)G]\, h_2(t, x, u, G) \tag{2.11}$$

where $h_i(t, x, u, G)(i = 1, 2)$ are known functions. Thus, we have obtained a particular solution of η which must yield an identity if replaced into the η-equation. The only unknowns are $V_2 = V_2(t, x, u)$ and $F = F(t, x, u)$. If any such solution is singular, i.e. does not form a group, then we have found the nonclassical symmetries, otherwise one obtains the classical symmetries [27].

3. Nonclassical symmetries of Eq. (1.1)

We use a simple MAPLE program to derive the Heir-equations. In particular the $G-$equation of (1.1) is:

$$G_t + BuG_u + A^2x\,(G_x + GG_u) + \frac{1}{2}A^2x^2G_{xx} = 0,$$

and the $\eta-$equation is

$$\begin{aligned}A^2x^2\left(G\eta\eta_{uG}+G\eta_{xu}+\eta\eta_{xG}\right) \quad &+\eta_t+B\eta+A^2\eta+\frac{A^2x^2}{2}\left(G^2\eta_{uu}+\eta_{xx}+\eta^2\eta_{GG}\right)\\ &+Bx\eta_x+Bu\eta_u+A^2x\left(\eta\eta_G+2G\eta_u+2\eta_x\right)=0. \qquad (3.1)\end{aligned}$$

The particular solution of the η-equation that we are looking for is

$$\eta(t,x,u,G)=\frac{2\left(V_2G-BxG+Bu-F\right)}{A^2x^2}, \qquad (3.2)$$

that replaced into (3.1) yields an overdetermined system in the unknowns F, V_2. Since we obtain a polynomial of third degree in G then we let MAPLE evaluate the four coefficients that we call d_i, $\;i=0,1,2,3$ where i stands for the corresponding power of G. We impose all of them to be zero. From $d_3=0$, we obtain

$$V_2(t,x,u)=\Theta_1(t,x)u+\Theta_2(t,x),$$

while d_2 yields

$$\begin{aligned}F(t,x,u) \quad &=\frac{1}{A^2x^2}\left(\frac{2\Theta_1^2u^3}{3}+2\Theta_1\Theta_2u^2-2B\Theta_1u^2\right)+\frac{\partial\Theta_1}{\partial x}u^2\\ &+\Theta_3(t,x)u+\Theta_4(t,x)\end{aligned}$$

with $\Theta_j(t,x)$, $\;j=1,\ldots,4$ arbitrary functions of t and x. Since d_1 is a polynomial of order 3 with respect to u, we set e_j, $j=0,\ldots,3$ as the coefficients of u^j, $j=0,\ldots,3$. From $e_3=0$ we get

$$\Theta_1(t,x)=0, \qquad (3.3)$$

which implies also $e_1=e_2=0$. Finally we have

$$\begin{aligned}e_0 \quad &=-3A^2x^3\Bigg[-x\Theta_2\left(B+4\frac{\partial\Theta_2}{\partial x}\right)+2Bx^2\frac{\partial\Theta_2}{\partial x}+A^3x^3\left(2\frac{\partial\Theta_3}{\partial x}-\frac{\partial^2\Theta_2}{\partial x^2}\right)\\ &-2x\frac{\partial\Theta_2}{\partial t}+4\Theta_2^2\Bigg].\end{aligned}$$

We consider the following special cases:

Case 1.: $\Theta_2(t,x)=\tilde{\Theta}_2(t)$.

Case 2.: $\Theta_2(t,x)=\tilde{\Theta}_2(x)$.

Case 1.

In this case by setting e_0 equal zero, one can obtain:

$$\tilde{\Theta}_2(t)=c_1e^{-Bt}, \quad \Theta_3(t,x)=\frac{c_1^2e^{-2Bt}}{A^2x^2}+\tilde{\Theta}_3(t).$$

One time differentiation of d_0 and setting it equal to zero yields

$$\tilde{\Theta}_3(t)=c_2, \quad c_1=0.$$

By using these values we can write d_0 as follows:

$$d_0 = -A^2x^3\left[A^2x^2\frac{\partial^2\Theta_4}{\partial x^2} + 2\frac{\partial\Theta_4}{\partial t} + 2Bx\frac{\partial\Theta_4}{\partial x} - 2B\Theta_4\right]. \tag{3.4}$$

From $d_0 = 0$ some subcases are considerable.

Subcase 1.1. $\Theta_4(t,x) = \tilde{\Theta}_4(x)$
In this subcase by using $d_0 = 0$ we get

$$\tilde{\Theta}_4(x) = c_3x + c_4x^{\frac{-2B}{A^2}},$$

where $c_3,\ c_4$ are arbitrary constants. In this step all of the unknowns are determined and we have

$$V_2(t,x,u) = 0, \quad F(t,x,u) = c_2u + c_3x + c_4x^{\frac{-2B}{A^2}},$$

and Eq. (3.2) becomes

$$\eta = \frac{2\left(-BxG + Bu - c_2u - c_3x - c_4x^{\frac{-2B}{A^2}}\right)}{A^2x^2},$$

namely

$$u_{xx} = \frac{2\left(-Bxu_x + Bu - c_2u - c_3x - c_4x^{\frac{-2B}{A^2}}\right)}{A^2x^2}. \tag{3.5}$$

Eq. (3.5) is a linear ordinary differential equation with respect to x and its solution is given by:

$$u(t,x) = \Psi_1(t)x^{\frac{A^2-2B+\lambda}{2A^2}} + \Psi_2(t)x^{\frac{A^2-2B-\lambda}{2A^2}} - \frac{c_3x^2 + c_4x^{\frac{A^2-2B}{A^2}}}{c_2x}, \tag{3.6}$$

where $\lambda = \sqrt{A^4 + 4B^2 + 4BA^2 - 8c_2A^2}$. Substituting (3.6) into (1.1) yields the following nonclassical symmetry solution

$$u(t,x) = c_5e^{c_2t}x^{\frac{A^2-2B+\lambda}{2A^2}} + c_6e^{c_2t}x^{\frac{A^2-2B-\lambda}{2A^2}} - \frac{c_3x^2 + c_4x^{\frac{A^2-2B}{A^2}}}{c_2x}, \tag{3.7}$$

with $c_k, k = 2, \ldots, 6$ arbitrary constants.

Subcase 1.2. $\Theta_4(t,x) = \tilde{\Theta}_4(t)$
From $d_0 = 0$ we get

$$\tilde{\Theta}_4(t) = c_3e^{Bt}.$$

Therefore

$$V_2(t,x,u) = 0, \quad F(t,x,u) = c_2u + c_3e^{Bt},$$

and form (3.2), η-equation is as follows:

$$\eta = \frac{2\left(Bu - BxG - c_2u - c_3e^{Bt}\right)}{A^2x^2},$$

which is equivalent to

$$u_{xx} = \frac{2\left(Bu - Bxu_x - c_2 u - c_3 e^{Bt}\right)}{A^2 x^2}.$$

Obtained equation is an ODE with respect to x and its solution is given by

$$u(t,x) = \Psi_1(t) x^{\frac{B+1-\lambda}{2}} + \Psi_2(t) x^{\frac{B+\lambda}{2}} + \frac{c_3 e^{Bt}}{B - c_2}, \tag{3.8}$$

where $\lambda = \sqrt{(B-1)^2 + 4c_2}$. Substituting (3.8) into (1.1) yields another non-classical symmetry solution of (1.1) with

$$\Psi_1(t) = c_4 e^{-\frac{t\left(\left(2+A^2\right)\left(B^2 - B\lambda - B\right) + 2A^2 c_2\right)}{4}}, \quad \Psi_2(t) = c_5 e^{-\frac{t\left(\left(2+A^2\right)\left(B^2 + B\lambda - B\right) + 2A^2 c_2\right)}{4}}, \tag{3.9}$$

where $c_k,\ k = 2, \ldots, 5$ are arbitrary constants and λ is defined as before.

Case 2.

In this case $e_0 = 0$ becomes as follows:

$$e_0 = 3A^2 x^3 \left[-2A^2 x^3 \frac{\partial \Theta_3}{\partial x} + A^2 x^3 \frac{d^2 \tilde{\Theta}_2}{dx^2} + (4x\tilde{\Theta}_2 - 2Bx^2) \frac{d\tilde{\Theta}_2}{dx} - 4\tilde{\Theta}_2^2 + 2Bx\tilde{\Theta}_2 \right] = 0. \tag{3.10}$$

To solve this equation, we consider the following subcases.

Subcase 2.1. $\Theta_3(t,x) = \frac{1}{2} \frac{d\tilde{\Theta}_2(x)}{dx}$

Eq. (3.10) yields $\tilde{\Theta}_2(x) = c_1 x$ and therefore (3.4) becomes

$$d_0 = x\left[A^2 x^2 \frac{\partial^2 \Theta_4}{\partial x^2} + 2Bx \frac{\partial \Theta_4}{\partial x} + 2\frac{\partial \Theta_4}{\partial t} - 2B\Theta_4 \right]. \tag{3.11}$$

By setting $\Theta_4(t,x) = \tilde{\Theta}_4(x)$ in (3.11), and solving $d_0 = 0$ we get

$$\tilde{\Theta}_4(x) = A_1 x + A_2 x^{-\frac{2B}{A^2}}.$$

Therefore

$$V_2(t,x,u) = c_1 x, \quad F(t,x,u) = \frac{1}{2} c_1 u + A_1 x + A_2 x^{-\frac{2B}{A^2}},$$

and η-equation becomes

$$\eta = \frac{2(c_1 - B)xG + 2Bu - c_1 u - 2A_1 x - 2A_2 x^{-\frac{2B}{A^2}}}{A^2 x^2},$$

in other words:

$$u_{xx} = \frac{2(c_1 - B)xu_x + 2Bu - c_1 u - 2A_1 x - 2A_2 x^{-\frac{2B}{A^2}}}{A^2 x^2}, \tag{3.12}$$

Eq. (3.5) is obtainable from Eq. (3.12) by setting $c_1 = \frac{B}{2} = 2c_2$. Thus we scape from the calculation of solutions for this case.
However, setting $\Theta_4(t,x) = \tilde{\Theta}_4(t)$ in (3.11), and solving $d_0 = 0$ yields

$$\tilde{\Theta}_4(t) = A_1 e^{Bt}.$$

Therefore

$$V_2(t,x,u) = c_1 x, \quad F(t,x,u) = \frac{1}{2} c_1 u + A_1 e^{Bt}.$$

Thus η-equation becomes

$$\eta = \frac{2(c_1 - B)xG + 2Bu - c_1 u - 2A_1 e^{Bt}}{A^2 x^2},$$

in other words:

$$u_{xx} = \frac{2(c_1 - B)x u_x + (2B - c_1)u - 2A_1 e^{Bt}}{A^2 x^2}. \tag{3.13}$$

Solving the obtained ODE yields:

$$u(t,x) = \Psi_1(t) x^{\frac{2c_1 - 2B + A^2 - \lambda}{2A^2}} + \Psi_2(t) x^{\frac{2c_1 - 2B + A^2 + \lambda}{2A^2}} + \frac{2A_1 e^{Bt}}{2B - c_1}, \tag{3.14}$$

where $\lambda = \sqrt{4(c_1 - B)^2 + 4A^2 B + A^4}$. Another nonclassical symmetry solution of (1.1) is obtainable as following, by substituting (3.14) into (1.1)

$$\begin{aligned}
&\Psi_1(t) = A_2 e^{c_1 t \left(\frac{2B - 2c_1 + \lambda}{2A^2} \right)}, \\
&\Psi_2(t) = A_3 e^{c_1 t \left(\frac{2B - 2c_1 + \lambda}{2A^2} \right)} + e^{c_1 t \left(\frac{2B - 2c_1 + \lambda}{2A^2} \right)} \times \\
&\frac{4A_1 B(2B - 2c_1 + \lambda) x^{\frac{2B - 2c_1 - \lambda - A^2}{2A^2}} e^{t \left(\frac{2BA^2 - 2Bc_1 + 2c_1^2 + c_1 \lambda}{2A^2} \right)}}{c_1 (4B + A^2)(2B - c_1)},
\end{aligned}$$

where $A_k,\ k = 1,2,3$ and c_1 are arbitrary constants and λ is defined as before.

Subcase 2.2. $\Theta_3(t,x) = \tilde{\Theta}_3(x)$
From Eq. (3.10) we have

$$\tilde{\Theta}_3(x) = \frac{\tilde{\Theta}_2(x)^2 - B\tilde{\Theta}_2(x)}{A^2 x^2} + \frac{1}{2} \frac{d\tilde{\Theta}_2(x)}{dx} + A_1.$$

By this value of $\tilde{\Theta}_3(x)$, equation $d_0 = 0$ becomes a polynomial of order one with respect to u. A special solution of coefficient of u in $d_0 = 0$ is

$$\tilde{\Theta}_2(x) = \frac{x \left(A^2 + \sqrt{A^4 + 8A^2 B + 8B^2 - 16A^2 A_1} \right)}{4}.$$

Thus $d_0 = 0$ becomes an equation without u, which by setting $\Theta_4(t,x) = \tilde{\Theta}_4(t)$ we get

$$\tilde{\Theta}_4(t) = k_1 e^{Bt}.$$

Therefore

$$\begin{aligned} V_2(t,x,u) &= \frac{(A^2+\lambda)x}{4}, \\ F(t,x,u) &= \frac{(A^2B+A^4+2B^2)u-\lambda(Bu-A^2u^2)+4k_1A^2e^{Bt}}{4A^2}, \end{aligned}$$

where $\lambda=\sqrt{A^4+8A^2B+8B^2-16A_1A^2}$ and η-equation becomes

(3.15)

$$\eta=\frac{A^4xG+\lambda(A^2xG+Bu-A^2u)-4A^2BxG}{2A^4x^2}+\frac{3A^2Bu-A^4u-2B^2u-4k_1A^2e^{Bt}}{2A^4x^2},$$

or equivalently

$$\begin{aligned} u_{xx} &= \frac{A^4xu_x+\lambda(A^2xu_x+Bu-A^2u)-4A^2Bxu_x}{2A^4x^2} \\ &+\frac{3A^2Bu-A^4u-2B^2u-4k_1A^2e^{Bt}}{2A^4x^2}. \end{aligned} \tag{3.16}$$

Another nonclassical symmetry solution of Eq. (1.1) can be found by solving Eq. (3.16) which is as follows:

$$u(t,x)=\Psi_1(t)x^{\frac{A^2-2B+\lambda}{2A^2}}+\Psi_2(t)x^{\frac{A^2-B}{A^2}}-\frac{4k_1A^2e^{Bt}}{(A^2-B)(A^2-2B+\lambda)}, \tag{3.17}$$

where

$$\Psi_1(t)=e^{t\left(\frac{4A_1A^2-A^2B-B^2}{2A^2}\right)},\quad \Psi_2(t)=e^{tB\left(\frac{A^2+B}{2A^2}\right)},$$

and $\lambda=\sqrt{A^4+8A^2B+8B^2-16A_1A^2}$.
Also, if in $d_0=0$ we set $\Theta_4(t,x)=\tilde{\Theta}_4(x)$, then

$$\tilde{\Theta}_4(x)=k_1x+k_2x^{-\frac{2B}{A^2}},$$

and therefore

$$\begin{aligned} V_2(t,x,u) &= \frac{x\left(A^2+\lambda\right)}{4}, \\ F(t,x,u) &= \frac{A^2Bx^2u-\lambda(Bx^2-A^2X^2)u+A^4x^2u}{4A^2x^2} \\ &+\frac{4k_1A^2x^3+2B^2x^2u+4k_2A^2x^{2\left(\frac{A^2-B}{A^2}\right)}}{4A^2x^2}, \end{aligned} \tag{3.18}$$

where $\lambda=\sqrt{A^4+8A^2B+8B^2-16A_1A^2}$. Hence, η-equation is as follows:

$$\begin{aligned} \eta &= \frac{A^4x^3G+\lambda(A^2x^3G+Bux^2-A^2ux^2)-4A^2Bx^3G}{2A^4x^4} \\ &+\frac{3A^2Bux^2-A^4ux^2-2B^2ux^2-4k_1A^2x^3-4k_2A^2x^{2\left(\frac{A^2-B}{A^2}\right)}}{2A^4x^2}, \end{aligned}$$

in other words

$$\begin{aligned} u_{xx} &= \frac{A^4x^3u_x + \lambda(A^2x^3u_x + Bux^2 - A^2ux^2) - 4A^2Bx^3u_x}{2A^4x^4} \\ &+ \frac{3A^2Bux^2 - A^4ux^2 - 2B^2ux^2 - 4k_1A^2x^3 - 4k_2A^2x^{\frac{2A^2-2B}{A^2}}}{2A^4x^2}, \end{aligned}$$

which its solution is given by

$$\begin{aligned} u(t,x) &= \Psi_1(t)x^{\frac{A^2-2B+\lambda}{2A^2}} + \Psi_2(t)x^{\frac{A^2-B}{A^2}} \\ &+ \frac{2A^2(A^2+B)(k_1x^{\frac{A^2+2B}{A^2}}\mu - \lambda B(k_1x^{\frac{A^2+2B}{A^2}} + k_2))}{x^{\frac{2B}{A^2}}(4A_1A^2 - A^2B - B^2)(\lambda - A^2)(A^2B + B^2)} \\ &+ \frac{k_2B(5A^2B + 4B^2 + A^4 - 8A_1A^2)}{x^{\frac{2B}{A^2}}(4A_1A^2 - A^2B - B^2)(\lambda - A^2)(A^2B + B^2)} \end{aligned}$$

where $\mu = 8A^2A_1 - 3A^2B - 4B^2$ and similar to previous

$$\Psi_1(t) = k_3e^{t\left(\frac{4A_1A^2 - A^2B - B^2}{2A^2}\right)}, \quad \Psi_2(t) = k_4e^{tB\left(\frac{A^2+B}{2A^2}\right)}.$$

Comparison of presented solutions of Eq. (1.1) in literature with nonclassical solutions shows that reported solutions in this section are new.

4. Nonlinear self-adjointness and construction of conservation laws

In this section, after some preliminaries, we obtain conservation laws for the Eq. (1.1) using the new conservation theorem introduced in [20].

4.1. **Preliminary.** Consider a k^{th}-order PDE of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and dependent variable u, viz.,

$$F(x, u, u, u_{(1)}, \ldots, u_{(k)}) = 0, \tag{4.1}$$

where $u_{(1)} = \{u_i\},\ u_{(2)} = \{u_{ij}\}, \ldots$ and $u_i = \mathcal{D}_i(u),\ u_{ij} = \mathcal{D}_j\mathcal{D}_i(u)$, where

$$\mathcal{D}_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \ldots, n,$$

are the total derivative operators with respect to x^is.
The Euler-Lagrange operator, by formal sum, is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s\geq 1}(-1)^s\mathcal{D}_{i_1}\cdots\mathcal{D}_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}. \tag{4.2}$$

Also, if $\mathcal{A}$ be the set of all differential functions of all finite orders, and $\xi^i, \eta \in \mathcal{A}$, then *Lie-Bäcklund* operator is

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{i_1i_2}\frac{\partial}{\partial u_{i_1i_2}} + \cdots, \tag{4.3}$$

where

$$\zeta_i = \mathcal{D}_i\big(\eta\big) - u_j \mathcal{D}_i\big(\xi^j\big), \tag{4.4}$$

$$\zeta_{i_1\ldots i_s} = \mathcal{D}_{i_s}\big(\zeta_{i_1\ldots i_{s-1}}\big) - u_{j i_1\ldots i_{s-1}} \mathcal{D}_{i_s}\big(\xi^j\big), \quad s > 1. \tag{4.5}$$

One can write the *Lie-Bäcklund* operator (4.3) in characteristic form

$$X = \xi^i \mathcal{D}_i + W \frac{\partial}{\partial u} + \sum_{s\geq 1} \mathcal{D}_{i_1} \ldots \mathcal{D}_{i_s}(W) \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}},$$

where

$$W = \eta - \xi^j u_j, \tag{4.6}$$

is the characteristic function.

Euler-Lagrange operators with respect to derivatives of u are obtained by replacing u and the corresponding derivatives in (4.2), e.g.

$$\frac{\delta}{\delta u_i} = \frac{\partial}{\partial u_i} + \sum_{s\geq 1} (-1)^s \mathcal{D}_{j_1} \cdots \mathcal{D}_{j_s} \frac{\partial}{\partial u_{i j_1 \cdots j_s}}. \tag{4.7}$$

There is a connection between the Euler-Lagrange, *Lie-Bäcklund* and the associated operators by the following identity:

$$X + \mathcal{D}_i(\xi^i) = W \frac{\delta}{\delta u} + \mathcal{D}_i \mathcal{N}^i,$$

where

$$\mathcal{N}^i = \xi^i + W \frac{\delta}{\delta u_i} + \sum_{s\geq 1} \mathcal{D}_{i_1} \cdots \mathcal{D}_{i_s}(W) \frac{\delta}{\delta u_{i i_1 \cdots i_s}}, \quad i = 1, \ldots, n,$$

are the Noether operators associated with a *Lie-Bäcklund* symmetry operator. The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^i \in \mathcal{A}, i = 1, \ldots, n$, is a conserved vector of Eq. (4.1) if

$$\mathcal{D}_i(T^i) = 0, \tag{4.8}$$

on the solution space of (4.1). The expression (4.8) is a local conservation law of Eq. (4.1) and $T^i \in \mathcal{A}$ are called the fluxes of the conservation law.

Definition 4.1. A local conservation law (4.8) of the PDE (4.1) is trivial if its fluxes are of the form $T^i = M^i + H^i$, where M^i and H^i are functions of x, u and derivatives of u such that M^i vanishes on the solutions of the system (4.1), and $\mathcal{D}_i H^i = 0$ is identically divergence-free.

In particular, a trivial conservation law contains no information about a given PDE (4.1) and arises in two cases:

1. Each of its fluxes vanishes identically on the solutions of the given PDE.

2. The conservation law vanishes identically as a differential identity. In particular, this second type of trivial conservation law is simply an identity holding for arbitrary fluxes. These $T = (T^1, T^2, ..., T^n)$ are called null divergences.

The adjoint equation to the k^{th}-order differential Eq. (4.1) is defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(k)}, v_{(k)}) = 0, \tag{4.9}$$

where

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(k)}, v_{(k)}) = \frac{\delta(v^\beta F_\beta)}{\delta u}, \quad v = v(x),$$

and $v = (v^1, v^2, \dots, v^m)$ are new dependent variables.
We recall here the following results as given in Ibragimov's paper [20].

Definition 4.2. ([20]) Eq. (4.1) is said to be self-adjoint if the substitution of $v = u$ into adjoint Eq. (4.9) yields the same Eq. (4.1).

Definition 4.3. ([21]) Eq. (4.1) is said to be quasi self-adjoint if the equation obtained from the adjoint Eq. (4.9) by the substitution $v = h(u)$, with a certain function $h(u)$ such that $h'(u) \neq 0$, is identical to the original equation.

Definition 4.4. ([12]) Eq. (4.1) is said to be weakly self-adjoint if the equation obtained from the adjoint Eq. (4.9) by the substitution $v = h(t, x, u)$, with a certain function $h(t, x, u)$ such that $h_t(t, x, u) \neq 0$, (*or* $h_x(t, x, u) \neq 0$) and $h_u(t, x, u) \neq 0$ is identical to the original equation.

Definition 4.5. ([22]) Eq. (4.1) is said to be nonlinearly self-adjoint if the equation obtained from the adjoint Eq. (4.9) by the substitution $v = h(x, u, u_{(1)}, \dots)$, with a certain function $h(x, u, u_{(1)}, \dots)$ such that $h(x, u, u_{(1)}, \dots) \neq$ *constant* is identical to the original equation (4.1).

Main theorem which in this paper is used to construct the conservation laws is given as follows:

Theorem 4.6. ([20]) *Every Lie point, Lie Bäcklund, and non local symmetry admitted by the Eq. (4.1) gives rise to a conservation law for the system consisting of the Eq. (4.1) and the adjoint Eq. (4.9) where the components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by*

$$T^i = \xi^i \mathcal{L} + W \frac{\delta \mathcal{L}}{\delta u_i} + \sum_{s \geq 1} \mathcal{D}_{i_1} \dots \mathcal{D}_{i_s}(W) \frac{\delta \mathcal{L}}{\delta u_{i i_1 i_2 \dots i_s}}, \quad i = 1, \dots n, \tag{4.10}$$

with Lagrangian given by

$$\mathcal{L} = vF(x, u, \dots, u_{(k)}).$$

4.2. **Construction of conservation laws for Eq. (1.1).** Adjoint equation for Eq. (1.1) is as follows:

$$F^* = \frac{\delta(vF)}{\delta u} = \frac{\delta\big(v\big[u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Bu\big]\big)}{\delta u},$$

which by some simplifications we get

$$F^* = -2Bv - v_t - Bxv_x + A^2 v + 2A^2 x v_x + \frac{1}{2}A^2 x^2 v_{xx} = 0. \tag{4.11}$$

By setting $t = x^1$ and $x = x^2$, the conservation law will be written

$$\mathcal{D}_t(T_i^t) + \mathcal{D}_x(T_i^x) = 0, \quad i = 1, \ldots, 7.$$

Now, we discuss about self-adjointness of Eq. (1.1) by the following theorem.

Theorem 4.7. *Eq .(1.1) is neither quasi self-adjoint nor weakly self-adjoint, however Eq. (1.1) is nonlinearly self-adjoint for*

$$h(t,x,u) = e^{c_1 t}\left(c_2 x^{\frac{2B-3A^2+\chi}{2A^2}} + c_3 x^{\frac{2B-3A^2-\chi}{2A^2}}\right), \tag{4.12}$$

where $\chi = \sqrt{4B^2 + 4BA^2 + A^4 + 8A^2 c_1}$.

Proof. By few computations we can show that Eq. (1.1) is neither quasi self-adjoint nor weakly self-adjoint. To demonstrate the nonlinear self-adjointness, setting $v = h(t,x,u)$ in Eq. (4.11) we get

$$\begin{aligned}&-2Bh + A^2 h - Bx(h_x + h_u u_x) + 2A^2 x\,(h_x + h_u u_x) - h_t - h_u u_t \\ &+\frac{1}{2}A^2 x^2 (h_{xx} + 2h_{xu}u_x + h_{uu}u_x^2 + h_u u_{xx}) = 0,\end{aligned}$$

which yields:

$$\begin{aligned}&F^* - \lambda\left(u_t + \frac{1}{2}A^2 x^2 u_{xx} + Bxu_x - Bu\right) = -\lambda u_t - \lambda Bxu_x + \lambda Bu - 2hB - h_t \\ &+A^2 h - h_u u_t + \frac{1}{2}A^2 x^2 (h_{xx} + h_{uu}u_x^2 + h_u u_{xx} - \lambda u_{xx} + 2h_{xu}u_x) - Bxh_x \\ &-Bxh_u u_x + 2A^2 x(h_x + h_u u_x) = 0.\end{aligned}$$

Comparing the coefficients for the different derivatives of u we obtain some conditions which one of them is $\lambda + h_u = 0$. Thus by setting $\lambda = -h_u$ in (4.13) we get

$$\begin{aligned}&A^2 x^2 (h_u u_{xx} + h_{xu}u_x) - Bh_u u - 2Bh - h_t - Bxh_x + A^2 h + 2A^2 x(h_x + h_u u_x) \\ &+\frac{1}{2}A^2 x^2 (h_{xx} + h_{uu}u_x^2) = 0.\end{aligned}$$

As previous, comparing the coefficients for the different derivatives of u, we have the following condition:

$$\frac{A^2 x^2}{2} h_{xx} + (2A^2 - B)xh_x + (A^2 - 2B)h - h_t = 0, \tag{4.13}$$

which solving this system completes the Proof. □

Here infinite dimensional Lie algebras of Eq. (1.1) presented in [24] are used to construct the infinite number of conservation laws.

Eq. (1.1) admits six-dimensional Lie algebras, thus we consider the following three cases:

(i) We first consider the Lie point symmetry generator $X_1 = \frac{\partial}{\partial t}$. By using (4.10), the components of the conserved vector are given by

$$\begin{aligned} T_1^t = & \quad \frac{1}{2}A^2x^2vu_{xx} + Bxvu_x - Bvu, \\ T_1^x = & \quad -Bxvu_t + A^2xvu_t + \frac{1}{2}A^2x^2(u_tv_x - vu_{tx}). \end{aligned}$$

By setting $c_1 = c_3 = 0$ and $c_2 = 1$ in Theorem 2, we have

$$\begin{aligned} T_1^t|_{v=\frac{1}{x^2}} = & \quad \frac{A^2}{2}u_{xx} + \frac{B}{x}u_x - \frac{B}{x^2}u = \mathcal{D}_x\left(\frac{A^2}{2}u_x + \frac{B}{x}u\right), \\ T_1^x|_{v=\frac{1}{x^2}} = & \quad -\frac{A^2}{2}u_{tx} - \frac{B}{x}u_t = -\mathcal{D}_t\left(\frac{A^2}{2}u_x + \frac{B}{x}u\right). \end{aligned}$$

Then transferring the terms $\mathcal{D}_x(\cdots)$ from T_1^t to T_1^x, provides the null divergence $T_1 = (T_1^t, T_1^x) = (0, 0)$.

(ii) Using Lie point symmetry generator $X_2 = x\frac{\partial}{\partial x}$ and (4.10), the components of the conserved vector are given by

$$\begin{aligned} T_2^t = & \quad -xvu_x, \\ T_2^x = & \quad xvu_t - Bxvu + \frac{1}{2}A^2x^2(vu_x + xu_xv_x). \end{aligned}$$

Setting $v = h(t, x, u) = \frac{1}{x^2}$ into T_2^t, T_2^x and after reckoning, we have

$$\begin{aligned} T_2^t|_{v=\frac{1}{x^2}} = & \quad -\frac{u}{x^2} + \mathcal{D}_x\left(-\frac{u}{x}\right), \\ T_2^x|_{v=\frac{1}{x^2}} = & \quad -\frac{B}{x}u - \frac{A^2}{2}u_x - \mathcal{D}_t\left(-\frac{u}{x}\right). \end{aligned}$$

Therefore $T_2 = (T_2^t, T_2^x) = (-\frac{u}{x^2}, -\frac{B}{x}u - \frac{A^2}{2}u_x)$.

(iii) Using Lie point symmetry generator $X_3 = u\frac{\partial}{\partial u}$ and (4.10), one can obtain the conserved vector whose components are

$$\begin{aligned} T_3^t = & \quad uv, \\ T_3^x = & \quad \frac{1}{2}A^2x^2(vu_x - uv_x) + Bxvu - A^2xuv. \end{aligned}$$

Setting $v=\frac{1}{x^2}$ concludes the previous conserved vectors, however $c_1=c_2=0$ and $c_3=1$ in (4.12) yields

$$
\begin{aligned}
T_3^t\big|_{v=x^{\left(\frac{2B}{A^2}-1\right)}} &= x^{\left(\frac{2B}{A^2}-1\right)}u,\\
T_3^x\big|_{v=x^{\left(\frac{2B}{A^2}-1\right)}} &= \frac{A^2}{2}\left(x^{\left(\frac{2B}{A^2}+1\right)}u_x - x^{\left(\frac{2B}{A^2}\right)}u\right).
\end{aligned}
$$

(iv) Using Lie point symmetry generator $X_4=2tx\frac{\partial}{\partial x}+\left(tu-\frac{2Btu}{A^2}+\frac{2u\ln(x)}{A^2}\right)\frac{\partial}{\partial u}$ and (4.10), one can obtain the conserved vector whose components are

$$
\begin{aligned}
T_4^t &= \left(A^2tu-2Btu+2u\ln(x)-2A^2txu_x\right)v,\\
T_4^x &= -\frac{1}{2}x\Big(-4A^2tvu_t+2A^2Btxvu_x-2A^2Btvu-2A^2vu-2BA^2xtuv_x\\
&\quad +A^4t(2uv+xuv_x-2x^2u_xv_x-3xu_xv)+4B^2tuv\\
&\quad +\ln(x)(4A^2uv-4Buv+2A^2xuv_x-2A^2xvu_x)\Big).
\end{aligned}
$$

Substituting $v=\frac{1}{x^2}$ into the components above, we obtain

$$
\begin{aligned}
T_4^t|_{v=\frac{1}{x^2}} &= \left(\frac{2\ln(x)-A^2t-2Bt}{x^2}\right)u+\mathcal{D}_x\left(\frac{-2A^2tu}{x}\right),\\
T_4^x|_{v=\frac{1}{x^2}} &= \frac{-2A^2Btxu_x-2A^2Btu-4B^2tu+4Bu\ln(x)-A^4txu_x}{2x}\\
&\quad +\frac{2A^2x\ln(x)u_x-2A^2u}{2x}-\mathcal{D}_t\left(\frac{-2A^2tu}{x}\right).
\end{aligned}
$$

Then transferring the terms $\mathcal{D}_x(\cdots)$ from T_4^t to T_4^x, provides

$$
\begin{aligned}
T_4^t|_{v=\frac{1}{x^2}} &= \left(\frac{2\ln(x)-A^2t-2Bt}{x^2}\right)u,\\
T_4^x|_{v=\frac{1}{x^2}} &= \frac{-2A^2Btxu_x-2A^2Btu-4B^2tu+4Bu\ln(x)-A^4txu_x}{2x}\\
&\quad +\frac{2A^2x\ln(x)u_x-2A^2u}{2x}.
\end{aligned}
\tag{4.14}
$$

(v) Using Lie point symmetry generator

$$X_5=8t\frac{\partial}{\partial t}+4x\ln(x)\frac{\partial}{\partial x}+\left(A^2tu+4Btu+\frac{4B^2tu}{A^2}+2u\ln(x)-\frac{4Bu\ln(x)}{A^2}\right)\frac{\partial}{\partial u}$$

and (4.10), one can obtain the conserved vector whose components are

$$T_5^t = v\Big(4A^4tx^2u_{xx} + 8A^2Btxu_x - 4A^2Btu + A^4tu + 4B^2tu + \ln(x)(2A^2u - 4Bu - 4A^2xu_x)\Big),$$

$$\begin{aligned}T_5^x = -\frac{1}{2}x\Big(&6A^4Btuv + A^6tx(uv_x - vu_x) + 16A^2Btvu_t + 4B^2A^2txuv_x + 2A^6tuv\\ &+A^4tx(8vu_{tx} - 8u_tv_x - 4Bvu_x + 4Buv_x)\\ &+A^4\ln(x)(2xuv_x - 6xvu_x - 4x^2u_xv_x + 4uv)\\ &-4A^2Bvu\ln(x) - 4A^2B^2txvu_x + 4A^2Bx\ln(x)(vu_x - uv_x)\\ &-8A^2\ln(x)vu_t - 16A^4tu_tv\\ &-8B^3tuv + 4A^2Bvu + 4A^4xvu_x + 8B^2u\ln(x)v - 2A^4vu\Big),\end{aligned}$$

which by setting $v = \frac{1}{x^2}$ we get

$$\begin{aligned}T_5^t|_{v=\frac{1}{x^2}} =& \frac{4A^2Bt + 4A^2 - 2A^2\ln(x) + A^4t + 4B^2t - 4B\ln(x)}{x^2}u\\ &+\mathcal{D}_x\left(4A^4tu_x + (8A^2Bt - 4A^2\ln(x))\frac{u}{x}\right),\\ T_5^x|_{v=\frac{1}{x^2}} =& \frac{1}{2x}\Big(12A^2Bu + 4A^4xu_x + A^6txu_x + 2A^4Btu - 4A^2B\ln(x)u\\ &-2A^4x\ln(x)u_x + 8A^2B^2tu + 4A^4Btxu_x + 4A^2B^2txu_x - 4A^2Bx\ln(x)u_x\\ &+2A^4u + 8B^3tu - 8B^2\ln(x)u\Big) - \mathcal{D}_t\left(4A^4tu_x + (8A^2Bt - 4A^2\ln(x))\frac{u}{x}\right).\end{aligned}$$

Therefore

$$\begin{aligned}T_5^t|_{v=\frac{1}{x^2}} \quad&= \frac{4A^2Bt + 4A^2 - 2A^2\ln(x) + A^4t + 4B^2t - 4B\ln(x)}{x^2}u,\\ T_5^x|_{v=\frac{1}{x^2}} \quad&= \frac{1}{2x}\Big(12A^2Bu + 4A^4xu_x + A^6txu_x + 2A^4Btu - 4A^2B\ln(x)u\\ &-2A^4x\ln(x)u_x + 8A^2B^2tu + 4A^4Btxu_x + 4A^2B^2txu_x\\ &-4A^2Bx\ln(x)u_x + 2A^4u + 8B^3tu - 8B^2\ln(x)u\Big).\end{aligned}$$

(vi) Using Lie point symmetry generator

$$\begin{aligned}X_6 = \quad& 8t^2\frac{\partial}{\partial t} + 8tx\ln(x)\frac{\partial}{\partial x} + \Big(A^2t^2u - 4tu + 4Bt^2u + 4tu\ln(x)\\ + \quad& \frac{4B^2t^2u}{A^2} - \frac{8Btu\ln(x)}{A^2} + \frac{4u\ln^2(x)}{A^2}\Big)\frac{\partial}{\partial u}\end{aligned}$$

and (4.10), one can obtain the conserved vector whose components are

$$T_6^t = v\Big(4A^4t^2x^2u_{xx} + 8A^2Bt^2xu_x - 4A^2Bt^2u + A^4t^2u - 4A^2tu + 4B^2t^2u$$
$$+ \ln(x)(4A^2tu - 8Btu - 8A^2txu_x) + 4u\ln(x)^2\Big),$$
$$T_6^x = -\frac{1}{2}x\Big(16A^2Btvu - 4A^4txuv_x + 12A^4txu_xv + 8A^4tu\ln(x)v - 16A^2t\ln(x)vu_t$$
$$+A^4t^2(6Buv - 8xu_tv_x + 8xvu_{tx} - 16u_tv - 4Bxvu_x) + 16A^2Bt^2u_tv$$
$$-4A^2B^2xt^2vu_x + \ln(x)^2(4A^2xuv_x + 8A^2uv - 4A^2xvu_x - 8Buv) + A^6t^2xuv_x$$
$$+16B^2tu\ln(x)v + \ln(x)(-8A^2Btvu + 4A^4txuv_x - 12A^4txu_xv)$$
$$-8A^4tx^2\ln(x)u_xv_x + 4A^4Bxt^2uv_x + 4A^2B^2t^2xuv_x + \ln(x)(8A^2Bxtvu_x$$
$$-8A^2Bxtuv_x - 8A^2uv) + 2A^6t^2uv + 12A^4tuv - 8B^3t^2uv - A^6t^2xvu_x\Big).$$

Setting $v = \frac{1}{x^2}$ into the components of T_6^t and T_6^x we get

$$T_6^t|_{v=\frac{1}{x^2}} = \frac{4A^2Bt^2 + A^4t^2 + 4A^2t + 4B^2t^2 - 4A^2t\ln(x) - 8Bt\ln(x) + 4\ln^2(x)}{x^2}u$$
$$+\mathcal{D}_x\left(4A^4t^2u_x + (8A^2Bt^2 - 8A^2t\ln(x))\frac{u}{x}\right),$$
$$T_6^x|_{v=\frac{1}{x^2}} = \frac{1}{2x}\Big(-8A^2Btx\ln(x)u_x + A^6t^2xu_x + 16A^2Btu + 4A^4txu_x + 2A^4Bt^2u$$
$$-16B^2t\ln(x)u + 4A^2x\ln^2(x)u_x + 8A^2B^2t^2u + 4A^4Bt^2xu_x + 4A^2B^2t^2xu_x$$
$$-8A^2Bt\ln(x)u - 4A^4tx\ln(x)u_x + 4A^4tu - 8A^2\ln(x)u + 8B^3t^2u + 8B\ln^2(x)u\Big)$$
$$-\mathcal{D}_t\left(4A^4t^2u_x + (8A^2Bt^2 - 8A^2t\ln(x))\frac{u}{x}\right),$$

thus

$$T_6^t|_{v=\frac{1}{x^2}} = \frac{4A^2Bt^2 + A^4t^2 + 4A^2t + 4B^2t^2 - 4A^2t\ln(x) - 8Bt\ln(x) + 4\ln^2(x)}{x^2}u,$$
$$T_6^x|_{v=\frac{1}{x^2}} = \frac{1}{2x}\Big(-8A^2Btx\ln(x)u_x + A^6t^2xu_x + 16A^2Btu + 4A^4txu_x + 2A^4Bt^2u$$
$$-16B^2t\ln(x)u + 4A^2x\ln^2(x)u_x + 8A^2B^2t^2u + 4A^4Bt^2xu_x + 4A^2B^2t^2xu_x$$
$$-8A^2Bt\ln(x)u - 4A^4tx\ln(x)u_x + 4A^4tu - 8A^2\ln(x)u + 8B^3t^2u + 8B\ln^2(x)u\Big).$$

(vii) Using Lie point symmetry generator $X_7 = \varphi(t,x)\frac{\partial}{\partial u}$ where $\varphi(t,x)$ satisfies following equation:

$$2\varphi_t - 2B\varphi + 2Bx\varphi_x + A^2x^2\varphi_{xx} = 0$$

and (4.10), one can obtain the conserved vector whose components are

$$T_7^t = v\varphi,$$
$$T_7^x = \frac{1}{2}x\bigg((2Bv - 2A^2v - A^2xv_x)\varphi + A^2xv\varphi_x\bigg).$$

Substituting $v = \frac{1}{x^2}$ into the components above, we obtain

$$T_7^t|_{v=\frac{1}{x^2}} = \frac{\varphi}{x^2},$$
$$T_7^x|_{v=\frac{1}{x^2}} = \frac{2B\varphi + A^2x\varphi_x}{2x}.$$

Since

$$\mathcal{D}_t\left(\frac{\varphi}{x^2}\right) + \mathcal{D}_x\left(\frac{2B\varphi + A^2x\varphi_x}{2x}\right) = 0, \tag{4.15}$$

it follows that the vector $T_7 = (T_7^t, T_7^x)$ is a local conserved current for equation (1.1).

5. **Final remarks**

The application of the nonclassical symmetry method to equation (1.1) yields some of it exact solutions by the application of the Heir-equations. We investigated the nonlinear self-adjointness of (1.1) and we found the nontrivial conservation laws, using the Ibragimov's conservation theorem.As mentioned before, Edelstein and Govinder in [7] have found the conservation laws of Eq. (1.1), by a method which is based upon the point symmetries. Their applied method, directly calculate the conservation laws using the symmetries whereas in this paper the adjoint equation of Eq. (1.1) was firstly obtained and then its Lagrangian before determining the conservation laws has been determined. Six conserved vectors for Eq. (1.1) have been reported in [7], whereas in this paper we present infinite number of conservation laws which are different. However, since two approaches are complementary, it is not possible to compare them.

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