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# ON THE ORDER OF A MODULE

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ABSTRACT. Let $(R, P)$ be a Noetherian unique factorization domain (UFD) and $M$ be a finitely generated $R$-module. Let $I(M)$ be the first nonzero Fitting ideal of $M$ and the order of $M$, denoted $ord_R(M)$, be the largest integer $n$ such that $I(M) \subseteq P^n$. In this paper, we show that if $M$ is a module of order one, then either $M$ is isomorphic with direct sum of a free module and a cyclic module or $M$ is isomorphic with a special module represented in the text. We also assert some properties of $M$ while $ord_R(M) = 2$.

**Keywords:** Fitting ideals, minimal free presentation, order of a module.
**MSC(2010):** Primary: 13C05; Secondary: 13D05, 11Y50.

## 1. Introduction

Throughout this paper $R$ denotes a Noetherian commutative ring with identity and all modules are unital. Let $M$ be a finitely generated $R$-module. For a set $\{x_1, \ldots, x_n\}$ of generators of M there exists a complex $R^s \xrightarrow{\varphi} R^r \xrightarrow{\psi} M \longrightarrow 0$ , where $R^r$ and $R^s$ are free $R$-modules and the set $\{e_1, \ldots, e_r\}$ is a basis for $R^r$ and the $R$-homomorphism $\psi$ is defined by $\psi(e_j) = x_j$. The complex $R^s \xrightarrow{\varphi} R^r \xrightarrow{\psi} M \longrightarrow 0$ is called a free presentation of $M$. Let the kernel of $\psi$ be generated by $u_i = a_{1i}e_1 + \ldots + a_{ri}e_r$, $1 \leq i \leq s$ and $A = (a_{ij}) \in M_{r\times s}(R)$ be the matrix presentation of $\varphi$ and $I_j(\varphi)$ be the ideal of $R$ generated by the minors of size $j$ of matrix $A$. By convention, the determinant of the $0 \times 0$ matrix is 1. In general, we set $I_j(\varphi) = R$ if $j \leq 0$.

By Fitting's Lemma [3, Corollary 20.4], the ideals $I_{r-j}(\varphi), 0 \leq j < \infty$, are independent of the choice of free presentation of $M$. So, we define the $j$th Fitting ideal of $M$ to be the ideal $\mathrm{Fitt}_j(M) = I_{r-j}(\varphi)$. The most important Fitting ideal of $M$ is the first of the $\mathrm{Fitt}_j(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$. Thus, $I(M) = I_{rank\varphi}(\varphi)$. Hence, we have $R =$

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$I_0(\varphi) \supseteq I_1(\varphi) \supseteq \cdots \supseteq I_{rank\varphi}(\varphi) \supsetneq 0$. Let $P$ be a prime ideal of $R$ and $M_P$ and $\varphi_P$ be the localization of $M$ and $\varphi$ in $P$, respectively. Note that if $I(M)$ contains a nonzerodivisor, then $rank(\varphi) = rank(\varphi_P)$ and so $I(M_P) = I(M)_P$. Let $I$ be an ideal of a Noetherian local ring $(R, P)$, by definition, $ord_R(I)$ is the largest integer $n$ such that $I \subseteq P^n$. We define the order of $M$, denoted $ord_R(M)$, to be $ord_R(I(M))$ [5].

## 2. **Module of order one**

A complex $\mathcal{F}\colon \ldots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \ldots$ of free modules $F_i$, over a local ring $(R, P)$ is called minimal if the maps in the complex $\mathcal{F} \otimes R/P$ are all 0. This simply means that any matrix representing $\varphi_n$ has all its entries in $P$. By [3, Theorem 20.2], there is, up to isomorphism, only one minimal free resolution of $M$.

**Theorem 2.1.** *Let $(R, P)$ be a Noetherian local ring and $I$ be an ideal of $R$. Then, $ord_R(I) = 1$ if and only if, for every finitely generated $R$-module $M$, $I \subseteq Fitt_0(M)$ implies that $M$ is cyclic.*

*Proof.* Let $F \xrightarrow{\varphi} G \longrightarrow M \longrightarrow 0$ be a free presentation of $M$ and $(a_{ij}) \in M_{m\times n}(R)$ be a matrix presentation of $\varphi$. By [3, Corollary 20.4], we can assume that $F \xrightarrow{\varphi} G \longrightarrow M \longrightarrow 0$ is a minimal free presentation of $M$. Thus, $a_{ij} \in P$, for all $i, j$. Let $ord_R(I) = 1$ and $M$ be a finitely generated $R$-module such that $I \subseteq \mathrm{Fitt}_0(M)$. Therefore, $ord_R(\mathrm{Fitt}_0(M)) = 1$. Let $M$ be generated by $r$ elements. So, $\mathrm{Fitt}_0(M) = I_r(M) \subseteq P^r$. Hence, $r = 1$ and $M$ is cyclic. Conversely, assume that $I \subseteq \mathrm{Fitt}_0(M)$ implies that $M$ is cyclic, for every finitely generated $R$-module $M$. If $I \subseteq P^2$, put $M = R/P \oplus R/P$. Then, $I \subseteq \mathrm{Fitt}_0(M) = P^2$ and $M$ is not cyclic, a contradiction. □

**Proposition 2.2.** *Let $(R, P)$ be a Noetherian local ring and let $M$ be a finitely generated $R$-module with $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \longrightarrow 0$ as a minimal free presentation of $M$. If $ord_R(M) = n$, then $rank(\varphi) \le n$.*

*Proof.* If $rank(\varphi) \ge n + 1$, then $I(M) \subseteq P^{n+1}$, a contradiction. □

Note that in a unique factorization domain (UFD), a greatest common divisor (GCD) of any collection of elements always exists. Also, for every $a, b, c$ in a UFD, if $a \mid bc$ and $a, b$ are relatively prime, then $a \mid c$.

**Theorem 2.3.** *Let $(R, P)$ be a Noetherian local UFD and let $M$ be a finitely generated $R$-module. If $ord_R(M) = 1$ then*
*(i) $M$ is isomorphic to $R^n / < (a_1, \ldots, a_n)^t >$, where $I(M) =< a_1, \ldots, a_n >$ and $n$ is a positive integer if $M$ is torsionfree, and*
*(ii) $M$ is isomorphic to $R^n \oplus R/I(M)$, for some positive integer $n$ if $M$ is not torsionfree.*

*Proof.* Let $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \longrightarrow 0$ be a free presentation of $M$ and $(a_{ij}) \in M_{m\times n}(R)$ be a matrix presentation of $\varphi$. By [3, Corollary 20.4], we can assume that $F \xrightarrow{\varphi} G \xrightarrow{\psi} M \longrightarrow 0$ is a minimal free presentation of $M$. Thus, $a_{ij} \in P$, for all $i, j$. Without loss of generality, we may assume that $a_{i1} \neq 0$, $1 \le i \le t$ and $a_{(t+1)1} = \ldots = a_{m1} = 0$. Put $d_i = GCD(a_{i1}, a_{(i+1)1})$, $1 \le i \le t$ and, for the moment, fix $j$, $2 \le j \le n$. Since $rank(\varphi) = 1$, for $i = 1, \ldots, t$, then we have $a_{i1}a_{(i+1)j} = a_{ij}a_{(i+1)1}$. Thus, $\frac{a_{i1}}{d_i}a_{(i+1)j} = a_{ij}\frac{a_{(i+1)1}}{d_i}$ and so $\frac{a_{i1}}{d_i} \mid a_{ij}$ which implies that there exists $r_{ij} \in R$ such that $a_{ij} = \frac{a_{i1}}{d_i}r_{ij}$ and so $a_{(i+1)j} = \frac{a_{(i+1)1}}{d_i}r_{ij}, 1 \le i \le t$. Therefore, $a_{ij} = \frac{a_{i1}}{d_i}r_{ij} = \frac{a_{i1}}{d_{i-1}}r_{(i-1)j}$, $2 \le i \le t$ and $a_{1j} = \frac{a_{11}}{d_1}r_{1j}$. Hence, $r_{ij}d_{i-1} = r_{(i-1)j}d_i$. Now, by induction on $i$, we show that $\frac{d_i}{GCD(d_1,\ldots,d_i)} \mid r_{ij}$, $1 \le i \le t$. For $i = 2$, since $r_{2j}d_1 = r_{1j}d_2$, then $\frac{d_2}{GCD(d_1,d_2)} \mid r_{2j}$. Assume that $d_is_i = r_{ij}GCD(d_1,\ldots,d_i)$, for some $s_i \in R$. We have $r_{(i+1)j}d_is_i = r_{ij}d_{i+1}s_i$. Thus,

$$r_{(i+1)j}GCD(d_1,\ldots,d_i) = d_{i+1}s_i. \tag{2.1}$$

On the other hand, from $r_{(i+1)j}d_i = r_{ij}d_{i+1}$ we obtain $\frac{d_{i+1}}{GCD(d_i,d_{i+1})} \mid r_{(i+1)j}$ and so there exists $s'_i \in R$ such that

$$r_{(i+1)j}GCD(d_i,d_{i+1}) = d_{i+1}s'_i. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$\begin{aligned} d_{i+1}s_iGCD(d_i,d_{i+1}) &= r_{(i+1)j}GCD(d_1,\ldots,d_i)GCD(d_i,d_{i+1}) \\ &= d_{i+1}s'_iGCD(d_1,\ldots,d_i). \end{aligned}$$

Thus, $s_iGCD(d_i,d_{i+1}) = s'_iGCD(d_1,\ldots,d_i)$ and so $\frac{GCD(d_i,d_{i+1})}{GCD(d_1,\ldots,d_{i+1})} \mid s'_i$. Now, by (2.2), we have $\frac{d_{i+1}GCD(d_i,d_{i+1})}{GCD(d_1,\ldots,d_{i+1})} \mid r_{(i+1)j}GCD(d_i,d_{i+1})$ and hence $\frac{d_{i+1}}{GCD(d_1,\ldots,d_{i+1})} \mid r_{(i+1)j}$ which completes the induction. If $i = t$, then

$$\frac{d_t}{GCD(d_1,\ldots,d_t)} \mid r_{tj}. \tag{2.3}$$

Now, we consider two cases.
Case 1: Suppose that $GCD(a_{11},\ldots,a_{m1}) = 1$. By (2.3), $d_t \mid r_{tj}$. Since $r_{tj}d_{t-1} = r_{(t-1)j}d_t$, then $d_{t-1} \mid r_{(t-1)j}$. Continuing this process, we have $d_i \mid r_{ij}$, $1 \le i \le t$. As a consequence, $a_{ij} = \frac{a_{i1}}{d_i}r_{ij} = a_{i1}\frac{r_{ij}}{d_i}$, $1 \le i \le t$,

$2 \leq j \leq n$. So, $I(M) =< a_{11}, \ldots, a_{m1} >$. It is easily seen that $ker\psi = Im\varphi =< (a_{11}, \ldots, a_{m1})^t >$. This means there exists $0 \longrightarrow R \longrightarrow G \longrightarrow M \longrightarrow 0$, a free resolution of $M$. Therefore, $M$ is isomorphic to $R^m / < (a_{11}, \ldots, a_{m1})^t >$ and $pd_R(M) = 1$.

Case 2: Suppose that $GCD(a_{11}, \ldots, a_{m1}) = x_0 \in P$ and, for the moment, fix $j$, $2 \leq j \leq n$. By (2.3), $\frac{d_t}{GCD(d_1, \ldots, d_t)} \mid r_{tj}$ which implies that $\frac{d_t}{x_0} \mid r_{tj}$. Therefore, there exists $r'_{tj} \in R$ such that $r_{tj} = \frac{d_t}{x_0} r'_{tj}$. Thus, $a_{tj} = \frac{a_{t1}}{x_0} r'_{tj}$. On the other hand, $r_{(t-1)j} d_t = r_{tj} d_{t-1}$. Hence, $r_{(t-1)j} d_t = \frac{d_t}{x_0} r'_{tj} d_{t-1}$ and so $r_{(t-1)j} = r'_{tj} \frac{d_{t-1}}{x_0}$. Therefore, $a_{(t-1)j} = \frac{a_{(t-1)1}}{d_{(t-1)j}} r_{(t-1)j} = \frac{a_{(t-1)1}}{x_0} r'_{tj}$. Continuing this process we obtain $r_{ij} = r'_{tj} \frac{d_i}{x_0}$ and so $a_{ij} = \frac{a_{i1}}{x_0} r'_{tj}$, $1 \leq i \leq t, 2 \leq j \leq n$. If there exists some $r'_{tj}, 2 \leq j \leq n$, such that $r'_{tj} \notin P$, then $I(M) =< \frac{a_{11}}{x_0}, \ldots, \frac{a_{m1}}{x_0} >$ and as in Case 1, $ker\psi = Im\varphi =< (\frac{a_{11}}{x_0}, \ldots, \frac{a_{m1}}{x_0})^t >$ which implies that $M$ is isomorphic to $R^m / < (\frac{a_{11}}{x_0}, \ldots, \frac{a_{m1}}{x_0})^t >$ and $pd_R(M) = 1$. Now, suppose $r'_{tj} \in P$, $2 \leq j \leq n$. Since $a_{ij} = \frac{a_{i1}}{x_0} r'_{tj}$, then $I =< a_{ij} : 1 \leq i \leq m, 1 \leq j \leq n >=< x_o, r'_{tj} : 2 \leq j \leq n >$. If for all $i$, $1 \leq i \leq t$, $\frac{a_{i1}}{x_0} \in P$, then $I(M) \subseteq P^2$ a contradiction. Without loss of generality, suppose $\frac{a_{11}}{x_0} \notin P$. Put $d_{i_1 \ldots i_j} := GCD(a_{i_1 1}, \ldots, a_{i_j 1})$. Define $\theta : R^m \longrightarrow R^{m-1} \oplus R/I(M)$ by $\theta(x_1, \ldots, x_m)^t = (\frac{a_{21}}{x_0} x_1 - \frac{a_{11}}{x_0} x_2, \ldots, \frac{a_{m1}}{x_0} x_1 - \frac{a_{11}}{x_0} x_m, x_1 + I(M))^t$. Let $(x_1, \ldots, x_m)^t \in ker\theta$, so that $\frac{a_{i1}}{x_o} x_1 = \frac{a_{11}}{x_0} x_i$, $2 \leq i \leq m$ and $x_1 \in I(M)$. Therefore, $\frac{a_{i1}}{d_{1i}} \mid x_i$, which implies that there exists $s_{1i} \in R$ such that $x_1 = \frac{a_{11}}{d_{1i}} s_{1i}$ and $x_i = \frac{a_{i1}}{d_{1i}} s_{1i}, 2 \leq i \leq m$. Therefore, $\frac{a_{11}}{d_{12}} s_{12} = \frac{a_{11}}{d_{13}} s_{13}$. ¡Thus¿, $s_{12} \frac{d_{13}}{d_{123}} = s_{13} \frac{d_{12}}{d_{123}}$. Hence, there exists $s_{123} \in R$ such that $s_{12} = \frac{d_{12}}{d_{123}} s_{123}$ and $s_{13} = \frac{d_{13}}{d_{123}} s_{123}$. Again, by induction on $i$, we show that $s_{1i} = \frac{d_{1i}}{d_{12\ldots i}} s_{12\ldots i}$ and $s_{12\ldots i} = \frac{d_{12\ldots i}}{d_{12\ldots (i+1)}} s_{12\ldots (i+1)}$, for some $s_{12\ldots i} \in R, 2 \leq i \leq t$. Let $s_{12\ldots i} = \frac{d_{12\ldots i}}{d_{12\ldots (i+1)}} s_{12\ldots (i+1)}$ and $s_{1i} = \frac{d_{1i}}{d_{12\ldots i}} s_{12\ldots i}$. We have $\frac{a_{11}}{d_{1i}} s_{1i} = \frac{a_{11}}{d_{1(i+1)}} s_{1(i+1)}$.

So, $s_{1i}d_{1(i+1)} = s_{1(i+1)}d_{1i}$. Therefore, $\frac{d_{1i}}{d_{12\ldots i}}d_{1(i+1)}s_{12\ldots i} = s_{1(i+1)}d_{1i}$. Hence, $s_{12\ldots i}\frac{d_{1(i+1)}}{d_{12\ldots(i+1)}} = s_{1(i+1)}\frac{d_{12\ldots i}}{d_{12\ldots(i+1)}}$. So, there exists $s_{12\ldots(i+1)} \in R$ such that $s_{1(i+1)} = \frac{d_{1(i+1)}}{d_{12\ldots(i+1)}}s_{12\ldots(i+1)}$ and $s_{12\ldots i} = \frac{d_{12\ldots i}}{d_{12\ldots(i+1)}}s_{12\ldots(i+1)}$. This completes the induction. Hence, $s_{1t} = \frac{d_{1t}}{d_{12\ldots t}}s_{12\ldots t}$. Put $s = s_{12\ldots t}$. Therefore, $s_{1t} = \frac{d_{1t}}{x_0}s$. So, $x_1 = \frac{a_{11}}{d_{1t}}s_{1t} = \frac{a_{11}}{x_0}s$. Also, $x_i = \frac{a_{i1}}{d_{1i}}s_{1i} = \frac{a_{i1}}{d_{12\ldots i}}s_{12\ldots i} = \frac{a_{i1}}{d_{12\ldots(i+1)}}s_{12\ldots(i+1)} = \ldots = \frac{a_{i1}}{d_{12\ldots t}}s_{12\ldots t} = \frac{a_{i1}}{x_0}s$, $2 \leq i \leq t$. On the other hand, $\frac{a_{11}}{x_0}s = x_1 \in I(M)$ and $\frac{a_{11}}{x_0} \notin P$. Therefore, $s \in I(M)$. Since $I(M) =< x_o, r'_{tj} : 2 \leq j \leq n >$, then there exists $r_i \in R$, $0 \leq i \leq n$, $i \neq 1$, such that $s = r_0x_0 + r_2r'_{t2} + \cdots + r_nr'_{tn}$. So, $x_i = \frac{a_{i1}}{x_0}s = \frac{a_{i1}}{x_0}(r_0x_0 + r_2r'_{t2} + \cdots + r_nr'_{tn}) = r_0a_{i1} + r_2a_{i2} + \cdots + r_na_{in}$. So, $(x_1, \ldots, x_m)^t \in Im\varphi$. This means $ker\theta \subseteq Im\varphi$. It is clear that $Im\varphi \subseteq ker\theta$. Since $\frac{a_{11}}{x_0} \notin P$, it is easily seen that $\theta$ is an epimorphism. Therefore, $M$ is isomorphic to $R^{m-1} \oplus R/I(M)$ in this case. □

**Lemma 2.4.** *Let $R$ be a Noetherian ring and $P_1, ..., P_n$ be distinct maximal ideals of $R$. Suppose that $M$ be a finitely generated $R$-module such that $M_{P_i} \cong R_{P_i}/P_i^{t_i}R_{P_i}$, for some $t_i \in \mathbb{N}$, $1 \leq i \leq n$, and for every maximal ideal $Q \neq P_i$, $1 \leq i \leq n$, $M_Q = 0$. Then, $M \cong R/P_1^{t_1} \oplus ... \oplus R/P_n^{t_n}$.*

*Proof.* Put $A_i = \{\mathrm{ann}_R(y) : M_{P_i} = \langle y/1\rangle\}$, for $i = 1, ..., n$. Let $M_{P_i} = \langle x_i/1\rangle$ such that $\mathrm{ann}_R(x_i)$ is maximal in $A_i$. Assume that $i$, $1 \leq i \leq n$, be arbitrary and fixed. Let $r \in P_i^{t_i}$. Then, $\frac{r}{1}\frac{x_i}{1} = 0$. So, there exists $s \in R \setminus P_i$ such that $rsx_i = 0$. Since $M_{P_i} = \langle sx_i/1\rangle$ and $\mathrm{ann}_R(x_i)$ is maximal in $A_i$, then $r \in \mathrm{ann}_R(sx_i) = \mathrm{ann}_R(x_i)$. Now, let $r \in \mathrm{ann}_R(x_i)$. So, $\frac{r}{1} \in P_i^{t_i}R_{P_i}$. Since $P_i^{t_i}$ is $P_i$-primary, then $r \in P_i^{t_i}$. Hence, $\mathrm{ann}_R(x_i) = P_i^{t_i}$. Put $P = \cap_{i=1}^{n}P_i^{t_i}$ and define $f : R/P \longrightarrow M; f(r + P) = r(x_1 + ... + x_n)$. For every $j \neq i$, let $s_j \in P_j^{t_j} \setminus P_i$. Then, $\frac{x_j}{1} = \frac{s_jx_j}{s_j} = 0$ in $M_{P_i}$. On the other hand for every maximal ideal $Q \neq P_i$, $f_Q = 0$ is an isomorphism between two zero modules. Thus, $f_q$ is an isomorphism for every maximal ideal $q$ of $R$. Hence, $M \cong R/P \cong R/P_1^{t_1} \oplus ... \oplus R/P_n^{t_n}$. □

Recall that an $R$-module $M$ is projective of constant rank $n$ if and only if $M_P$ is free of rank $n$ over $R_P$, for every prime ideal $P$ of $R$.

**Proposition 2.5.** *Let $R$ be Noetherian UFD and $M$ be a finitely generated nontorsionfree $R$-module. Suppose that $P_1, ..., P_n$ be distinct maximal ideals of*

*$R$ such that $I(M) = P_1 ... P_n$. Then, $M \cong P \oplus R/P_1 \oplus ... \oplus R/P_n$ where $P$ is a projective $R$-module (of constant rank).*

*Proof.* Since $I(M_{P_i}) = P_i R_{P_i}$, then by Theorem 2.3, $M_{P_i} \cong R_{P_i}^{m_i} \oplus (R/P_i)_{P_i}$ for some positive integers $m_i$, $1 \leq i \leq n$. Let $Q \neq P_i$ be a maximal ideal of $R$. Then, $I(M_Q) = R_Q$. Hence, by [2, Lemma1], $M_Q$ is a free $R_Q$-module, for every maximal ideal $Q \neq P_i$, $1 \leq i \leq n$. Thus, for every maximal ideal $q$ of $R$, $(M/T(M))_q$ is free. Since $R$ is a domain, then by [1, Remark, p. 112], $(M/T(M))$ is projective of constant rank. Hence, $M \cong (M/T(M)) \oplus T(M)$. On the other hand $T(M)_{P_i} = T(M_{P_i}) \cong R_{P_i}/P_i R_{P_i}$ and $T(M)_Q = 0$, for every maximal ideal $Q \neq P_i, 1 \leq i \leq n$. Therefore, by Lemma 2.4, $T(M) \cong R/P_1 \oplus ... \oplus R/P_n$. Hence, $M \cong P \oplus R/P_1 \oplus ... \oplus R/P_n$, for some projective $R$-module $P$. □

**Lemma 2.6.** *Let $(R, P)$ be a regular local ring and $M$ be a finitely generated $R$-module. If $dim(R) \leq 2$ and $ord_R(ann_R(M)) = 1$, then $M \cong R/I_1 \oplus ... \oplus R/I_k$ for some ideals $I_i$, $1 \leq i \leq k$.*

*Proof.* If $dim(R) = 1$, then $R$ is a PID. Hence, $M \cong R/I_1 \oplus ... \oplus R/I_k$, for some ideals $I_i$, $1 \leq i \leq k$. Let $dim(R) = 2$ and $\mathrm{ann}_R(M) \nsubseteq P^2$. Then, there is an element $y \in \mathrm{ann}_R(M) \backslash P^2$. Put $\overline{R} = R/\langle y \rangle$. Then, $\overline{R}$ is a regular local ring of dimension 1. Hence, $M \cong \overline{R}/\overline{I_1} \oplus ... \oplus \overline{R}/\overline{I_k} = R/J_1 \oplus ... \oplus R/J_k$, for some ideals $I_i$ and $J_i$, $1 \leq i \leq k$. □

**Theorem 2.7.** *Let $(R, P)$ be a regular local ring and $M$ be a finitely generated $R$-module. If $dim(R) = n \geq 3$ and there exist $x_1, x_2, ..., x_{n-1} \in ann_R(M) \setminus P^2$ such that $ann_R(M) \nsubseteq P^2 + \langle x_1, ..., x_{n-2} \rangle$, then $M \cong R/I_1 \oplus ... \oplus R/I_k$, for some ideals $I_i$, $1 \leq i \leq k$.*

*Proof.* ¡The proof is¿ by induction on $n$. Let $n = 3$. Put $\overline{R} = R/\langle x_1 \rangle$. Then, $\overline{R}$ is a regular local ring of dimension 2. Since $\mathrm{ann}_R(M) \nsubseteq P^2 + \langle x_1 \rangle$, then $\mathrm{ann}_{\overline{R}}(M) \nsubseteq \overline{P^2}$ then by Lemma 2.6, $M \cong R/I_1 \oplus ... \oplus R/I_k$. Assume that $dim(R) = n$. Put $\overline{R} = R/\langle x_1 \rangle$. Then, $\overline{R}$ is a regular local ring of dimension $n-1$. Since $\mathrm{ann}_R(M) \nsubseteq P^2 + \langle x_1, ..., x_{n-2} \rangle$, then $\mathrm{ann}_{\overline{R}}(M) \nsubseteq \overline{P^2} + \overline{\langle x_2, ..., x_{n-2} \rangle}$. By induction hypothesis $M \cong \overline{R}/\overline{I_1} \oplus ... \oplus \overline{R}/\overline{I_k} = R/J_1 \oplus ... \oplus R/J_k$, for some ideals $J_i$, $1 \leq i \leq k$. □

## 3. Some othre cases

**Theorem 3.1.** *Let $(R, P)$ be a Noetherian local UFD and let $M$ be a finitely generated $R$-module. Assume that there exists a free presentation*

$R^n \xrightarrow{\varphi} R^m \xrightarrow{\psi} M \longrightarrow 0$ *of $M$ such that $rank(\varphi) = 1$. If $ord_R(M) = 2$, then*

*(i) If $M$ is torsionfree, then $M \cong R^m / < (a_1, \ldots, a_m)^t >$, for some $a_i \in R$, $1 \leq i \leq m$.*

*(ii) If $M$ is not torsionfree, then $M \cong R^{m-1} \oplus R/I(M)$ or $M \cong R^m/J < (a_1, \ldots, a_m)^t >$ , for some ideal $J \neq R$ and for some $a_i \in R$, $1 \leq i \leq m$.*

*Proof.* By [3, Corollary 20.4], we can assume that

$R^n \xrightarrow{\varphi} R^m \xrightarrow{\psi} M \longrightarrow 0$ is a minimal free presentation of $M$. Let $(a_{ij}) \in M_{m\times n}(R)$ be a matrix presentation of $\varphi$. Thus, $a_{ij} \in P$, for all $i, j$. Without loss of generality, we may assume that $a_{i1} \neq 0$, $1 \leq i \leq t$ and $a_{(t+1)1} = \ldots = a_{m1} = 0$. Put $d_i = GCD(a_{i1}, a_{(i+1)1})$, $1 \leq i \leq t$ and, for the moment, fix $j$, $2 \leq j \leq n$. Since $rank(\varphi) = 1$, then for $i = 1, \ldots, t$, we have $a_{i1}a_{(i+1)j} = a_{ij}a_{(i+1)1}$. Similar to the proof of Theorem 2.3, we have $\dfrac{d_t}{GCD(d_1, \ldots, d_t)} \mid r_{tj}$. We consider two cases.

Case 1: Suppose that $GCD(a_{11}, \ldots, a_{m1}) = 1$. Therefore, $M$ is isomorphic to $R^m/ < (a_{11}, \ldots, a_{m1})^t >$ and $pd_R(M) = 1$.

Case 2: Suppose that $GCD(a_{11}, \ldots, a_{m1}) = x_0 \in P$ and, for the moment, fix $j$, $2 \leq j \leq n$. By the same argument and notation as in Case 1, we have $a_{ij} = \dfrac{a_{i1}}{x_0} r'_{tj}$, for some $r'_{tj}, 1 \leq i \leq t, 2 \leq j \leq n$. If there exists some $r'_{tj}, 2 \leq j \leq n$, such that $r'_{tj} \notin P$, then $I(M) =< \dfrac{a_{11}}{x_0}, \ldots, \dfrac{a_{m1}}{x_0} >$ which implies that $M$ is isomorphic to $R^m/ < (\dfrac{a_{11}}{x_0}, \ldots, \dfrac{a_{m1}}{x_0})^t >$. Now, suppose $r'_{tj} \in P$, for all $j$, $2 \leq j \leq n$. Put $J =< x_o, r'_{tj} : 2 \leq j \leq n >$. If for all $i$, $1 \leq i \leq t$, $\dfrac{a_{i1}}{x_0} \in I(M)$, then $I(M) \subseteq P^3$, a contradiction. Without loss of generality suppose $\dfrac{a_{11}}{x_0} \notin I(M)$. If $\dfrac{a_{11}}{x_0} \notin P$, then $J = I(M)$ and $M$ is isomorphic to $R^{m-1} \oplus R/I(M)$ in this case. Now, assume that $\dfrac{a_{i1}}{x_0} \in P$ for all $1 \leq i \leq m$. It is easily seen that $M \cong R^m/J < (a_1, \ldots, a_m)^t >$, where $ord_R(J) = ord_R\langle a_1, ..., a_m\rangle = 1$ and $J\langle a_1, ..., a_m\rangle = I(M)$. □

The following theorem represents some properties of module $M$ with $\mathrm{Fitt}_0(M) = P^n$.

**Theorem 3.2.** *Let $(R, P)$ be a Noetherian local ring and $M$ be a finitely generated $R$-module with $Fitt_0(M) = P^n$, for some positive integer $n$. Then,*
*(i) $M$ is generated by $n$ elements.*
*(ii) $M$ is an Artinian $R$-module.*
*(iii) Every submodule of $M$ is $P$-primary, particularly $Ass(M) = \{P\}$.*
*(iv) $M/P^{n-1}M$ is cyclic if and only if $M \cong R/P^n$.*
*(v) If $pd_R(M) < \infty$, then $pd_R(M) = depth(P, R)$.*

*Proof.* $(i)$ Let $\mathrm{Fitt}_0(M) = P^n$ and $M$ be generated by $r$ elements. Then, $P^n = \mathrm{Fitt}_0(M) = I_r(M) \subseteq P^r$. So, by Nakayama's Lemma $r \leq n$.
$(ii)$ Since $\mathrm{Fitt}_0(M) = P^n \subseteq \mathrm{ann}_R(M)$, then $P^{n-1}M$ is an $R/P$-module. So,

there exists positive integers $m$ such that $P^{n-1}M \cong (R/P)^m$. Hence, $P^{n-1}M$ is Artinian. Since $P^{n-1} \subseteq \mathrm{ann}_R(M/P^{n-1}M)$, then $M/P^{n-1}M$ is $(R/P^{n-1})$-module. Since $R/P^{n-1}$ is an Artinian ring and $M/P^{n-1}M$ is a finitely generated module, then $M/P^{n-1}M$ is Artinian. So, $0 \longrightarrow P^{n-1}M \longrightarrow M \longrightarrow M/P^{n-1}M \longrightarrow 0$ is an exact sequence of $R$-modules. Since $P^{n-1}M$ and $M/P^{n-1}M$ are Artinian $R$-modules, then $M$ is Artinian.
$(iii)$ Let $N$ be a proper submodule of $M$. Since $P^n = \mathrm{Fitt}_0(M) \subseteq \mathrm{ann}_R(M) \subseteq (N : M)$, then $\sqrt{(N : M)} = P$. It is easily seen that $N$ is $P$-primary submodule of $M$.
$(iv)$ If $n = 1$, then $\mathrm{Fitt}_0(M) = P \subseteq \mathrm{ann}(M)$. Thus, $M \cong R/P$. Let $n \geq 2$ and $M/P^{n-1}M$ is cyclic. By proof of $(ii)$ we have the exact sequence $0 \longrightarrow P^{n-1}M \xrightarrow{\varphi} M \xrightarrow{\psi} R/P \longrightarrow 0$ . By $(i)$ there exist $x_1, ..., x_n$ in M such that $M = \langle x_1, ..., x_n\rangle$. Since $\psi$ is onto, then there is $m \in M$ such that $\psi(m) = 1 + P$. Let $m = r_1x_1 + ... + r_nx_n$ for some $r_i \in R, 1 \leq i \leq n$. If $r_i \in P$, for every $i$, then $\psi(m) = r_1\psi(x_1) + ... + r_n\psi(x_n) = 0 = 1 + P$, a contradiction. Let $r_1 \notin P$. Then, $M = \langle m, x_2, ..., x_n\rangle$. On the other hand, for $i = 2, ..., n$ there exists $a_i \in R$ such that $\psi(x_i) = a_i + P = a_i\psi(m) = \psi(a_im)$. Hence, $x_i - a_im \in ker\psi = Im\varphi = P^{n-1}M$. We have $M = \langle m, x_2, ..., x_n\rangle = P^{n-1}M + \langle m\rangle$. So, $P^{n-1}M = P^{n-1}(P^{n-1}M + \langle m\rangle) = P^{n-1}m$ and $M = P^{n-1}m + \langle m\rangle = \langle m\rangle$. Hence, $M \cong R/\mathrm{ann}_R(M)$. Since $\mathrm{Fitt}_0(M) = P^n$, then $M \cong R/P^n$.
$(v)$ By Auslander-Buchsbaum formula we have,
$pd_R(M) = depth(P, R) - depth(P, M)$. By $(iii)$, $Ass(M) = \{P\}$, so $pd_R(M) = depth(P, R)$. □

**Example 3.3.** Let $R$ be the ring $K[[x, y]]$ of formal power series over a field $K$. It is known that $R = K[[x, y]]$ is a Noetherian local ring with maximal ideal $P = \langle x, y\rangle$. Consider $M = R^2/< \begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix} >$ as an $R$-module. Then, $\mathrm{Fitt}_0(M) = P^2$.

Let $M$ be an $R$-module. We say that $M$ is a prime module if $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ for every non-zero submodule $N$ of $M$ [8].
Let $(R, P)$ be a Noetherian local ring and $M$ be a finitely generated prime $R$-module with $\mathrm{Fitt}_0(M) = P^n$, for some positive integer $n$. By Theorem 3.2, part $(iii)$, $P \in Ass(M)$. So, there exists an element $x \in M$ such that $P = \mathrm{ann}_R(x)$. Since $M$ is prime, then $\mathrm{ann}_R(M) = P$, consequently $M \cong (R/P)^n$.
An $R$-module $M$ is called a multiplication module if for each submodule $N$ of $M$, $N = IM$ for some ideal $I$ of $R$. In this case we can take $I = (N : M)$ [8].

Let $(R, P)$ be a Noetherian local ring and $M$ be a finitely generated $R$-module. Let $M$ be a multiplication module with $\mathrm{Fitt}_0(M) = P^n$, for some positive integer $n$. Let $N$ be a submodule of $M$ such that $PM \subseteq N$. There exists an ideal $I$ of $R$ such that $N = IM$. So, $PM \subseteq N = IM \subseteq PM$. This implies that $PM$ is a maximal submodule of $M$. Thus, $M/PM \cong R/P$. Hence, by Theorem 3.2, part $(iv)$, $M \cong R/P^n$.

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