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SUFFICIENT CONDITIONS FOR UNIVALENCE AND STARLIKENESS

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Abstract. It is known that the condition $\mathfrak{Re}\{zf'(z)/f(z)\} > 0$, $|z| < 1$ is a sufficient condition for f, $f(0) = f'(0)-1 = 0$ to be starlike in $|z| < 1$. The purpose of this work is to present some new sufficient conditions for univalence and starlikeness.
Keywords: Analytic functions, convex functions, starlike functions, univalent functions.
MSC(2010): Primary: 30C45; Secondary: 30C80.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. We will use the following notations:

$$\left\{ \begin{array}{lll} J_{\mathcal{CV}}(f;z) & := & 1 + \frac{zf''(z)}{f'(z)}, \\ J_{\mathcal{ST}}(f;z) & := & \frac{zf'(z)}{f(z)}. \end{array} \right. \tag{1.1}$$

Let the function $f \in \mathcal{H}$ be univalent in the unit disc $\mathbb{D}$ with the normalization $f(0) = 0$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to $w_0 = 0$ if and only if [3]

$$\mathfrak{Re} J_{\mathcal{ST}}(f;z) > 0, \quad z \in \mathbb{D}, \tag{1.2}$$

and such a function f is said to be starlike in $\mathbb{D}$ with respect to $w_0 = 0$ (or briefly starlike). Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear

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segment joining any two points of E lies entirely in E. An univalent function f maps $\mathbb{D}$ onto a convex domain E if and only if [13]

$$\mathfrak{Re} J_{\mathcal{CV}}(f;z) > 0, \quad z \in \mathbb{D}, \tag{1.3}$$

and then f is said to be convex in $\mathbb{D}$ (or briefly convex). It is well known that if an analytic function f satisfies (1.2) and $f(0) = 0$, $f'(0) \neq 0$ then f is univalent and starlike in $\mathbb{D}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathcal{S}^*$, and the set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$ by $\mathcal{K}$.

To prove the main results, we also need the following generalization of the Nunokawa's lemma, [4,5].

Lemma 1.1. *[4], [5] Let $p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n$, $c_m \neq 0$ be an analytic function in $\mathbb{D}$ with $p(z) \neq 0$. If there exists a point z_0, $|z_0| < 1$, such that*

$$|\arg\{p(z)\}| < \frac{\pi\beta}{2} \quad for \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\beta}{2}$$

for some $\beta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},$$

for some $k \geq m(a + a^{-1})/2 > m$, where

$$\{p(z_0)\}^{1/\beta} = \pm ia, \quad and \quad a > 0.$$

2. Main results

Theorem 2.1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. If f satisfies the condition*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right| + \frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right| < \infty, \quad z \in \mathbb{D} \tag{2.1}$$

then f is starlike in $|z| \leq 1$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}.$$

If there exists a point $z = \alpha$, $|\alpha| < 1$ for which $p(\alpha) = 0$, then we can put $f'(z) = (z - \alpha)^n g(z)$, where n is a positive integer and $g(z)$ is analytic in $|z| < 1$ and $g(\alpha) \neq 0$. Moreover, it follows that

$$1 + zf''(z)/f'(z) = 1 + nz/(z - \alpha) + zg'(z)/g(z).$$

As $z \to \alpha$, the right hand side of the above equation becomes infinite. This is in contradiction with (2.1). Since, we have proved

$$\frac{zf'(z)}{f(z)} \neq 0, \quad z \in \mathbb{D}. \tag{2.2}$$

Therefore, we have $p(0) = 1$, and by (2.2) we have $p(z) \neq 0$ in $\mathbb{D}$. Moreover, (2.1) becomes

$$\begin{aligned} & \left|1 + \frac{zf''(z)}{f'(z)}\right| - \frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right| - \frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right| \\ = & \left|p(z) + \frac{zp'(z)}{p(z)}\right| - \frac{3}{2}|p(z)| - \frac{1}{2}\left|\frac{1}{p(z)}\right| < 0, \quad z \in \mathbb{D}. \end{aligned} \tag{2.3}$$

We want to show the starlikeness of f or equivalently $\mathfrak{Re}\{p(z)\} > 0$ in the unit disc $\mathbb{D}$. Assume on contrary, that there exists a point $z_0 \in \mathbb{D}$ such that

$$p(z_0) = \pm ia, \; a > 0. \tag{2.4}$$

Then by Lemma 1.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi},$$

for some

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right). \tag{2.5}$$

Then applying (2.4) and (2.5) in (2.3), we have

$$\begin{aligned} & \left|1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right| - \frac{3}{2}\left|\frac{z_0 f'(z_0)}{f(z_0)}\right| - \frac{1}{2}\left|\frac{f(z_0)}{z_0 f'(z_0)}\right| \\ = & \left|p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right| - \frac{3}{2}|p(z_0)| - \frac{1}{2}\left|\frac{1}{p(z_0)}\right| \\ = & \left|\pm ia + \frac{2ik \arg\{p(z_0)\}}{\pi}\right| - \frac{3}{2}|ia| - \frac{1}{2}\left|\frac{1}{ia}\right| \\ = & |\pm ia \pm ki| - \frac{3}{2}a - \frac{1}{2a} \\ \geq & a + \frac{1}{2}\left(a + \frac{1}{a}\right) - \frac{3}{2}a - \frac{1}{2a} \\ = & 0 \end{aligned}$$

and this contradicts (2.3). Therefore, $\mathfrak{Re}\{p(z)\} > 0$ in the whole unit disc $\mathbb{D}$ or equivalently f is a starlike function. □

We have $3x/2 + 1/(2x) \geq \sqrt{3}$ for $x \geq 0$, since

$$\frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right| + \frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right| \geq \sqrt{3}, \quad z \in \mathbb{D},$$

and therefore, we obtain the following corollary.

Corollary 2.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. If f satisfies the condition*

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \sqrt{3}, \quad z \in \mathbb{D} \tag{2.6}$$

then f is starlike in $|z| \le 1$.

For example, let $g(z) = \log(1 - xz) = z + \ldots$, where $x = (3 - \sqrt{3})/3$. Then

$$\left| 1 + \frac{zg''(z)}{g'(z)} \right| = \left| \frac{1}{1 - xz} \right| < \left| \frac{1}{1 - x} \right| = \sqrt{3}, \quad z \in \mathbb{D}.$$

Therefore, by Corollary 2.2, $g(z) = \log(1 - xz)$ is a starlike function. The starlikeness implies the univalence thus the above theorems are also certain univalence conditions. Recall that Umezawa [14] proved that

$$\left| \frac{f''(z)}{f'(z)} \right| \le \sqrt{6}, \quad |z| \le 1, \tag{2.7}$$

implies the univalence of $f(z)$ in $|z| \le 1$. Notice also here that in [6] Ozaki proved that if $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is analytic in $\mathbb{D}$, with $f(z)f'(z)/z \ne 0$ there, and if either

$$\mathfrak{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \ge -\frac{1}{2}$$

or

$$\mathfrak{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) \le \frac{3}{2} \tag{2.8}$$

holds throughout $\mathbb{D}$, then f is univalent and convex in at least one direction in $\mathbb{D}$. Moreover, R. Singh and S. Singh [8] proved that (2.8) implies that $f(z)$ is close-to-convex and bounded in $\mathbb{D}$. Recall here that an analytic function f is said to be a close-to-convex function of order β, $\beta \in [0, 1)$, if and only if there exist a number $\varphi \in \mathbb{R}$ and a function $g \in \mathcal{K}$, such that

$$\mathfrak{Re} \left(e^{i\varphi} \frac{f'(z)}{g'(z)} \right) > \beta, \qquad z \in \mathbb{D}. \tag{2.9}$$

The the number $\sqrt{6}$ in (2.7), was improved to $3.05\ldots$ in [1]. In [7], the following interesting result is proved. Let $f \in \mathcal{A}$ with $f(z)f'(z)/z \ne 0$ in $|z| < 1$. Then, for each $\alpha \in [-1/2, 0)$, there exists a function f which satisfies

$$1 + \mathfrak{Re} \frac{zf''(z)}{f'(z)} > \alpha, \quad z \in \mathbb{D},$$

but f is not starlike in $|z| < 1$. Another type sufficient conditions for starlikeness are contained in the recent papers [9–11] and [12].

One can consider the maximum value of λ such that the condition

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<\lambda, \quad |z|\le 1, \tag{2.10}$$

implies that f is univalent in the unit disc. The radius of univalence of the function $g(z)=(e^{\pi z}-1)/\pi$ is $r=1$ and

$$\left|1+\frac{z_0 g''(z_0)}{g'(z_0)}\right| = 1+\pi \;\text{ at } z_0=1.$$

Therefore, $\lambda \le 1+\pi$.

Conjecture. The maximum value of λ such that the condition (2.10) implies univalence of f is $\lambda = 1+\pi$.

Theorem 2.3. *Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ be analytic in $\mathbb{D}$. If f satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \sqrt{1+\left\{\frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right|+\frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right|\right\}^2} < \infty, \quad z\in\mathbb{D} \tag{2.11}$$

then f is starlike in $|z|\le 1$.

Proof. Let us put

$$p(z)=\frac{zf'(z)}{f(z)}, \quad z\in\mathbb{D}.$$

Then $p(0)=1$, and as in the proof of Theorem 2.1 we obtain $p(z)\neq 0$ in $\mathbb{D}$. Moreover, by (2.11) we have

$$\begin{aligned}&\left|\frac{zf''(z)}{f'(z)}\right| - \sqrt{1+\left\{\frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right|+\frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right|\right\}^2}\\ (2.12)= \quad &\left|p(z)-1+\frac{zp'(z)}{p(z)}\right| - \sqrt{1+\left\{\frac{3}{2}|p(z)|+\frac{1}{2}\left|\frac{1}{p(z)}\right|\right\}^2} < 0, \quad z\in\mathbb{D}.\end{aligned}$$

We want to show the starlikeness of f or equivalently that $\mathfrak{Re}\{p(z)\}>0$ in the unit disc $\mathbb{D}$. Assume on contrary, that there exists a point $z_0\in\mathbb{D}$ such that

$$p(z_0)=\pm ia, \;\; a>0. \tag{2.13}$$

Then by Lemma 1.1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik\arg\{p(z_0)\}}{\pi},$$

for some

$$k \ge \frac{1}{2}\left(a+\frac{1}{a}\right). \tag{2.14}$$

Then applying (2.13) and (2.14) in (2.12), we have

$$\begin{aligned}
&\left|\frac{z_0 f''(z_0)}{f'(z_0)}\right| - \sqrt{1+\left\{\frac{3}{2}\left|\frac{z_0 f'(z_0)}{f(z_0)}\right| + \frac{1}{2}\left|\frac{f(z_0)}{z_0 f'(z_0)}\right|\right\}^2} \\
= \;& \left|p(z_0) - 1 + \frac{z_0 p'(z_0)}{p(z_0)}\right| - \sqrt{1+\left\{\frac{3}{2}|p(z_0)| + \frac{1}{2}\left|\frac{1}{p(z_0)}\right|\right\}^2} \\
= \;& \left|\pm ia - 1 + \frac{2ik\arg\{p(z_0)\}}{\pi}\right| - \sqrt{1+\left\{\frac{3}{2}|\pm ia| + \frac{1}{2}\left|\frac{1}{\pm ia}\right|\right\}^2} \\
= \;& |-1 \pm ia \pm ki| - \sqrt{1+\left\{\frac{3a}{2} + \frac{1}{2a}\right\}^2} \\
\geq \;& \sqrt{1+\left(a + \frac{1}{2}\left(a+\frac{1}{a}\right)\right)^2} - \sqrt{1+\left\{\frac{3a}{2} + \frac{1}{2a}\right\}^2} \\
= \;& 0
\end{aligned}$$

and this contradicts (2.12). Therefore, $\mathfrak{Re}\{p(z)\} > 0$ in the whole unit disc $\mathbb{D}$ or equivalently f is a starlike function. □

We have $\sqrt{1+(3x/2+1/(2x))^2} \geq 2$ for $x \geq 0$ since

$$\sqrt{1+\left\{\frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right| + \frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right|\right\}^2} \geq 2, \quad z \in \mathbb{D},$$

and therefore, we obtain the following corollary.

Corollary 2.4. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. If f satisfies the condition*

$$\left|\frac{f''(z)}{f'(z)}\right| < 2, \quad z \in \mathbb{D} \tag{2.15}$$

then f is starlike in $|z| \leq 1$.

Corollary 2.5. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. If f satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \sqrt{1+\left\{\frac{3}{2}\left|\frac{zf'(z)}{f(z)}\right|\right\}^2} < \infty, \quad z \in \mathbb{D}$$

then f is starlike in $|z| \leq 1$.

Corollary 2.6. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$. If f satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \sqrt{1+\left\{\frac{1}{2}\left|\frac{f(z)}{zf'(z)}\right|\right\}^2} < \infty, \quad z \in \mathbb{D}$$

then f is starlike in $|z| \leq 1$.

Remark 2.7. The above Corollary 2.4 is the result proved earlier by Miller and Mocanu [2]. Therefore, Theorem 2.3 is a generalization of their result.

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