Title:

Sufficient conditions for univalence and starlikeness

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SUFFICIENT CONDITIONS FOR UNIVALENCE AND STARLIKENESS

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Abstract. It is known that the condition \( \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad |z| < 1 \) is a sufficient condition for \( f, f(0) = f'(0) - 1 = 0 \) to be starlike in \( |z| < 1 \). The purpose of this work is to present some new sufficient conditions for univalence and starlikeness.

Keywords: Analytic functions, convex functions, starlike functions, univalent functions.


1. Introduction

Let \( \mathcal{H} \) denote the class of analytic functions in the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) on the complex plane \( \mathbb{C} \). We will use the following notations:

\[
\begin{align*}
J_{CV}(f; z) & := 1 + \frac{zf''(z)}{f'(z)}, \\
J_{ST}(f; z) & := \frac{zf'(z)}{f(z)}.
\end{align*}
\]

(1.1)

Let the function \( f \in \mathcal{H} \) be univalent in the unit disc \( \mathbb{D} \) with the normalization \( f(0) = 0 \). Then \( f \) maps \( \mathbb{D} \) onto a starlike domain with respect to \( w_0 = 0 \) if and only if \( |z| < 1 \)

\[
\Re J_{ST}(f; z) > 0, \quad z \in \mathbb{D},
\]

(1.2)

and such a function \( f \) is said to be starlike in \( \mathbb{D} \) with respect to \( w_0 = 0 \) (or briefly starlike). Recall that a set \( E \subset \mathbb{C} \) is said to be starlike with respect to a point \( w_0 \in E \) if and only if the linear segment joining \( w_0 \) to every other point \( w \in E \) lies entirely in \( E \), while a set \( E \) is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment...
segment joining any two points of $E$ lies entirely in $E$. An univalent function $f$ maps $D$ onto a convex domain $E$ if and only if [13]

\begin{equation}
\Re J_{CV}(f; z) > 0, \quad z \in D,
\end{equation}

and then $f$ is said to be convex in $D$ (or briefly convex). It is well known that if an analytic function $f$ satisfies (1.2) and $f(0) = 0$, $f'(0) \neq 0$ then $f$ is univalent and starlike in $D$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $D$ will be denoted by $\mathcal{S}$, and the set of all functions $f \in \mathcal{A}$ that are convex univalent in $D$ by $\mathcal{K}$.

To prove the main results, we also need the following generalization of the Nunokawa’s lemma, [4,5].

**Lemma 1.1.** [4, 5] Let $p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n$, $c_m \neq 0$ be an analytic function in $D$ with $p(z) \neq 0$. If there exists a point $z_0$, $|z_0| < 1$, such that

\[ |\arg \{p(z_0)\}| < \frac{\pi \beta}{2} \quad \text{for } |z| < |z_0| \]

and

\[ |\arg \{p(z_0)\}| = \frac{\pi \beta}{2} \]

for some $\beta > 0$, then we have

\[ \frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg \{p(z_0)\}}{\pi}, \]

for some $k \geq m(\alpha + \alpha^{-1})/2 > m$, where

\[ \{p(z_0)\}^{1/\beta} = \pm ia, \quad \text{and } a > 0. \]

2. Main results

**Theorem 2.1.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $D$. If $f$ satisfies the condition

\begin{equation}
1 + \frac{zf''(z)}{f'(z)} < \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| < \infty, \quad z \in D
\end{equation}

then $f$ is starlike in $|z| \leq 1$.

**Proof.** Let us put

\[ p(z) = \frac{zf'(z)}{f(z)}, \quad z \in D. \]

If there exists a point $z = \alpha$, $|\alpha| < 1$ for which $p(\alpha) = 0$, then we can put $f'(z) = (z - \alpha)^n g(z)$, where $n$ is a positive integer and $g(z)$ is analytic in $|z| < 1$ and $g(\alpha) \neq 0$. Moreover, it follows that

\[ 1 + z f''(z)/f'(z) = 1 + nz/(z - \alpha) + zg'(z)/g(z). \]
As \( z \to \alpha \), the right hand side of the above equation becomes infinite. This is in contradiction with (2.1). Since, we have proved

\[
(2.2) \quad \frac{zf'(z)}{f(z)} \neq 0, \quad z \in \mathbb{D}.
\]

Therefore, we have \( p(0) = 1 \), and by (2.2) we have \( p(z) \neq 0 \) in \( \mathbb{D} \). Moreover, (2.1) becomes

\[
(2.3) \quad \left| 1 + \frac{zf''(z)}{f'(z)} \right| - \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| - \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| = \left| p(z) + \frac{zp'(z)}{p(z)} \right| - \frac{3}{2} \left| p(z) \right| - \frac{1}{2} \left| \frac{1}{p(z)} \right| < 0, \quad z \in \mathbb{D}.
\]

We want to show the starlikeness of \( f \) or equivalently \( \mathcal{R} \{ p(z) \} > 0 \) in the unit disc \( \mathbb{D} \). Assume on contrary, that there exists a point \( z_0 \in \mathbb{D} \) such that

\[
(2.4) \quad p(z_0) = \pm ia, \quad a > 0.
\]

Then by Lemma 1.1, we have

\[
\frac{z_0p'(z_0)}{p(z_0)} = \frac{2ik \arg \{ p(z_0) \}}{\pi},
\]

for some

\[
(2.5) \quad k \geq \frac{1}{2} \left( a + \frac{1}{a} \right).
\]

Then applying (2.4) and (2.5) in (2.3), we have

\[
\left| 1 + \frac{z_0f''(z_0)}{f'(z_0)} \right| - \frac{3}{2} \left| \frac{z_0f'(z_0)}{f(z_0)} \right| - \frac{1}{2} \left| \frac{f(z_0)}{z_0f'(z_0)} \right| = \left| p(z_0) + \frac{zp'(z_0)}{p(z_0)} \right| - \frac{3}{2} \left| p(z_0) \right| - \frac{1}{2} \left| \frac{1}{p(z_0)} \right|
\]

\[
= \left| \pm ia + \frac{2ik \arg \{ p(z_0) \}}{\pi} \right| - \frac{3}{2} |ia| - \frac{1}{2} \left| \frac{1}{ia} \right|
\]

\[
= |\pm ia \pm ki| - \frac{3}{2} a - \frac{1}{2a}
\]

\[
\geq a + \frac{1}{2} \left( a + \frac{1}{a} \right) - \frac{3}{2} a - \frac{1}{2a}
\]

\[
= 0
\]

and this contradicts (2.3). Therefore, \( \mathcal{R} \{ p(z) \} > 0 \) in the whole unit disc \( \mathbb{D} \) or equivalently \( f \) is a starlike function. \( \square \)

We have \( 3x/2 + 1/(2x) \geq \sqrt{3} \) for \( x \geq 0 \), since

\[
\frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| \geq \sqrt{3}, \quad z \in \mathbb{D},
\]
and therefore, we obtain the following corollary.

**Corollary 2.2.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \). If \( f \) satisfies the condition
\[
1 + \frac{zf''(z)}{f'(z)} < \sqrt{3}, \quad z \in \mathbb{D}
\]
then \( f \) is starlike in \( |z| < 1 \).

For example, let \( g(z) = \log(1-xz) = z + \ldots \), where \( x = (3-\sqrt{3})/3 \). Then
\[
1 + \frac{zg''(z)}{g'(z)} = \left| \frac{1}{1-xz} \right| < \frac{1}{1-x} = \sqrt{3}, \quad z \in \mathbb{D}.
\]
Therefore, by Corollary 2.2, \( g(z) = \log(1-xz) \) is a starlike function. The starlikeness implies the univalence thus the above theorems are also certain univalence conditions. Recall that Umezawa [14] proved that
\[
1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) \geq -\frac{1}{2},
\]
or
\[
1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) \leq \frac{3}{2}
\]
holds throughout \( \mathbb{D} \), then \( f \) is univalent and convex in at least one direction in \( \mathbb{D} \). Moreover, R. Singh and S. Singh [8] proved that (2.8) implies that \( f(z) \) is close-to-convex and bounded in \( \mathbb{D} \). Recall here that an analytic function \( f \) is said to be a close-to-convex function of order \( \beta, \beta \in [0,1) \), if and only if there exist a number \( \varphi \in \mathbb{R} \) and a function \( g \in \mathcal{K} \), such that
\[
\Re \left( e^{i\varphi} \frac{f'(z)}{g'(z)} \right) > \beta, \quad z \in \mathbb{D}.
\]
The number \( \sqrt{6} \) in (2.7), was improved to 3.05... in [1]. In [7], the following interesting result is proved. Let \( f \in \mathcal{A} \) with \( f(z)f'(z)/z \neq 0 \) in \( |z| < 1 \). Then, for each \( \alpha \in [-1/2,0) \), there exists a function \( f \) which satisfies
\[
1 + \Re \left( \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathbb{D},
\]
but \( f \) is not starlike in \( |z| < 1 \). Another type sufficient conditions for starlikeness are contained in the recent papers [9–11] and [12].
One can consider the maximum value of $\lambda$ such that the condition
\begin{equation}
1 + \left| \frac{zf''(z)}{f'(z)} \right| < \lambda, \quad |z| \leq 1,
\end{equation}
implies that $f$ is univalent in the unit disc. The radius of univalence of the function $g(z) = (e^{\pi z} - 1)/\pi$ is $r = 1$ and
\begin{equation}
1 + \left| \frac{z_0g''(z_0)}{g'(z_0)} \right| = 1 + \pi \quad \text{at } z_0 = 1.
\end{equation}
Therefore, $\lambda \leq 1 + \pi$.

**Conjecture.** The maximum value of $\lambda$ such that the condition (2.10) implies univalence of $f$ is $\lambda = 1 + \pi$.

**Theorem 2.3.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $D$. If $f$ satisfies the condition
\begin{equation}
\left| \frac{zf''(z)}{f'(z)} \right| < \sqrt{1 + \left\{ \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| \right\}^2} < \infty, \quad z \in D
\end{equation}
then $f$ is starlike in $|z| \leq 1$.

**Proof.** Let us put
\begin{equation}
p(z) = \frac{zf'(z)}{f(z)}, \quad z \in D.
\end{equation}
Then $p(0) = 1$, and as in the proof of Theorem 2.1 we obtain $p(z) \neq 0$ in $D$. Moreover, by (2.11) we have
\begin{equation}
\left| \frac{zf''(z)}{f'(z)} \right| - \sqrt{1 + \left\{ \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| \right\}^2} < 0, \quad z \in D.
\end{equation}
(2.12)

We want to show the starlikeness of $f$ or equivalently that $\Re \{ p(z) \} > 0$ in the unit disc $D$. Assume on contrary, that there exists a point $z_0 \in D$ such that
\begin{equation}
p(z_0) = \pm ia, \quad a > 0.
\end{equation}
(2.13)
Then by Lemma 1.1, we have
\begin{equation}
\frac{z_0p'(z_0)}{p(z_0)} = \frac{2ik \arg \{ p(z_0) \}}{\pi},
\end{equation}
for some
\begin{equation}
k \geq \frac{1}{2} \left( a + \frac{1}{a} \right).
\end{equation}
(2.14)
Then applying (2.13) and (2.14) in (2.12), we have
\[
\frac{z_0 f''(z_0)}{f'(z_0)} - \sqrt{1 + \left\{ \frac{3}{2} \left| \frac{z_0 f'(z_0)}{f(z_0)} \right| + \frac{1}{2} \left| \frac{f(z_0)}{z_0 f'(z_0)} \right| \right\}^2} = p(z_0) - 1 + \frac{z_0 p'(z_0)}{p(z_0)} - \sqrt{1 + \left\{ \frac{3}{2} |p(z_0)| + \frac{1}{2} \left| \frac{1}{p(z_0)} \right| \right\}^2} = \pm ia - 1 + \frac{2i \arg \{p(z_0)\}}{\pi} - \sqrt{1 + \left\{ \frac{3}{2} |\pm ia| + \frac{1}{2} \left| \frac{1}{\pm ia} \right| \right\}^2} = |-1 \pm ia \pm ki| - \sqrt{1 + \left\{ \frac{3a}{2} + \frac{1}{2a} \right\}^2} \geq \sqrt{1 + \left( a + \frac{1}{2} \left( a + \frac{1}{a} \right) \right)^2} - \sqrt{1 + \left\{ \frac{3a}{2} + \frac{1}{2a} \right\}^2} = 0
\]
and this contradicts (2.12). Therefore, \( \Re\{p(z)\} > 0 \) in the whole unit disc \( \mathbb{D} \) or equivalently \( f \) is a starlike function. \( \square \)

We have \( \sqrt{1 + (3x/2 + 1/(2x))^2} \geq 2 \) for \( x \geq 0 \) since
\[
\sqrt{1 + \left\{ \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| + \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| \right\}^2} \geq 2, \quad z \in \mathbb{D},
\]
and therefore, we obtain the following corollary.

**Corollary 2.4.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \). If \( f \) satisfies the condition
\[
(2.15) \quad \left| \frac{f''(z)}{f'(z)} \right| < 2, \quad z \in \mathbb{D}
\]
then \( f \) is starlike in \( |z| \leq 1 \).

**Corollary 2.5.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \). If \( f \) satisfies the condition
\[
\left| \frac{zf''(z)}{f'(z)} \right| < \sqrt{1 + \left\{ \frac{3}{2} \left| \frac{zf'(z)}{f(z)} \right| \right\}^2} < \infty, \quad z \in \mathbb{D}
\]
then \( f \) is starlike in \( |z| \leq 1 \).

**Corollary 2.6.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be analytic in \( \mathbb{D} \). If \( f \) satisfies the condition
\[
\left| \frac{zf''(z)}{f'(z)} \right| < \sqrt{1 + \left\{ \frac{1}{2} \left| \frac{f(z)}{zf'(z)} \right| \right\}^2} < \infty, \quad z \in \mathbb{D}
\]
then $f$ is starlike in $|z| \leq 1$.

**Remark 2.7.** The above Corollary 2.4 is the result proved earlier by Miller and Mocanu [2]. Therefore, Theorem 2.3 is a generalization of their result.

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