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ON TWO PROBLEMS CONCERNING THE ZARISKI TOPOLOGY OF MODULES

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Abstract. Let R be an associative ring and let M be a left R-module. Let $Spec_R(M)$ be the collection of all prime submodules of M (equipped with classical Zariski topology). It is conjectured that every irreducible closed subset of $Spec_R(M)$ has a generic point. In this article we give an affirmative answer to this conjecture and show that if M has a Noetherian spectrum, then $Spec_R(M)$ is a spectral space.
Keywords: Prime spectrum, classical Zariski topology, spectral space.
MSC(2010): Primary: 13C13; Secondary: 13C99.

1. Introduction

Throughout this article, all rings are associative rings with identity elements, and all modules are unital left modules. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ will denote respectively the natural numbers, the ring of integers, and the field of quotients of $\mathbb{Z}$. We use $\subseteq$ and $\subset$ for weak and strong inclusion, respectively. Also for a topological space X, $dim(X)$ denotes the *combinatorial dimension* of X.

Let M be a left R-module. For any submodule N of M we denote the annihilator of the module M/N by $(N :_R M)$, i.e. $(N :_R M) = \{r \in R | rM \subseteq N\}$. A submodule P of M is called *prime* if $P \neq M$ and whenever $r \in R$ and $e \in M$ satisfy $re \in P$ then $r \in (P :_R M)$ or $e \in P$.

Let $Spec_R(M)$ be the collection of all prime submodules of M. Put $V(N) = \{P \in Spec_R(M) | P \supseteq N\}$ and $\Omega = \{V(N) | N$ is a submodule of $M\}$. Then there exists a topology τ on $Spec_R(M)$ having Ω as the set of its closed subsets if and only if Ω is closed under the finite union. When this is the case, M is called a *top R-module* [15].

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Put

$$\Omega^c = \left\{ \bigcap_{i \in I} \left(\bigcup_{j=1}^{n_i} V(N_{i,j}) \right) | \, V(N_{i,j}) \in \Omega, \, n_i \in \mathbb{N}, \, I \text{ is an index set} \right\}.$$

Then the elements of Ω^c satisfy the axioms for closed sets in a topological space on $Spec_R(M)$ ([5] and [6]). This topology is called the *classical Zariski topology* of M. We denote this topology by τ^c.

X is irreducible if $X \neq \emptyset$ and for every decomposition $X = A_1 \cup A_2$ with closed subsets $A_i \subseteq X, i = 1, 2$, we have $A_1 = X$ or $A_2 = X$. A subset T of X is irreducible if T is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets F, G which are closed in X and satisfy $T \subseteq F \cup G$, $T \subseteq F$ or $T \subseteq G$. An irreducible component of X is a maximal irreducible subset of X. Every irreducible subset of X is contained in an irreducible component of X, and X is the union of its irreducible components.

A topological space X is a *spectral space* if X is homeomorphic to $Spec(S)$ for some ring S. This concept plays an important role in studying algebraic properties of an R-module M when we have a related topology. For an example, when $Spec_R(M)$ (with classical Zariski topology) is homeomorphic to $Spec(S)$, where S is a commutative ring, we can transfer some of known topological properties of $Spec(S)$ to $Spec_R(M)$ and then by using these properties explore some of the algebraic properties of M.

Spectral spaces have been characterized by M. Hochster as quasi-compact T_0-spaces X having a quasi-compact open base closed under finite intersection and each irreducible closed subset of X has a generic point [8, p. 52, Proposition. 4].

The concept of prime submodule has led to the development of topologies on the spectrum of modules. A brief history of this is given by Lu in [13]. More information in this regard can be found in [1, 2, 17].

In [5, p. 126], there is a conjecture which says that every irreducible closed subset of $Spec_R(M)$ has a generic point. Also in [6, Question 3.5], there is a question as follows: Let M be an R-module with Noetherian spectrum. Is $(Spec_R(M), \tau^c)$ a spectral space? In Section 2, we will give affirmative answers to both conjecture and open question (see Theorem 2.2 and Corollary 2.4). Also we define (FC) property for classical Zariski topology and obtain its relation with minimal prime divisors of submodules of M.

2. Main results

Let Z be a subset of a topological space W. Then the notion $\overline{Z}$ will denote the closure of Z in W.

Lemma 2.1. *Let M be a left R-module and Y be a nonempty subset of $Spec_R(M)$. Then $\overline{Y} = \overline{\bigcup_{P \in Y} V(P)}$. In particular, when Y is closed we have $Y = \bigcup_{P \in Y} V(P)$.*

Proof. This is straightforward. □

In [3, Theorem 3.8 (a)], the authors showed that if M is an X-injective module over a PID ring R, then every irreducible closed subset of $Spec_R(M)$ has a generic point. In below we drop the restrictions and prove this conjecture in a general case.

Theorem 2.2. *For any R-module M, every irreducible closed subset of $Spec_R(M)$ has a generic point. In particular, this is true when M is a top module.*

Proof. Let Y be an irreducible closed subset of $Spec_R(M)$ and $\bigcap_{P\in Y} P = Q$. By [5, Theorem 3.4], Q is a prime submodule of M . It is enough to prove that $Y = V(Q)$. We show that $\bigcup_{P\in Y} V(P) = V(Q)$. Clearly $\bigcup_{P\in Y} V(P) \subseteq V(Q)$. To see the reverse inclusion, let F be a closed subset of $Spec_R(M)$ containing $\bigcup_{P\in Y} V(P)$. Since F is closed, $F = \bigcap_{i\in\Lambda} \bigcup_{j=1}^{n_i} V(N_{i,j})$ for some submodules $N_{i,j}$ of M. We may assume (without loss of generality) that $F = V(E) \cup V(L)$, where E and L are submodules of M. By Lemma 2.1, $\bigcup_{P\in Y} V(P) = Y$ is irreducible. Since $\bigcup_{P\in Y} V(P) \subseteq V(E) \cup V(L)$, it follows that $\bigcup_{P\in Y} V(P) \subseteq V(E)$ or $\bigcup_{P\in Y} V(P) \subseteq V(L)$. This implies that $\bigcap_{P\in Y} P \supseteq E$ or $\bigcap_{P\in Y} P \supseteq L$. Thus $V(Q) \subseteq V(E)$ or $V(Q) \subseteq V(L)$, so $V(Q) \subseteq F$. By the above arguments, we have $V(Q) \subseteq \bigcup_{P\in Y} V(P)$. □

Definition 2.3. Let M be an R-module and let N be a proper submodule of M. $P \in V(N)$ is called a *minimal prime submodule* over N if there does not exist $Q \in V(N)$ such that $Q \subset P$. If $V(N) \neq \emptyset$, then the existence of minimal prime submodules over N can be verified easily by using Zorn's lemma.We say P is a *prime divisor* (resp. *minimal prime divisor*) of N if $P \in V(N)$ (resp. $P \in Min(V(N))$).

The *prime dimension* of an R-module M, denoted by $dim(M)$, is defined to be the supremum of the length of the strictly chains of prime submodules of M if $Specs(M) \neq \emptyset$ and -1 otherwise [14].

Corollary 2.4. *Let M be a left R-module. Then the following hold.*

(a) *Let Y be a closed subset of $Spec_R(M)$. Then Z is an irreducible component of Y if and only if $Z = V(P)$ for some minimal element P of Y.*
(b) *$dim(Spec_R(M)) = dim(M)$.*
(c) *If M has a Noetherian spectrum, then $Spec_R(M)$ is a spectral space. In particular, this is true when M is a top module.*

Proof. (a) Let Y be a closed subset of $Spec_R(M)$ and let Z be an irreducible component of Y. Since every irreducible components is closed, Z is closed in Y. But every irreducible closed subset of Y is an irreducible closed subset of $Spec_R(M)$. Therefore $Z = V(P)$ for some prime submodule P of M by

Theorem 2.2. Since $Z = V(P) \subseteq Y$ is a component of Y, we conclude that P is a minimal element of Y by [5, Lemma 3.3]. Conversely, let P be a minimal element of Y. Then $V(P)$ is an irreducible closed subset of $Spec_R(M)$ by [5, Lemma 3.3]. Since $P \in Y$, then $V(P) \subseteq \overline{Y}$ by Lemma 2.1. This implies that $V(P) \subseteq Y$. But $V(P)$ is an irreducible closed subset of $Spec_R(M)$. It follows that $V(P)$ is an irreducible closed subset of Y. Now let $V(P) \subseteq T$, where T is an irreducible subset of Y. Then $V(P) \subseteq T \subseteq \overline{T}$. Since $\overline{T}$ is an irreducible closed subset of $Spec_R(M)$, then $\overline{T} = V(Q)$ for some prime submodule Q of M by Theorem 2.2. It follows that $Q = P$, whence $T = V(P)$. By the above arguments, $V(P)$ is an irreducible component of Y.

(b) Let $Z_0 \subset Z_1 \subset ... \subset Z_t$ be a strictly increasing chain of irreducible closed subsets Z_i of $Spec_R(M)$ of length t. By Theorem 2.2, for each i we have $Z_i = V(P_i)$ for some $P_i \in Spec_R(M)$. On the other hand $V(P_i) \subset V(P_j)$ if and only if $P_i \supset P_j$. Hence $P_0 \supset P_1 \supset ... \supset P_t$, is a strictly decreasing chain of prime submodules of M. Conversely, for every strictly decreasing chain $P_0 \supset P_1 \supset ... \supset P_t$ of prime submodules of M of length t, $V(P_0) \subset V(P_1) \subset ... \subset V(P_t)$ is a strictly increasing chain of irreducible closed subsets of $Spec_R(M)$ of length t. This in turn implies that $dim(Spec_R(M)) = dim(M)$ and this completes the proof.

(c) Since $Spec_R(M)$ is Noetherian, it is quasi-compact and the quasi-compact open subsets of $Spec_R(M)$ are closed under finite intersection and form an open base [4, p. 79, Exer. 6]. Also, $Spec_R(M)$ is a T_0-space by [5, Proposition 3.8(i)]. Now the result follows from Theorem 2.2 and the Hochster's characterizations. □

Definition 2.5. Let M be an R-module.

(a) We say that M has property (FC) if every closed subset of $(Spec_R(M), \tau^c)$ has a finite number of irreducible components.
(b) We say that M has property (FP) if every submodule of M has a finite number of minimal prime divisors.

It is well known that every closed subset of the prime spectrum of R has a finite number of irreducible components if and only if every ideal of R has a finite number of minimal prime divisors. Is it possible to extend this result to module? This question is answered in the following theorem.

Theorem 2.6. *Let M be a left R-module. Then we have the following.*

(a) *If M has property (FC), then M has property (FP). However, the converse is not true in general.*
(b) *M has property (FC) if and only if M has property (FP) and $\gamma(M) = \{\bigcap_{i=1}^{m} F_i | F_i \in \beta(M), m \in \mathbb{N}\}$ is closed under arbitrary intersection, where $\beta(M) = \{V(N_1) \cup V(N_2) \cup ... \cup V(N_k) | N_i$ is a submodule of M and $k \in \mathbb{N}\}$.*

Proof. (a) Let N be a submodule of M and let P be a minimal prime divisor of N. Then $V(P)$ is an irreducible component of $V(N)$ by Corollary 2.4. This in turn implies that every submodule of M has a finite number of minimal prime divisors. To see the second assertion, set $M = \mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}\oplus\cdots$ and regarding M as $\mathbb{Q}$ module. Since the prime submodules of a vector space are just the proper submodules, every submodule of M has a finite number of minimal prime divisors, namely itself. Now let $P_1 = (\mathbf{0}) \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots$, $P_2 = \mathbb{Q} \oplus (\mathbf{0}) \oplus \mathbb{Q} \oplus \cdots$, $P_3 = \mathbb{Q}\oplus\mathbb{Q}\oplus(\mathbf{0})\oplus\mathbb{Q}\oplus\cdots, \cdots$, $Q_1 = \mathbb{Q}\oplus(\mathbf{0})\oplus(\mathbf{0})\oplus\cdots$, $Q_2 = (\mathbf{0})\oplus\mathbb{Q}\oplus(\mathbf{0})\oplus\cdots$, $Q_3 = (\mathbf{0})\oplus(\mathbf{0})\oplus\mathbb{Q}\oplus(\mathbf{0})\oplus\cdots, \cdots$, and $Y = \bigcap_{i\in\mathbb{N}}(V(P_i)\cup V(Q_i))$. One can see that $P_1, P_2, \cdots$ are minimal elements of Y. Hence Y is a closed subset of $Spec_R(M)$ with infinitely many irreducible components by Corollary 2.4 (a).

(b) This follows from part (a), [7, Propositon 5, p. 95], and the fact that $V(N) \cap V(K) = V(N + K)$ for every submodules N and K of M.

□

Corollary 2.7. *Let M be an R-module with $dim(M) < \infty$. Then the following hold.*

(a) *If M has property (FC), then $Spec_R(M)$ with classical Zariski topology is a spectral space.*

(b) *If M is a top R-module with property (FP), then $Spec_R(M)$ with classical Zariski topology is a spectral space.*

Proof. Use Corollary 2.4, Theorem 2.6, and [16, Proposition 1.1].

□

Example 2.8. Set $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $K_s = \{(sn, n) : n \in \mathbb{Z}\}$, where $s \in \mathbb{Q}^*$, and $M = \mathbb{Q} \oplus \mathbb{Z}$ as $\mathbb{Z}$-module. Then we have the following.

(a) $Max_{\mathbb{Z}}(M) = \{\mathbb{Q} \oplus p_i\mathbb{Z} : i \in \mathbb{N}\}$, where p_i is a prime number.

(b) $Spec_{\mathbb{Z}}(M) = Max_{\mathbb{Z}}(M) \cup \{(\mathbf{0}) \oplus \mathbb{Z}, \mathbb{Q} \oplus (\mathbf{0}), (\mathbf{0}) \oplus (\mathbf{0})\} \cup \{K_s : s \in \mathbb{Q}^*\}$.

(c) $\Omega = \{\emptyset, Spec_{\mathbb{Z}}(M), V(\mathbb{Q}\oplus(\mathbf{0})), V((\mathbf{0})\oplus\mathbb{Z})\} \cup \{V(K_s), V(\mathbb{Q}\oplus\bigcap_{i\in\Lambda} p_i\mathbb{Z}), V((\mathbf{0}) \oplus \bigcap_{i\in\Lambda} p_i\mathbb{Z}), V((\mathbb{Q} \oplus \bigcap_{i\in\Lambda} p_i\mathbb{Z}) \cap K_s) |\ s \in \mathbb{Q}^*$ *and* Λ *is a finite subset of* $\mathbb{N}\}$, where Ω is a sub-basis of $(Spec_R(M), \Omega^c)$.

(d) M has both (FC) and (FP) properties. Further M is not a top module and $Spec_R(M)$ is a spectral space.

Proof. Let M be an R-module and N be a submodule of M. First we recall the following facts:

(A_1) If $(N :_R M) \in Max(R)$, then $N \in Spec_R(M)$. Also every maximal submodule is prime submodule [9].

(A_2) If $M = M_1 \oplus M_2$ and $P_1 \in Spec_p(M_1)$ with $p \in V(Ann_R(M))$, then $P_1 \oplus M_2 \in Spec_p(M)$ [12, Lemma 4.6]. (Here, for an R-module L, $Spec_p(L)$ denotes the set of all p-prime submodules of L.)

(A_3) Let R be an integral domain. If $T(M)$ (i.e., the torsion submodule of M) is a proper submodule of M, then $T(M)$ is a (0)-prime submodule of M [11, Lemma 4.5].

(A_4) Let R be an integral domain which is not a field and k the field of quotients of R. Then the R-module k has $Max(k) = \emptyset$ and $Spec_R(k) = \{(\mathbf{0})\}$ [10, Theorem 1].

(A_5) Let p be a prime ideal of R and $K \in Spec_p(M)$, then $K \cap N = N$ or $K \cap N \in Spec_p(N)$ [15, Lemma 1.6].

(a) Let $i \in \mathbb{N}$. Then $M/p_iM \cong \mathbb{Z}/p_i\mathbb{Z}$. This implies that

$$\{p_iM : i \in \mathbb{N}\} = \{\mathbb{Q} \oplus p_i\mathbb{Z} : i \in \mathbb{N}\} \subseteq Max(M).$$

Now if $K \in Max(M)$, then $(K : M) = p_j$ for some $j \in \mathbb{N}$ by A_1, whence $p_jM \subseteq K$ and, therefore, $p_jM = K$. It follows that

$$Max(M) = \{\mathbb{Q} \oplus p_i\mathbb{Z} : i \in \mathbb{N}\}.$$

(b) By A_4, $Spec_{\mathbb{Z}}(\mathbb{Q}) = \{(\mathbf{0})\}$. Thus $(\mathbf{0}) \oplus \mathbb{Z}$, $\mathbb{Q} \oplus (\mathbf{0}) \in Spec_{(0)}(M)$ by A_2. Also $(\mathbf{0}) \oplus (\mathbf{0}) \in Spec_{(0)}(M)$ by A_3. It is not difficult to see that K_s is a (0)-prime submodule of M for every $s \in \mathbb{Q}^*$. Now let K be a (0)-prime submodule of M. We can assume that $K \neq (\mathbf{0}) \oplus (\mathbf{0})$. Now we consider the following cases:

(I) $\mathbb{Q} \oplus (\mathbf{0}) \nsubseteq K$ and $(\mathbf{0}) \oplus \mathbb{Z} \subseteq K$. In this case we have $(\mathbb{Q} \oplus (\mathbf{0})) \cap K \in Spec_{(0)}(\mathbb{Q} \oplus (\mathbf{0}))$ by A_5. Hence $(\mathbb{Q} \oplus (\mathbf{0})) \cap K = (\mathbf{0}) \oplus (\mathbf{0})$. Thus we have $K = K \cap M = (K \cap (\mathbb{Q} \oplus (\mathbf{0}))) \oplus (K \cap ((\mathbf{0}) \oplus \mathbb{Z})) = (\mathbf{0}) \oplus \mathbb{Z}$.

(II) $\mathbb{Q} \oplus (\mathbf{0}) \subseteq K$ and $(\mathbf{0}) \oplus \mathbb{Z} \nsubseteq K$. Similarly we have $K = \mathbb{Q} \oplus (\mathbf{0})$.

(III) $\mathbb{Q} \oplus (\mathbf{0}) \nsubseteq K$ and $(\mathbf{0}) \oplus \mathbb{Z} \nsubseteq K$. In this case we have $(\mathbb{Q} \oplus (\mathbf{0})) \cap K = ((\mathbf{0}) \oplus \mathbb{Z}) \cap K = (\mathbf{0}) \oplus (\mathbf{0})$. This implies that for every non-zero element (x, y) of K, we have $x \neq 0$ and $y \neq 0$. Now we show that $K = K_t$ for some $t \in \mathbb{Q}^*$. To see this, set $A = \{x/y | \, 0 \neq (x, y) \in K\}$ and $K_A = \bigcup_{s \in A} K_s$. It is enough to prove that $K = K_A$ and A is a singleton subset of $\mathbb{Q}^*$. To see the first assumption, let (x, y) be a non-zero element of K. By the above mentioned, we have $y \neq 0$ and therefore $(x, y) = ((x/y)y, y) \in K_{(x/y)}$. Hence $K \subseteq K_A$. Now let $(x, y) \in K_A$. Then $(x, y) \in K_s$ for some $s \in A$. Consequently, $x/y = s = x_1/y_1$ with $(x_1, y_1) \in K$. Set $x = r/s$ and $x_1 = r_1/s_1$ where $r, s, r_1, s_1 \in \mathbb{Z} \setminus \{0\}$. It is easy to see that $syr_1(r, sy) \in K$. Since K is a (0)-prime submodule of M, $(r, sy) \in K$, whence $s(r/s, y) \in K$. This implies that $(x, y) \in K$ as desired. To prove the next assertion let a/b and c/d be two elements of A. Then there exists the elements r, s, p, and q of $\mathbb{Z}$ such that $a = r/s$ and $c = p/q$. It is not difficult to see that $(0, psbqdsb - rqdqdsb) \in K_A$. This in turn implies that $a/b = c/d$ as desired.

(c) We can see that the following assertions:

(B_1) $\forall s, t \in \mathbb{Q}^* : s \neq t \Longrightarrow K_s \cap K_t = (\mathbf{0}) \oplus (\mathbf{0})$.

(B_2) $\forall s \in \mathbb{Q}^* : K_s \cap (\mathbb{Q} \oplus (\mathbf{0})) = K_s \cap (\mathbb{Z} \oplus (\mathbf{0})) = (\mathbf{0}) \oplus (\mathbf{0})$.

(B_3) $\forall s \in \mathbb{Q}^*, \forall \Lambda \subseteq \mathbb{N}$: if Λ is finite, then

$$V((\mathbb{Q} \oplus \bigcap_{i\in\Lambda} p_i\mathbb{Z}) \cap K_s)) = \{K_s, \mathbb{Q} \oplus p_i\mathbb{Z} \,|\, i \in \Lambda\}.$$

(B_4) $V(N) = V(rad(N))$ for every submodule N of M.

Let N be a submodule of M. If $N = (\mathbf{0}) \oplus (\mathbf{0})$, there is nothing to prove. Hence we assume that $N \neq (\mathbf{0}) \oplus (\mathbf{0})$. If $V(N) \subseteq Max(M)$, then we consider two cases : (1) $|V(N)| = \infty$, (2) $|V(N)| < \infty$. In case (1), $N \subseteq \mathbb{Q} \oplus (\mathbf{0})$. Therefore $V(N) = V(\mathbb{Q} \oplus (\mathbf{0})) = Max(M)$ by item B_4. In case (2), we may assume that $V(N) = \{\mathbb{Q} \oplus p_1\mathbb{Z}, \mathbb{Q} \oplus p_2\mathbb{Z}, ..., \mathbb{Q} \oplus p_k\mathbb{Z}\}$. Hence $V(N) = V(\mathbb{Q} \oplus \bigcap_{i=1}^k p_i\mathbb{Z})$ by item B_4. If $V(N) \nsubseteq Max(M)$, then we can assume $V(N) = \{\mathbb{Q} \oplus p_1\mathbb{Z}, \mathbb{Q} \oplus p_2\mathbb{Z}, ..., \mathbb{Q} \oplus p_k\mathbb{Z}, K_s\}$ for some $s \in \mathbb{Q}^*$ by items B_1 and B_2. Now items B_3 and B_4 implies that $V(N) = V((\mathbb{Q}\oplus\bigcap_{i=1}^k p_i\mathbb{Z})\cap K_s)$.

(d) By [7, p. 97, Proposition 9], it is enough to show that every subspace of $(Spec_{\mathbb{Z}}(M), \tau^c)$ is quasi-compact. To see this, let Y be a subspace of $Spec_{\mathbb{Z}}(M)$ and let $(F_i)_{i\in I}$ be a family of closed subset of Y such that $\bigcap_{i\in I} F_i = \emptyset$. Without loss of generality we may assume that I is an infinite set. If for every $i \in I$, F_i is infinite, then since $F_i \supseteq V(\mathbb{Q} \oplus (\mathbf{0})) \cap Y$, we have $\bigcap_{i\in I} F_i \neq \emptyset$ which is a contradiction. Hence there exists $i_0 \in I$ such that F_{i_0} is finite. Set $F_{i_0} = \{a_1, a_2, ..., a_n\}$. Thus we can choose $i_1, i_2, ..., i_n$ of $I \setminus \{i_0\}$ with $a_1 \notin F_{i_1}, a_2 \notin F_{i_2}, ..., a_n \notin F_n$. This implies that $\bigcap_{k=0}^n F_{i_k} = \emptyset$, as desired. Therefore, $Spec_{\mathbb{Z}}(M)$ is Noetherian and so M has (FC) and (FP) properties by [7, p. 98, Proposition 10] and Theorem 2.6. Moreover, $Spec_{\mathbb{Z}}(M)$ is a spectral space by Corollary 2.4 and M is not a top R-module by [15, Theorem 5.1]. □

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