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COMPLEXES OF C-PROJECTIVE MODULES

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Abstract. Inspired by a recent work of Buchweitz and Flenner, we show that, for a semidualizing bimodule C, C–perfect complexes have the ability to detect when a ring is strongly regular. It is shown that there exists a class of modules which admit minimal resolutions of C–projective modules.

Keywords: Semidualizing, C–projective, $\mathcal{P}_C$–resolution, C–perfect complex, strongly regular.
MSC(2010): Primary: 13D05; Secondary: 16E05, 16E10.

1. **Introduction**

Let R be a left and right noetherian ring (not necessarily commutative), all modules left R–modules and C a semidualizing (R, R)–bimodule (Definition 2.1). A complex $X_\bullet$ of R–modules is said to be C*–perfect* if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \longrightarrow C \otimes_R P_n \longrightarrow C \otimes_R P_{n-1} \longrightarrow \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow 0,$$

where each P_i is a finite (i.e. finitely generated) projective R–module. The *width* of such a C–perfect complex $X_\bullet$, denoted by $\mathrm{wd}(X_\bullet)$, is defined to be the minimal length n of a complex $T_\bullet$ satisfying the above conditions. Recall from [3], a ring R is called *strongly regular* whenever there exists a non-negative integer r such that every R–perfect complex is quasiisomorphic to a direct sum of R–perfect complexes of width $\leqslant r$. Buchweitz and Flenner, in [3], characterize the commutative noetherian rings which are strongly regular.

Our first objective is to detect when a ring is strongly regular by means of C–perfect complexes (Theorem 3.8). We also prove that C–projective modules (i.e., modules of the form $C \otimes_R P$ with P projective) have the ability to detect when a ring is hereditary (Proposition 3.1).

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Our second goal is to find a class of R–modules which admit minimal resolutions of C–projective modules (see Theorem 3.10).

2. Preliminaries

Throughout, R is a left and right noetherian ring (not necessarily commutative) and let all R–modules be left R–modules. Right R–modules are identified with left modules over the opposite ring R^{op}. An (R,R)–*bimodule* M is both left and right R–module with compatible structures.

Definition 2.1. [9, Definition 2.1] An (R,R)–bimodule C is *semidualizing* if it is a finite R–module, finite R^{op}–module, and the following conditions hold.

(1) The homothety map $R \xrightarrow{{}^{R}\gamma} \mathrm{Hom}_{R^{\mathrm{op}}}(C,C)$ is an isomorphism.
(2) The homothety map $R \xrightarrow{\gamma^{R}} \mathrm{Hom}_{R}(C,C)$ is an isomorphism.
(3) $\mathrm{Ext}_{R}^{\geqslant 1}(C,C)=0$.
(4) $\mathrm{Ext}_{R^{\mathrm{op}}}^{\geqslant 1}(C,C)=0$.

Assume that R is a commutative noetherian ring, then the above definition agrees with the definition of semidualizing R–module (see e.g. [9, 2.1]). Also, every finite projective R–module of rank 1 is semidualizing (see [11, Corollary 2.2.5]).

Definition 2.2. [9, Definition 3.1] A semidualizing (R,R)–bimodule C is said to be *faithfully semidualizing* if it satisfies the following conditions
(a) If $\mathrm{Hom}_R(C,M)=0$, then $M=0$ for any R–module M;
(b) If $\mathrm{Hom}_{R^{\mathrm{op}}}(C,N)=0$, then $N=0$ for any R^{op}–module N.

Note that over a commutative noetherian ring, all semidualizing modules are faithfully semidualizing, by [9, Proposition 3.1].

For the remainder of this section C denotes a semidualizing (R,R)–bimodule. The following class of modules, is already appeared in, for example, [8], [9], and [13].

Definition 2.3. An R–module is called C–*projective* if it has the form $C\otimes_R P$ for some projective R–module P. The class of (resp. finite) C–projective modules is denoted by $\mathcal{P}_C$ (resp. $\mathcal{P}_C^f$).

A complex A of R–modules is called $\mathrm{Hom}_R(\mathcal{P}_C,-)$–exact if $\mathrm{Hom}_R(C\otimes_R P,A)$ is exact for each projective R–module P. The term $\mathrm{Hom}_R(-,\mathcal{P}_C)$–exact is defined dually.

For the notations in the next fact one may see [12, Definitions 1.4 and 1.5].
Fact 2.1. A $\mathcal{P}_C$–*resolution* of an R–module M is a complex X in $\mathcal{P}_C$ with $X_{-n}=0=\mathrm{H}_n(X)$ for all $n>0$ and $M\cong \mathrm{H}_0(X)$. The following exact sequence is the *augmented* $\mathcal{P}_C$–*resolution* of M associated to X:

$$X^{+}=\cdots \xrightarrow{\partial_2^X} C\otimes_R P_1 \xrightarrow{\partial_1^X} C\otimes_R P_0 \longrightarrow M \longrightarrow 0.$$

A $\mathcal{P}_C$–resolution X of M is called *proper* if in addition X^+ is $\mathrm{Hom}_R(\mathcal{P}_C, -)$–exact.

The *$\mathcal{P}_C$–projective dimension* of M is the quantity

$$\mathcal{P}_C - \mathrm{pd}(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{ is an} \mathcal{P}_C - \text{resolution of} M\}.$$

The objects of $\mathcal{P}_C$–projective dimension 0 are exactly C–projective R–modules.

The notion (*proper*) *$\mathcal{P}_C$–coresolution* is defined dually. The *augmented $\mathcal{P}_C$–coresolution* associated to a $\mathcal{P}_C$–coresolution Y is denoted by ${}^+Y$.

In [13], the authors proved the following proposition for a commutative ring R. However, by an easy inspection, one can see that it is true even if R is non-commutative.

Proposition 2.4. *Assume that C is a faithfully semidualizing (R, R)–bimodule and that M is an R–module. The following statements hold true.*

(a) *[13, Corollary 2.10(a)] The inequality $\mathcal{P}_C$–$\mathrm{pd}(M) \leqslant n$ holds if and only if there is a complex*

$$0 \longrightarrow C \otimes_R P_n \longrightarrow \cdots \longrightarrow C \otimes_R P_0 \longrightarrow M \longrightarrow 0$$

which is $\mathrm{Hom}_R(\mathcal{P}_C, -)$–exact.

(b) *[13, Theorem 2.11(a)] $\mathrm{pd}_R(M) = \mathcal{P}_C$–$\mathrm{pd}_R(C \otimes_R M)$.*

(c) *[13, Theorem 2.11(c)] $\mathcal{P}_C$–$\mathrm{pd}_R(M) = \mathrm{pd}_R(\mathrm{Hom}_R(C, M))$.*

Remark 2.5. By [9, Proposition 5.3] the class $\mathcal{P}_C$ is precovering, that is, for an R–module M, there exists a projective R–module P and a homomorphism $\phi : C \otimes_R P \to M$ such that, for every projective Q, the induced map

$$\mathrm{Hom}_R(C \otimes_R Q, C \otimes_R P) \xrightarrow{\mathrm{Hom}_R(C\otimes_R Q,\phi)} \mathrm{Hom}_R(C \otimes_R Q, M)$$

is surjective. Then one can iteratively take precovers to construct a complex

$$W = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \longrightarrow 0 \qquad (2.5.1)$$

such that W^+ is $\mathrm{Hom}_R(\mathcal{P}_C, -)$–exact, where

$$W^+ = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \xrightarrow{\phi} M \longrightarrow 0.$$

For the notions precovering, covering, preenveloping and enveloping one can see [6].

Note that if C is faithfully semidualizing (R, R)–bimodule and M is an R–module, then, by Proposition 2.4(a), $\mathcal{P}_C$–$\mathrm{pd}(M)$ is equal to the length of the shortest complex as (2.5.1). Thus for any R–module M, the quantity $\mathcal{P}_C$–projective dimension of M, defined in [9] and [13], is equal to $\mathcal{P}_C$–$\mathrm{pd}(M)$ in Fact 2.1.

3. Results

A ring R is (left) hereditary if every left ideal is projective. The Cartan-Eilenberg theorem [10, Theorem 4.19] shows that R is hereditary if and only if every submodule of a projective module is projective. We show that the quality of being hereditary can be detected by C–projective modules, which is interesting on its own.

Proposition 3.1. *Assume that C runs through the class of faithfully semidualizing (R, R)–bimodules. The following statements are equivalent.*

(i) *R is left hereditary.*

(ii) *For any C, every submodule of a C–projective R–module is also C–projective.*

(iii) *There exists a C such that every submodule of a C–projective R–module is also C–projective.*

Proof. (i)⇒(ii). Let C be a faithfully semidualizing bimodule and N a submodule of $C \otimes_R P$, where P is a projective R–module. Then one gets the exact sequence $0 \longrightarrow \mathrm{Hom}_R(C, N) \longrightarrow P$. As R is left hereditary, $\mathrm{Hom}_R(C, N)$ is a projective R–module. By Proposition 2.4(c), $\mathcal{P}_C\text{–pd}(N) = \mathrm{pd}(\mathrm{Hom}_R(C, N)) = 0$.

(ii)⇒(iii) is immediate.

(iii)⇒(i). As every submodule of a C–projective R–module is C–projective, for any R–module M one has $\mathcal{P}_C\text{–pd}(M) \leqslant 1$. Then for any R–module N one gets $\mathrm{pd}(N) = \mathcal{P}_C\text{–pd}(C \otimes_R N) \leqslant 1$, by Proposition 2.4(b). It follows that every submodule of a projective is projective and so, by [10, Theorem 4.19], R is left hereditary. □

Definition 3.2. A complex $X_\bullet$ of R–modules is called *C–perfect* if it is quasi-isomorphic to a finite complex

$$T_\bullet = 0 \longrightarrow C \otimes_R P_n \longrightarrow C \otimes_R P_{n-1} \longrightarrow \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow 0,$$

where P_i are finite projective R–modules. The *width* of such a C–perfect complex $X_\bullet$, denoted by $\mathrm{wd}(X_\bullet)$, is defined to be the minimal length n of a complex $T_\bullet$ satisfying the above conditions. A C–perfect complex $X_\bullet$ is called *indecomposable* if it is not quasiisomorphic to a direct sum of two non-trivial C–perfect complexes.

Definition 3.3. [3, Definition 1.1] A ring R is called *strongly r–regular* if every perfect complex over R is quasiisomorphic to a direct sum of perfect complexes of width $\leqslant r$. If R is strongly r–regular for some r then it will be called *strongly regular*.

Remark 3.4. As Professor Ragnar-Olaf Buchweitz kindly pointed out in his personal communication with the authors, in [3] it should be added the blanket statement that rings are noetherian and modules are finite. Thus Definition 3.3

agrees with [3, Definition 1.1]. Indeed, over a noetherian ring every perfect complex has bounded and finite homology.

Note that a hereditary ring R is strongly 1-regular, see [3, Remark 1.2].

In order to bring the results Theorem 3.8 and Proposition 3.9, we quote some preliminaries.

Definition 3.5. [7, III.3.2(b)] and [4, Definition 2.2.8] Let $\alpha : A \to B$ be a morphism of R–complexes. The *mapping cone* of α, $\mathrm{Cone}(\alpha)$, is a complex which is given by

$$(\mathrm{Cone}(\alpha))_n = B_n \oplus A_{n-1} \quad \text{and} \quad \partial_n^{\mathrm{Cone}(\alpha)} = \begin{pmatrix} \partial_n^B & \alpha_{n-1} \\ 0 & -\partial_{n-1}^A \end{pmatrix}.$$

It easy to see that the following lemma is also true if R is non-commutative.

Lemma 3.6. *Let $\alpha : A \to B$ be a morphism of R–complexes and M be an R–module. The following statements hold true.*

(a) *[4, Lemma 2.2.10] The morphism α is a quasiisomorphism if and only if $\mathrm{Cone}(\alpha)$ is acyclic.*
(b) *[4, Lemma 2.3.11]* $\mathrm{Cone}(\mathrm{Hom}_R(M, \alpha)) \cong \mathrm{Hom}_R(M, \mathrm{Cone}(\alpha))$.
(c) *[4, Lemma 2.4.11]* $\mathrm{Cone}(M \otimes_R \alpha) \cong M \otimes_R \mathrm{Cone}(\alpha)$.

Remark 3.7. Let C be a semidualizing (R,R)–bimodule. Assume that $X = 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0$ is an exact complex of R–modules.

(a) If each X_i is a projective R–module, then it is easy to see that the induced complex $C \otimes_R X$ is exact.
(b) If each X_i is a C–projective R–module, then the induced complex $\mathrm{Hom}_R(C, X)$ is exact, since $\mathrm{Ext}_R^{\geqslant 1}(C, X_i) = 0$.

Theorem 3.8. *The following statements are equivalent.*

(i) *R is strongly r–regular.*
(ii) *For any faithfully semidualizing bimodule C, every C–perfect complex is quasiisomorphic to a direct sum of C–perfect complexes of width $\leqslant r$.*
(iii) *There exists a faithfully semidualizing bimodule C such that every C–perfect complex is quasiisomorphic to a direct sum of C–perfect complexes of width $\leqslant r$.*

Proof. (i)$\Rightarrow$(ii). Let R be strongly r–regular, C a faithfully semidualizing bimodule. Assume that $X_\bullet$ is a C–perfect complex. Then, by Definition 3.2, there exists a finite complex

$$T_\bullet = 0 \longrightarrow C \otimes_R P_n \longrightarrow C \otimes_R P_{n-1} \longrightarrow \cdots \longrightarrow C \otimes_R P_0 \longrightarrow 0,$$

such that each P_i is a finite projective R–module and $X_\bullet$ is quasiisomorphic to $T_\bullet$. Therefore $\mathrm{Hom}_R(C, T_\bullet) \cong 0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$ is a perfect complex. By Definition 3.3, there is a quasiisomorphism α :

$\operatorname{Hom}_R(C, T_\bullet) \xrightarrow{\simeq} \bigoplus_{i=1}^{s} F_\bullet^{(i)}$, where each $F_\bullet^{(i)}$ is a perfect complex of width $\leqslant r$. We may assume that each $F_\bullet^{(i)}$ is a finite complex of finite projective R–modules. By Lemma 3.6(a), $\operatorname{Cone}(\alpha)$ is acyclic. As $\operatorname{Cone}(\alpha)$ is a finite complex of projective R–modules, Remark 3.7 implies that the complex $C \otimes_R \operatorname{Cone}(\alpha)$ is acyclic. By Lemma 3.6, the complex $\operatorname{Cone}(C \otimes_R \alpha)$ is acyclic too and so $C \otimes_R \alpha$ is quasiisomorphism. Therefore $T_\bullet$ is quasiisomorphic to $\bigoplus_{i=1}^{s} C \otimes_R F_\bullet^{(i)}$. Note that each $C \otimes_R F_\bullet^{(i)}$ is a C–perfect complex of width $\leqslant r$.

(ii)⇒(iii) is immediate.

(iii)⇒(i). Let $Y_\bullet$ be a perfect complex. Then, by Definition 3.2, there is a finite complex $F_\bullet = 0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$ of finite projective modules which is quasiisomorphic to $Y_\bullet$. As $C \otimes_R F_\bullet$ is a C–perfect complex, our assumption implies that there is a quasiisomorphism $\beta : C \otimes_R F_\bullet \xrightarrow{\simeq} \bigoplus_{i=1}^{s} T_\bullet^{(i)}$, where each $T_\bullet^{(i)}$ is a C–perfect complex of width $\leqslant r$. We may assume that, for each i,

$$T_\bullet^{(i)} = 0 \longrightarrow C \otimes_R P_{n_i}^{(i)} \longrightarrow \cdots \longrightarrow C \otimes_R P_0^{(i)} \longrightarrow 0$$

where each $P_j^{(i)}$ is a finite projective R–module. Similar to the proof of (i)⇒(ii), one observes that $\operatorname{Hom}_R(C, \beta)$ is a quasiisomorphism. Therefore $F_\bullet$ is quasiisomorphic to $\bigoplus_{i=1}^{s} \operatorname{Hom}_R(C, T_\bullet^{(i)})$. Note that each $\operatorname{Hom}_R(C, T_\bullet^{(i)})$ is a perfect complex of width $\leqslant r$. Thus R is strongly r–regular. □

In [2, Section 1], Avramov and Martsinkovsky define a general notion of minimality for complexes: A complex X is *minimal* if every homotopy equivalence $\sigma : X \longrightarrow X$ is an isomorphism. In [14, Lemma 4.8], it is proved that, over a commutative local ring R with maximal ideal $\mathfrak{m}$, a complex X consisting of modules in $\mathcal{P}_C^f$ is minimal if and only if $\partial^X(X) \subseteq \mathfrak{m}X$.

In consistent to [3, Lemma 1.6] we prove the following proposition.

Proposition 3.9. *Let R be a commutative noetherian local ring and C a semidualizing R–module. The following statements hold true.*

(a) *Every C–perfect complex $X_\bullet$ is quasiisomorphic to a minimal finite complex*

$$T_\bullet = 0 \longrightarrow C \otimes_R F_n \longrightarrow C \otimes_R F_{n-1} \longrightarrow \cdots \longrightarrow C \otimes_R F_1 \longrightarrow C \otimes_R F_0 \longrightarrow 0,$$

where each F_i is finite free R–module.

(b) *If two minimal finite complexes of modules of the form $C^m = \oplus^m C$ are quasiisomorphic, then they are isomorphic.*

Proof. (a). By Definition 3.2, a C–perfect complex $X_\bullet$ is quasiisomorphic to a finite complex

$$T_\bullet = 0 \longrightarrow C \otimes_R P_n \longrightarrow C \otimes_R P_{n-1} \longrightarrow \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow 0,$$

where each P_i is a finite free R–module. The complex $\operatorname{Hom}_R(C, T_\bullet)$ is a perfect complex and so, by [3, Lemma 1.6(1)], there exists a minimal finite complex

$F_\bullet$ of finite free R–modules and a quasiisomorphism $\alpha : \mathrm{Hom}_R(C, T_\bullet) \xrightarrow{\simeq} F_\bullet$. As in the proof of Theorem 3.8, it follows that $C \otimes_R \alpha : C \otimes_R \mathrm{Hom}_R(C, T_\bullet) \to C \otimes_R F_\bullet$ is a quasiisomorphism. As $C \otimes_R F_\bullet$ is a minimal finite complex, we are done.

(b). Let $T_\bullet$ and $L_\bullet$ be two minimal finite complexes of modules of the form C^m. Assume that $\alpha : T_\bullet \to L_\bullet$ is a quasiisomorphism. Then, by Remark 3.7 and Lemma 3.6, $\mathrm{Hom}_R(C, \alpha) : \mathrm{Hom}_R(C, T_\bullet) \to \mathrm{Hom}_R(C, L_\bullet)$ is a quasiisomorphism of minimal finite complexes of finite free R–modules. Thus, by the proof of [3, Lemma 1.6(2)], $\mathrm{Hom}_R(C, \alpha)$ is an isomorphism. Now, there is a commutative diagram of complexes and morphisms

$$\begin{array}{ccc} T_\bullet & \xrightarrow[\alpha]{\simeq} & L_\bullet \\ \uparrow \cong & & \uparrow \cong \\ C \otimes_R \mathrm{Hom}_R(C, T_\bullet) & \xrightarrow[C \otimes_R \mathrm{Hom}_R(C,\alpha)]{\cong} & C \otimes_R \mathrm{Hom}_R(C, L_\bullet), \end{array}$$

where the vertical morphisms are natural isomorphisms. This implies that α itself must be an isomorphism. □

It is proved in [14, Lemma 4.9] that every finite module M over a commutative noetherian local ring R with $\mathcal{P}_C^f$–pd$(M) < \infty$ admits a minimal $\mathcal{P}_C^f$–resolution. Now we show that every finite R–module which has a proper $\mathcal{P}_C$–resolution, admits a minimal proper one. Note that if $\mathcal{P}_C^f$–pd$(M) < \infty$ then M admits a proper $\mathcal{P}_C$–resolution (see proof of [13, Corollary 2.10]).

Theorem 3.10. *Assume that R is a commutative noetherian local ring and that C is a semidualizing R–module. Then $\mathcal{P}_C^f$ is covering in the category of finite R–modules. For any finite R–module M, there is a complex $X = \cdots \longrightarrow C^{n_1} \longrightarrow C^{n_0} \longrightarrow 0$ with the following properties.*

(1) *$X^+ = \cdots \longrightarrow C^{n_1} \longrightarrow C^{n_0} \longrightarrow M \longrightarrow 0$ is $\mathrm{Hom}_R(\mathcal{P}_C, -)$–exact.*

(2) *X is a minimal complex.*

If M admits a proper $\mathcal{P}_C$–resolution, then X^+ is exact and so X is a minimal proper $\mathcal{P}_C$–resolution of M.

Proof. Let M be a finite R–module. Assume that $n_0 = \nu(\mathrm{Hom}_R(C, M))$ denotes the number of a minimal set of generators of $\mathrm{Hom}_R(C, M)$ and that $\alpha : R^{n_0} \longrightarrow \mathrm{Hom}_R(C, M)$ is the natural epimorphism. As α is a $\mathcal{P}^f$–cover of $\mathrm{Hom}_R(C, M)$, the natural map $\beta = C \otimes_R R^{n_0} \xrightarrow{C \otimes_R \alpha} C \otimes_R \mathrm{Hom}_R(C, M) \xrightarrow{\nu_M} M$ is a $\mathcal{P}_C^f$–cover of M. Set $M_1 = \mathrm{Ker}\beta$ and $n_1 = \nu(\mathrm{Hom}_R(C, M_1))$. Thus there is a $\mathcal{P}_C^f$–cover $\beta_1 : C \otimes_R R^{n_1} \longrightarrow M_1$. Proceeding in this way one obtains a complex

$$X = \cdots \xrightarrow{\partial_2 = \epsilon_2 \beta_2} C \otimes_R R^{n_1} \xrightarrow{\partial_1 = \epsilon_1 \beta_1} C \otimes_R R^{n_0} \longrightarrow 0,$$

where $\epsilon_i : M_i \to C \otimes_R R^{n_{i-1}}$ is the inclusion map for all $i \geqslant 1$. As the maps in X are obtained by $\mathcal{P}_C^f$–covers, the complex X^+ is $\mathrm{Hom}_R(\mathcal{P}_C, -)$–exact. It is easy to see that $\mathrm{Hom}_R(C, X)$ is minimal free resolution of $\mathrm{Hom}_R(C, M)$. Now we show that X is a minimal complex. Let $f : X \to X$ be a morphism which is homotopic to id_X. It is easy to see that the morphism $\mathrm{Hom}_R(C, f)$ is homotopic to $\mathrm{id}_{\mathrm{Hom}_R(C,X)}$. As the complex $\mathrm{Hom}_R(C, X)$ is minimal, by [2, Proposition 1.7], the morphism $\mathrm{Hom}_R(C, f)$ is an isomorphism. The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ C \otimes_R \mathrm{Hom}_R(C, X) & \xrightarrow[C\otimes_R \mathrm{Hom}_R(C,f)]{\cong} & C \otimes_R \mathrm{Hom}_R(C, X), \end{array}$$

with vertical natural isomorphisms, implies that f is an isomorphism. Therefore, by [2, Proposition 1.7], X is minimal. If M admits a proper $\mathcal{P}_C$–resolution, then by [13, Corollary 2.3], X^+ is exact. □

The proof of the next lemma is similar to [13, Corollary 2.3].

Lemma 3.11. *Let R be a commutative noetherian ring and let M be a finite R–module. Assume that C is a semidualizing R–module. The following are equivalent.*

(i) *M admits a proper $\mathcal{P}_C^f$–coresolution.*

(ii) *Every $\mathrm{Hom}_R(-, \mathcal{P}_C^f)$–exact complex of the form*

$$0 \longrightarrow M \longrightarrow C \otimes_R Q_0 \longrightarrow C \otimes_R Q_{-1} \longrightarrow \cdots$$

is exact, where Q_i is an object of $\mathcal{P}^f$ for all $i \leqslant 0$.

(iii) *The natural homomorphism $M \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism and $\mathrm{Ext}_R^{\geqslant 1}(\mathrm{Hom}_R(M, C), C) = 0$.*

Proposition 3.12. *Assume that R is a commutative noetherian local ring and that C is a semidualizing R–module. Then $\mathcal{P}_C^f$ is enveloping in the category of finite R–modules. For any finite R–module M, there is a complex $Y = 0 \longrightarrow C^{m_0} \longrightarrow C^{m_1} \longrightarrow \cdots$ with the following properties.*

(1) *${}^+Y = 0 \longrightarrow M \longrightarrow C^{m_0} \longrightarrow C^{m_1} \longrightarrow \cdots$ is $\mathrm{Hom}_R(-, \mathcal{P}_C)$–exact.*

(2) *Y is a minimal complex.*

If M admits a proper $\mathcal{P}_C^f$–coresolution, then ${}^+Y$ is exact and so Y is a minimal proper $\mathcal{P}_C$–coresolution of M.

Proof. Let M be a finite R–module. Assume that $m_0 = \nu(\mathrm{Hom}_R(M, C))$ denotes the number of a minimal set of generators of $\mathrm{Hom}_R(M, C)$ and that $\alpha : R^{m_0} \longrightarrow \mathrm{Hom}_R(M, C)$ is the natural $\mathcal{P}^f$–cover of $\mathrm{Hom}_R(M, C)$. It follows that $\gamma = M \xrightarrow{\delta_M} \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C) \xrightarrow{\mathrm{Hom}_R(\alpha, C)} \mathrm{Hom}_R(R^{m_0}, C)$ is a $\mathcal{P}_C^f$–envelope of M. Set $M_{-1} = \mathrm{Coker}\gamma$ and $m_1 = \nu(\mathrm{Hom}_R(M_{-1}, C))$. As mentioned, there is a $\mathcal{P}_C^f$–envelope $\gamma_1 : M_{-1} \longrightarrow \mathrm{Hom}_R(R^{m_1}, C)$. Proceeding in this way one obtains a complex $Y = 0 \longrightarrow \mathrm{Hom}_R(R^{m_0}, C) \xrightarrow{\partial_0=\gamma_1\pi_1} \mathrm{Hom}_R(R^{m_1}, C) \xrightarrow{\partial_{-1}=\gamma_2\pi_2} \cdots$, where π_i is the natural epimorphism for all $i \geqslant 1$. Since the maps in Y are obtained by $\mathcal{P}_C^f$–envelopes, the complex ^+Y is $\mathrm{Hom}_R(-, \mathcal{P}_C)$–exact. It is easy to see that $\mathrm{Hom}_R(Y, C)$ is minimal free resolution of $\mathrm{Hom}_R(M, C)$. Similar to the proof of Theorem 3.10, we find that Y is a minimal complex. If M admits a proper $\mathcal{P}_C^f$–coresolution, then, by Lemma 3.11, ^+Y is exact. □

In the following example we find an R–module M with $\mathcal{P}_C$–pd$(M) = \infty$ which admits a minimal proper $\mathcal{P}_C$–resolution. This example shows that a commutative noetherian local ring which admits an exact zero-divisor is not a strongly regular ring.

Example 3.13. Let R be a commutative noetherian local ring and C a semidualizing R–module. Assume that x, y form a pair of exact zero-divisors on both R and C (e.g. see [1, Example 3.2]). Then $\mathcal{P}_C$–pd$(C/xC) = \mathrm{pd}(R/xR) = \infty$. The complex

$$T_\bullet = \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \longrightarrow 0 \text{ (resp. } L_\bullet = 0 \longrightarrow C \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} \cdots)$$

is a minimal $\mathcal{P}_C$–resolution (resp. $\mathcal{P}_C$–coresolution) of C/xC. By [1, Proposition 3.4], C/xC is a semidualizing R/xR–module. By [5, Proposition 2.13], there are isomorphisms

$$\mathrm{Hom}_R(C, C/xC) \cong \mathrm{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR,$$

$$\mathrm{Hom}_R(C/xC, C) \cong \mathrm{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR.$$

Applying $\mathrm{Hom}_R(C, -)$ and $\mathrm{Hom}_R(-, C)$ on the above complexes, respectively, would result the isomorphisms $\mathrm{Hom}_R(C, T_\bullet^+) \cong F_\bullet^+$ and $\mathrm{Hom}_R(^+L_\bullet, C) \cong F_\bullet^+$, where $F_\bullet^+$ is the exact complex $\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0$. Therefore $T_\bullet$ (resp. $L_\bullet$) is a minimal proper $\mathcal{P}_C$–resolution (resp. $\mathcal{P}_C$–coresolution) of C/xC.

For each n, one obtains a C–perfect complex of length n as

$$T_\bullet^{(n)} = 0 \longrightarrow C \longrightarrow C \longrightarrow \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \longrightarrow 0,$$

where $T_i^{(n)} = T_i$ for all $0 \leq i \leq n$ and $T_i^{(n)} = 0$ otherwise. Note that the induced map $\bar{d}_i : T_i^{(n)}/\mathrm{Ker}\, d_i \to T_{i-1}^{(n)}$ is injective, where $\mathrm{Ker}\, d_i$ is equal to yC or xC. As C is indecomposable R–module, $T_\bullet^{(n)}$ is indecomposable which has a similar proof to [3, Proposition 1.5].

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