Title:
Complexes of $C$-projective modules

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COMPLEXES OF $C$-PROJECTIVE MODULES

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Abstract. Inspired by a recent work of Buchweitz and Flenner, we show that, for a semidualizing bimodule $C$, $C$–perfect complexes have the ability to detect when a ring is strongly regular. It is shown that there exists a class of modules which admit minimal resolutions of $C$–projective modules.

Keywords: Semidualizing, $C$–projective, $P_C$–resolution, $C$–perfect complex, strongly regular.


1. Introduction

Let $R$ be a left and right noetherian ring (not necessarily commutative), all modules left $R$–modules and $C$ a semidualizing $(R, R)$–bimodule (Definition 2.1). A complex $X_\bullet$ of $R$–modules is said to be $C$–perfect if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to 0,$$

where each $P_i$ is a finite (i.e. finitely generated) projective $R$–module. The width of such a $C$–perfect complex $X_\bullet$, denoted by $\text{wd}(X_\bullet)$, is defined to be the minimal length $n$ of a complex $T_\bullet$ satisfying the above conditions. Recall from [3], a ring $R$ is called strongly regular whenever there exists a non-negative integer $r$ such that every $R$–perfect complex is quasiisomorphic to a direct sum of $R$–perfect complexes of width $\leq r$. Buchweitz and Flenner, in [3], characterize the commutative noetherian rings which are strongly regular.

Our first objective is to detect when a ring is strongly regular by means of $C$–perfect complexes (Theorem 3.8). We also prove that $C$–projective modules (i.e., modules of the form $C \otimes_R P$ with $P$ projective) have the ability to detect when a ring is hereditary (Proposition 3.1).
Our second goal is to find a class of \( R \)-modules which admit minimal resolutions of \( C \)-projective modules (see Theorem 3.10).

2. Preliminaries

Throughout, \( R \) is a left and right noetherian ring (not necessarily commutative) and let all \( R \)-modules be left \( R \)-modules. Right \( R \)-modules are identified with left modules over the opposite ring \( R^{op} \). An \((R, R)\)-bimodule \( M \) is both left and right \( R \)-module with compatible structures.

**Definition 2.1.** \([9, \	ext{Definition 2.1}]\) An \((R, R)\)-bimodule \( C \) is semidualizing if it is a finite \( R \)-module, finite \( R^{op} \)-module, and the following conditions hold.

1. The homothety map \( R \xrightarrow{R} \text{Hom}_{R^{op}}(C, C) \) is an isomorphism.
2. The homothety map \( R \xrightarrow{R} \text{Hom}_{R}(C, C) \) is an isomorphism.
3. \( \text{Ext}^{\geq 1}_{R}(C, C) = 0 \).
4. \( \text{Ext}^{\geq 1}_{R^{op}}(C, C) = 0 \).

Assume that \( R \) is a commutative noetherian ring, then the above definition agrees with the definition of semidualizing \( R \)-module (see e.g. \( [9, 2.1] \)). Also, every finite projective \( R \)-module of rank 1 is semidualizing (see \([11, \text{Corollary 2.2.5}] \)).

**Definition 2.2.** \([9, \text{Definition 3.1}]\) A semidualizing \((R, R)\)-bimodule \( C \) is said to be faithfully semidualizing if it satisfies the following conditions

(a) If \( \text{Hom}_{R}(C, M) = 0 \), then \( M = 0 \) for any \( R \)-module \( M \);
(b) If \( \text{Hom}_{R^{op}}(C, N) = 0 \), then \( N = 0 \) for any \( R^{op} \)-module \( N \).

Note that over a commutative noetherian ring, all semidualizing modules are faithfully semidualizing, by \([9, \text{Proposition 3.1}] \).

For the remainder of this section \( C \) denotes a semidualizing \((R, R)\)-bimodule. The following class of modules, is already appeared in, for example, \([8], [9], \) and \([13] \).

**Definition 2.3.** An \( R \)-module is called \( C \)-projective if it has the form \( C \otimes_{R} P \) for some projective \( R \)-module \( P \). The class of (resp. finite) \( C \)-projective modules is denoted by \( \mathcal{P}_{C} \) (resp. \( \mathcal{P}_{C}^{f} \)).

A complex \( A \) of \( R \)-modules is called \( \text{Hom}_{R}(\mathcal{P}_{C}, -) \)-exact if \( \text{Hom}_{R}(C \otimes_{R} P, A) \) is exact for each projective \( R \)-module \( P \). The term \( \text{Hom}_{R}(-, \mathcal{P}_{C}) \)-exact is defined dually.

For the notations in the next fact one may see \([12, \text{Definitions 1.4 and 1.5}] \).

**Fact 2.1.** A \( \mathcal{P}_{C} \)-resolution of an \( R \)-module \( M \) is a complex \( X \) in \( \mathcal{P}_{C} \) with \( X_{-n} = 0 = H_{n}(X) \) for all \( n > 0 \) and \( M \cong H_{0}(X) \). The following exact sequence is the augmented \( \mathcal{P}_{C} \)-resolution of \( M \) associated to \( X \):

\[
X^{+} = \cdots \xrightarrow{\partial^{X}_{n}} C \otimes_{R} P_{1} \xrightarrow{\partial^{X}_{0}} C \otimes_{R} P_{0} \rightarrow M \rightarrow 0.
\]
A $\mathcal{P}_C$–resolution $X$ of $M$ is called **proper** if in addition $X^+$ is $\text{Hom}_R(\mathcal{P}_C, -)$–exact.  

The $\mathcal{P}_C$–**projective dimension** of $M$ is the quantity 

$$\mathcal{P}_C - \text{pd}(M) = \inf \{ \sup \{ n \geq 0 : X_n \neq 0 \} \mid X \text{ is an } \mathcal{P}_C - \text{resolution of } M \}.$$  

The objects of $\mathcal{P}_C$–projective dimension 0 are exactly $\mathcal{P}_C$–projective $R$–modules.  

The notion (proper) $\mathcal{P}_C$–coresolution is defined dually. The **augmented $\mathcal{P}_C$–coresolution** associated to a $\mathcal{P}_C$–coresolution $Y$ is denoted by $^+Y$.

In [13], the authors proved the following proposition for a commutative ring $R$. However, by an easy inspection, one can see that it is true even if $R$ is non-commutative.

**Proposition 2.4.** Assume that $C$ is a faithfully semidualizing $(R, R)$–bimodule and that $M$ is an $R$–module. The following statements hold true.

(a) [13, Corollary 2.10(a)] The inequality $\mathcal{P}_C - \text{pd}(M) \leq n$ holds if and only if there is a complex

$$0 \rightarrow C \otimes_R P_n \rightarrow \cdots \rightarrow C \otimes_R P_0 \rightarrow M \rightarrow 0$$

which is $\text{Hom}_R(\mathcal{P}_C, -)$–exact.

(b) [13, Theorem 2.11(a)] $\text{pd}_R(M) = \mathcal{P}_C - \text{pd}_R(C \otimes_R M)$.

(c) [13, Theorem 2.11(c)] $\mathcal{P}_C - \text{pd}_R(M) = \text{pd}_R(\text{Hom}_R(C, M))$.

**Remark 2.5.** By [9, Proposition 5.3] the class $\mathcal{P}_C$ is precovering, that is, for an $R$–module $M$, there exists a projective $R$–module $P$ and a homomorphism $\phi : C \otimes_R P \rightarrow M$ such that, for every projective $Q$, the induced map

$$\text{Hom}_R(C \otimes_R Q, C \otimes_R P) \xrightarrow{\text{Hom}_R(C \otimes_R Q, \phi)} \text{Hom}_R(C \otimes_R Q, M)$$

is surjective. Then one can iteratively take precovers to construct a complex

$$W = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \rightarrow 0$$

such that $W^+$ is $\text{Hom}_R(\mathcal{P}_C, -)$–exact, where

$$W^+ = \cdots \xrightarrow{\partial_2^X} C \otimes_R P_1 \xrightarrow{\partial_1^X} C \otimes_R P_0 \xrightarrow{\phi} M \rightarrow 0.$$  

For the notions precovering, covering, preenveloping and enveloping one can see [6].

Note that if $C$ is faithfully semidualizing $(R, R)$–bimodule and $M$ is an $R$–module, then, by Proposition 2.4(a), $\mathcal{P}_C - \text{pd}(M)$ is equal to the length of the shortest complex as (2.5.1). Thus for any $R$–module $M$, the quantity $\mathcal{P}_C$–projective dimension of $M$, defined in [9] and [13], is equal to $\mathcal{P}_C - \text{pd}(M)$ in Fact 2.1.
3. Results

A ring $R$ is (left) hereditary if every left ideal is projective. The Cartan-Eilenberg theorem [10, Theorem 4.19] shows that $R$ is hereditary if and only if every submodule of a projective module is projective. We show that the quality of being hereditary can be detected by $C$–projective modules, which is interesting on its own.

**Proposition 3.1.** Assume that $C$ runs through the class of faithfully semidualizing $(R, R)$–bimodules. The following statements are equivalent.

(i) $R$ is left hereditary.

(ii) For any $C$, every submodule of a $C$–projective $R$–module is also $C$–projective.

(iii) There exists a $C$ such that every submodule of a $C$–projective $R$–module is also $C$–projective.

**Proof.** (i)$\Rightarrow$(ii). Let $C$ be a faithfully semidualizing bimodule and $N$ a submodule of $C \otimes_R P$, where $P$ is a projective $R$–module. Then one gets the exact sequence $0 \rightarrow \text{Hom}_R(C, N) \rightarrow P$. As $R$ is left hereditary, $\text{Hom}_R(C, N)$ is a projective $R$–module. By Proposition 2.4(c), $P_{C\text{-pd}}(N) = \text{pd}(\text{Hom}_R(C, N)) = 0$.

(ii)$\Rightarrow$(iii) is immediate.

(iii)$\Rightarrow$(i). As every submodule of a $C$–projective $R$–module is $C$–projective, for any $R$–module $M$ one has $P_{C\text{-pd}}(M) \leq 1$. Then for any $R$–module $N$ one gets $\text{pd}(N) = P_{C\text{-pd}}(C \otimes_R N) \leq 1$, by Proposition 2.4(b). It follows that every submodule of a projective is projective and so, by [10, Theorem 4.19], $R$ is left hereditary. □

**Definition 3.2.** A complex $X_\bullet$ of $R$–modules is called $C$–perfect if it is quasiisomorphic to a finite complex

$$T_\bullet = 0 \rightarrow C \otimes_R P_n \rightarrow C \otimes_R P_{n-1} \rightarrow \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow 0,$$

where $P_i$ are finite projective $R$–modules. The width of such a $C$–perfect complex $X_\bullet$, denoted by $\text{wd}(X_\bullet)$, is defined to be the minimal length $n$ of a complex $T_\bullet$ satisfying the above conditions. A $C$–perfect complex $X_\bullet$ is called indecomposable if it is not quasiisomorphic to a direct sum of two non-trivial $C$–perfect complexes.

**Definition 3.3.** [3, Definition 1.1] A ring $R$ is called strongly $r$–regular if every perfect complex over $R$ is quasiisomorphic to a direct sum of perfect complexes of width $\leq r$. If $R$ is strongly $r$–regular for some $r$ then it will be called strongly regular.

**Remark 3.4.** As Professor Ragnar-Olaf Buchweitz kindly pointed out in his personal communication with the authors, in [3] it should be added the blanket statement that rings are noetherian and modules are finite. Thus Definition 3.3
agrees with [3, Definition 1.1]. Indeed, over a noetherian ring every perfect complex has bounded and finite homology.

Note that a hereditary ring $R$ is strongly 1-regular, see [3, Remark 1.2].

In order to bring the results Theorem 3.8 and Proposition 3.9, we quote some preliminaries.

**Definition 3.5.** [7, III.3.2(b)] and [4, Definition 2.2.8] Let $\alpha : A \to B$ be a morphism of $R$–complexes. The mapping cone $\text{Cone}(\alpha)$, is a complex which is given by

$$(\text{Cone}(\alpha))_n = B_n \oplus A_{n-1} \quad \text{and} \quad \partial_n^{\text{Cone}(\alpha)} = \begin{pmatrix} \partial_n^B & \alpha_{n-1} \\ 0 & -\partial_{n-1}^A \end{pmatrix}.$$ 

It easy to see that the following lemma is also true if $R$ is non-commutative.

**Lemma 3.6.** Let $\alpha : A \to B$ be a morphism of $R$–complexes and $M$ be an $R$–module. The following statements hold true.

(a) [4, Lemma 2.2.10] The morphism $\alpha$ is a quasiisomorphism if and only if $\text{Cone}(\alpha)$ is acyclic.

(b) [4, Lemma 2.3.11] $\text{Cone}(\text{Hom}_R(M, \alpha)) \cong \text{Hom}_R(M, \text{Cone}(\alpha))$.

(c) [4, Lemma 2.4.11] $\text{Cone}(M \otimes_R \alpha) \cong M \otimes_R \text{Cone}(\alpha)$.

**Remark 3.7.** Let $C$ be a semidualizing $(R, R)$–bimodule. Assume that $X = 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0$ is an exact complex of $R$–modules.

(a) If each $X_i$ is a projective $R$–module, then it is easy to see that the induced complex $C \otimes_R X$ is exact.

(b) If each $X_i$ is a $C$–projective $R$–module, then the induced complex $\text{Hom}_R(C, X)$ is exact, since $\text{Ext}^{\geq 1}_R(C, X_i) = 0$.

**Theorem 3.8.** The following statements are equivalent.

(i) $R$ is strongly $r$–regular.

(ii) For any faithfully semidualizing bimodule $C$, every $C$–perfect complex is quasiisomorphic to a direct sum of $C$–perfect complexes of width $\leq r$.

(iii) There exists a faithfully semidualizing bimodule $C$ such that every $C$–perfect complex is quasiisomorphic to a direct sum of $C$–perfect complexes of width $\leq r$.

**Proof.** (i)$\Rightarrow$(ii). Let $R$ be strongly $r$–regular, $C$ a faithfully semidualizing bimodule. Assume that $X_\bullet$ is a $C$–perfect complex. Then, by Definition 3.2, there exists a finite complex $T_\bullet = 0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to C \otimes_R P_0 \to 0$, such that each $P_i$ is a finite projective $R$–module and $X_\bullet$ is quasiisomorphic to $T_\bullet$. Therefore $\text{Hom}_R(C, T_\bullet) \cong 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0$ is a perfect complex. By Definition 3.3, there is a quasiisomorphism $\alpha :
Hom$_R(C, T_\bullet) \xrightarrow{\cong} \bigoplus_{i=1}^r F^{(i)}_\bullet$, where each $F^{(i)}_\bullet$ is a perfect complex of width $\leq r$. We may assume that each $F^{(i)}_\bullet$ is a finite complex of finite projective $R$-modules. By Lemma 3.6(a), Cone($\alpha$) is acyclic. As Cone($\alpha$) is a finite complex of projective $R$-modules, Remark 3.7 implies that the complex $C \otimes_R$Cone($\alpha$) is acyclic. By Lemma 3.6, the complex Cone($C \otimes_R \alpha$) is acyclic too and so $C \otimes_R \alpha$ is quasiisomorphic. Therefore $T_\bullet$ is quasiisomorphic to $\bigoplus_{i=1}^r C \otimes_R F^{(i)}_\bullet$. Note that each $C \otimes_R F^{(i)}_\bullet$ is a $C$-perfect complex of width $\leq r$.

(ii)$\Rightarrow$(iii) is immediate.

(iii)$\Rightarrow$(i). Let $Y_\bullet$ be a perfect complex. Then, by Definition 3.2, there is a finite complex $F_\bullet = 0 \rightarrow P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_0 \rightarrow 0$ of finite projective modules which is quasiisomorphic to $Y_\bullet$. As $C \otimes_R F_\bullet$ is a $C$-perfect complex, our assumption implies that there is a quasiisomorphism $\beta : C \otimes_R F_\bullet \xrightarrow{\cong} \bigoplus_{i=1}^r T^{(i)}_\bullet$, where each $T^{(i)}_\bullet$ is a $C$-perfect complex of width $\leq r$. We may assume that, for each $i$,

$$T^{(i)}_\bullet = 0 \rightarrow C \otimes_R P^{(i)}_{n_i} \rightarrow \cdots \rightarrow C \otimes_R P^{(i)}_0 \rightarrow 0$$

where each $P^{(i)}_j$ is a finite projective $R$-module. Similar to the proof of (i)$\Rightarrow$(ii), one observes that $\text{Hom}_R(C, \beta)$ is a quasiisomorphism. Therefore $F_\bullet$ is quasiisomorphic to $\bigoplus_{i=1}^r \text{Hom}_R(C, T^{(i)}_\bullet)$. Note that each $\text{Hom}_R(C, T^{(i)}_\bullet)$ is a perfect complex of width $\leq r$. Thus $R$ is strongly $r$-regular. $\square$

In [2, Section 1], Avramov and Martsinkovsky define a general notion of minimality for complexes: A complex $X$ is minimal if every homotopy equivalence $\sigma : X \rightarrow X$ is an isomorphism. In [14, Lemma 4.8], it is proved that, over a commutative local ring $R$ with maximal ideal $m$, a complex $X$ consisting of modules in $\mathcal{P}^f_C$ is minimal if and only if $\partial^X(X) \subseteq mX$.

In consistent to [3, Lemma 1.6] we prove the following proposition.

**Proposition 3.9.** Let $R$ be a commutative noetherian local ring and $C$ a semidualizing $R$-module. The following statements hold true.

(a) Every $C$-perfect complex $X_\bullet$ is quasiisomorphic to a minimal finite complex

$$T_\bullet = 0 \rightarrow C \otimes_R F_n \rightarrow C \otimes_R F_{n-1} \rightarrow \cdots \rightarrow C \otimes_R F_1 \rightarrow C \otimes_R F_0 \rightarrow 0,$$

where each $F_i$ is finite free $R$-module.

(b) If two minimal finite complexes of modules of the form $C^m = \oplus^m C$ are quasiisomorphic, then they are isomorphic.

**Proof.** (a). By Definition 3.2, a $C$-perfect complex $X_\bullet$ is quasiisomorphic to a finite complex

$$T_\bullet = 0 \rightarrow C \otimes_R P_n \rightarrow C \otimes_R P_{n-1} \rightarrow \cdots \rightarrow C \otimes_R P_1 \rightarrow C \otimes_R P_0 \rightarrow 0,$$

where each $P_i$ is a finite free $R$-module. The complex $\text{Hom}_R(C, T_\bullet)$ is a perfect complex and so, by [3, Lemma 1.6(1)], there exists a minimal finite complex
Let \( F_\bullet \) of finite free \( R \)-modules and a quasiisomorphism \( \alpha : \text{Hom}_R(C, T_\bullet) \xrightarrow{\cong} F_\bullet \).

As in the proof of Theorem 3.8, it follows that \( C \otimes_R \alpha : C \otimes_R \text{Hom}_R(C, T_\bullet) \rightarrow C \otimes_R F_\bullet \) is a quasiisomorphism. As \( C \otimes_R F_\bullet \) is a minimal finite complex, we are done.

(b). Let \( T_\bullet \) and \( L_\bullet \) be two minimal finite complexes of modules of the form \( C^m \).

Assume that \( \alpha : T_\bullet \rightarrow L_\bullet \) is a quasiisomorphism. Then, by Remark 3.6 and Lemma 3.7, \( \text{Hom}_R(C, \alpha) : \text{Hom}_R(C, T_\bullet) \rightarrow \text{Hom}_R(C, L_\bullet) \) is a quasiisomorphism of minimal finite complexes of finite free \( R \)-modules.

Thus, by the proof of [3, Lemma 1.6(2)], \( \text{Hom}_R(C, \alpha) \) is an isomorphism. Now, there is a commutative diagram of complexes and morphisms

\[
\begin{array}{ccc}
T_\bullet & \xrightarrow{\alpha} & L_\bullet \\
\cong & & \cong \\
C \otimes_R \text{Hom}_R(C, T_\bullet) & \xrightarrow{\cong} & C \otimes_R \text{Hom}_R(C, L_\bullet),
\end{array}
\]

where the vertical morphisms are natural isomorphisms. This implies that \( \alpha \) itself must be an isomorphism. \( \square \)

It is proved in [14, Lemma 4.9] that every finite module \( M \) over a commutative noetherian local ring \( R \) with \( \mathcal{P}_C^f\)-pd\( (M) < \infty \) admits a minimal \( \mathcal{P}_C^f \)-resolution. Now we show that every finite \( R \)-module which has a proper \( \mathcal{P}_C^f \)-resolution, admits a minimal proper one. Note that if \( \mathcal{P}_C^f\)-pd\( (M) < \infty \) then \( M \) admits a proper \( \mathcal{P}_C^f \)-resolution (see proof of [13, Corollary 2.10]).

**Theorem 3.10.** Assume that \( R \) is a commutative noetherian local ring and that \( C \) is a semidualizing \( R \)-module. Then \( \mathcal{P}_C^f \) is covering in the category of finite \( R \)-modules. For any finite \( R \)-module \( M \), there is a complex \( X = \cdots \rightarrow C^{n_1} \rightarrow C^{n_0} \rightarrow 0 \) with the following properties.

1. \( X^+ = \cdots \rightarrow C^{n_1} \rightarrow C^{n_0} \rightarrow M \rightarrow 0 \) is \( \text{Hom}_R(\mathcal{P}_C, -) \)-exact.
2. \( X \) is a minimal complex.

If \( M \) admits a proper \( \mathcal{P}_C^f \)-resolution, then \( X^+ \) is exact and so \( X \) is a minimal proper \( \mathcal{P}_C^f \)-resolution of \( M \).

**Proof.** Let \( M \) be a finite \( R \)-module. Assume that \( n_0 = \nu(\text{Hom}_R(C, M)) \) denotes the number of a minimal set of generators of \( \text{Hom}_R(C, M) \) and that \( \alpha : R^{n_0} \rightarrow \text{Hom}_R(C, M) \) is the natural epimorphism. As \( \alpha \) is a \( \mathcal{P}_C^f \)-cover of \( \text{Hom}_R(C, M) \), the natural map \( \beta = C \otimes_R R^{n_0} \xrightarrow{\alpha \otimes_R} C \otimes_R \text{Hom}_R(C, M) \xrightarrow{\beta_1} M \) is a \( \mathcal{P}_C^f \)-cover of \( M \). Set \( M_1 = \text{Ker} \beta \) and \( n_1 = \nu(\text{Hom}_R(C, M_1)) \). Thus there is a \( \mathcal{P}_C^f \)-cover \( \beta_1 : C \otimes_R R^{n_1} \rightarrow M_1 \). Proceeding in this way one obtains a complex

\[
X = \cdots \xrightarrow{\partial_2 = \epsilon_2 \beta_2} C \otimes_R R^{n_1} \xrightarrow{\partial_1 = \epsilon_1 \beta_1} C \otimes_R R^{n_0} \rightarrow 0,
\]
where $\epsilon_i : M_i \to C \otimes_R R^{m_{i-1}}$ is the inclusion map for all $i \geq 1$. As the maps in $X$ are obtained by $P_C^i$-covers, the complex $X^+$ is $\text{Hom}_R(P_C, -)$-exact. It is easy to see that $\text{Hom}_R(C, X)$ is minimal free resolution of $\text{Hom}_R(C, M)$. Now we show that $X$ is a minimal complex. Let $f : X \to X$ be a morphism which is homotopic to $\text{id}_X$. It is easy to see that the morphism $\text{Hom}_R(C, f)$ is homotopic to $\text{id}_{\text{Hom}_R(C, X)}$. As the complex $\text{Hom}_R(C, X)$ is minimal, by [2, Proposition 1.7], the morphism $\text{Hom}_R(C, f)$ is an isomorphism. The commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{\cong} & & \downarrow{\cong} \\
C \otimes_R \text{Hom}_R(C, X) & \xrightarrow{\cong} & C \otimes_R \text{Hom}_R(C, X),
\end{array}
$$

with vertical natural isomorphisms, implies that $f$ is an isomorphism. Therefore, by [2, Proposition 1.7], $X$ is minimal. If $M$ admits a proper $P_C$-resolution, then by [13, Corollary 2.3], $X^+$ is exact. \(\square\)

The proof of the next lemma is similar to [13, Corollary 2.3].

**Lemma 3.11.** Let $R$ be a commutative noetherian ring and let $M$ be a finite $R$-module. Assume that $C$ is a semidualizing $R$-module. The following are equivalent.

(i) $M$ admits a proper $P_C^i$-coresolution.

(ii) Every $\text{Hom}_R(-, P_C^i)$-exact complex of the form

$$
0 \to M \to C \otimes_R Q_0 \to C \otimes_R Q_{-1} \to \cdots
$$

is exact, where $Q_i$ is an object of $P_C^i$ for all $i \leq 0$.

(iii) The natural homomorphism $M \to \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism and $\text{Ext}^1_R(\text{Hom}_R(M, C), C) = 0$.

**Proposition 3.12.** Assume that $R$ is a commutative noetherian local ring and that $C$ is a semidualizing $R$-module. Then $P_C^i$ is enveloping in the category of finite $R$-modules. For any finite $R$-module $M$, there is a complex $Y = 0 \to C^{m_0} \to C^{m_1} \to \cdots$ with the following properties.

(1) $Y = 0 \to M \to C^{m_0} \to C^{m_1} \to \cdots$ is $\text{Hom}_R(-, P_C)$-exact.

(2) $Y$ is a minimal complex.

If $M$ admits a proper $P_C^i$-coresolution, then $^+Y$ is exact and so $Y$ is a minimal proper $P_C$-coresolution of $M$.

**Proof.** Let $M$ be a finite $R$-module. Assume that $m_0 = \nu(\text{Hom}_R(M, C))$ denotes the number of a minimal set of generators of $\text{Hom}_R(M, C)$ and that $\alpha : R^{m_0} \to \text{Hom}_R(M, C)$ is the natural $P_C^i$-cover of $\text{Hom}_R(M, C)$. It follows that $\gamma = M \xrightarrow{\delta M} \text{Hom}_R(\text{Hom}_R(M, C), C) \xrightarrow{\text{Hom}_R(\alpha, C)} \text{Hom}_R(R^{m_0}, C)$ is a $P_C^i$-envelope of $M$. Set $M_{-1} = \text{Coker}\gamma$ and $m_1 = \nu(\text{Hom}_R(M_{-1}, C))$. As
mentioned, there is a $P_C^I$-envelope $\gamma_1 : M_{-1} \rightarrow \text{Hom}_R(R^{m_1}, C)$. Proceeding in this way one obtains a complex $Y = 0 \rightarrow \text{Hom}_R(R^{m_0}, C) \xrightarrow{\partial_0 = \gamma_1 \pi_1} \text{Hom}_R(R^{m_1}, C) \xrightarrow{\partial_1 = \gamma_2 \pi_2} \cdots$, where $\pi_i$ is the natural epimorphism for all $i \geq 1$. Since the maps in $Y$ are obtained by $P_C^I$-envelopes, the complex $+Y$ is $\text{Hom}_R(-, P_C)$-exact. It is easy to see that $\text{Hom}_R(Y, C)$ is minimal free resolution of $\text{Hom}_R(M, C)$. Similar to the proof of Theorem 3.10, we find that $Y$ is a minimal complex. If $M$ admits a proper $P_C^I$-coresolution, then, by Lemma 3.11, $+Y$ is exact. \□

In the following example we find an $R$–module $M$ with $P_C$–pd($M$) = $\infty$ which admits a minimal proper $P_C$–resolution. This example shows that a commutative noetherian local ring which admits an exact zero-divisor is not a strongly regular ring.

**Example 3.13.** Let $R$ be a commutative noetherian local ring and $C$ a semidualizing $R$–module. Assume that $x, y$ form a pair of exact zero-divisors on both $R$ and $C$ (e.g. see [1, Example 3.2]). Then $P_C$–pd($C/xC$) = pd($R/xR$) = $\infty$. The complex $T_\bullet = \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \rightarrow 0$ (resp. $L_\bullet = 0 \rightarrow C \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} \cdots$)

is a minimal $P_C$–resolution (resp. $P_C$–coresolution) of $C/xC$. By [1, Proposition 3.4], $C/xC$ is a semidualizing $R/xR$–module. By [5, Proposition 2.13], there are isomorphisms

$$\text{Hom}_R(C, C/xC) \cong \text{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR,$$

$$\text{Hom}_R(C/xC, C) \cong \text{Hom}_{R/xR}(C/xC, C/xC) \cong R/xR.$$

Applying $\text{Hom}_R(C, -)$ and $\text{Hom}_R(-, C)$ on the above complexes, respectively, would result the isomorphisms $\text{Hom}_R(C, T_\bullet^+) \cong F_\bullet^+$ and $\text{Hom}_R(+L_\bullet, C) \cong F_\bullet^+$, where $F_\bullet^+$ is the exact complex $\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \rightarrow R/xR \rightarrow 0$. Therefore $T_\bullet$ (resp. $L_\bullet$) is a minimal proper $P_C$–resolution (resp. $P_C$–coresolution) of $C/xC$.

For each $n$, one obtains a $C$–perfect complex of length $n$ as

$$T_{\bullet}^{(n)} = 0 \rightarrow C \xrightarrow{x} C \xrightarrow{x} \cdots \xrightarrow{x} C \xrightarrow{y} C \xrightarrow{x} C \rightarrow 0,$$

where $T_{\bullet}^{(n)} = T_i$ for all $0 \leq i \leq n$ and $T_{\bullet}^{(n)} = 0$ otherwise. Note that the induced map $d_i : T_i^{(n)}/\text{Ker} d_i \rightarrow T_{i-1}^{(n)}$ is injective, where $\text{Ker} d_i$ is equal to $yC$ or $xC$. As $C$ is indecomposable $R$–module, $T_{\bullet}^{(n)}$ is indecomposable which has a similar proof to [3, Proposition 1.5].
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