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HYPERSTABILITY OF SOME FUNCTIONAL EQUATION ON RESTRICTED DOMAIN: DIRECT AND FIXED POINT METHODS

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Abstract. The study of stability problems of functional equations was motivated by a question of S. M. Ulam asked in 1940. The first result giving answer to this question is due to D.H. Hyers. Subsequently, his result was extended and generalized in several ways. In this paper we prove some hyperstability results for the equation $g(ax+by)+g(cx+dy)=Ag(x)+Bg(y)$ on restricted domain. Namely, we show, under some weak natural assumptions, functions satisfying the above equation approximately (in some sense) must be actually solutions to it.
Keywords: hyperstability, linear equation, quadratic equation, p-Wright affine function, fixed point theorem.
MSC(2010): Primary: 39B82; Secondary: 39B62, 47H14, 47J20, 47H10.

1. Introduction

Let X and Y be linear spaces over fields $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, respectively, and $a, b, c, d \in \mathbb{F}$, $A, B \in \mathbb{K}$ be fixed. The functional equation

$$g(ax+by)+g(cx+dy)=Ag(x)+Bg(y), \qquad x, y \in X, \tag{1.1}$$

for function $g : X \to Y$, generalizes simultaneously three quite known equations. Namely, with $a = c$ and $b = d$, and $A = 2\alpha$, $B = 2\beta$, it is the linear equation

$$g(ax+by)=\alpha g(x)+\beta g(y), \qquad x, y \in X;$$

which for $a = b = \alpha = \beta = 1$, becomes Cauchy equation

$$g(x+y)=g(x)+g(y), \qquad x, y \in X; \tag{1.2}$$

and for $a = b = \alpha = \beta = \frac{1}{2}$, becomes Jensen equation

$$g\left(\frac{x+y}{2}\right)=\frac{g(x)+g(y)}{2}, \qquad x, y \in X.$$

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When $a = d =: p$, $b = c =: 1 - p$ and $A = B = 1$ the equation (1.1) takes the form

$$g(px + (1-p)y) + g((1-p)x + py) = g(x) + g(y), \qquad x, y \in X. \tag{1.3}$$

For $\mathbb{F} = \mathbb{R}$ and $p \in (0,1)$, solutions of the equation (1.3) are called p-Wright affine functions, which are both p-Wright convex and concave (see [11]). For $p = 1/3$ equation (1.3) takes the form

$$g(2x + y) + g(x + 2y) = g(3x) + g(3y), \qquad x, y \in X,$$

which has been studied in [24] (see also [7]) in connection with some investigations of the generalized (σ, τ)-Jordan derivations on Banach algebras. The cases of more arbitrary p have been studied in [11, 12, 22] (see also [16, 19]).

The third particular case of the equation (1.1) (with $\mathbb{F} = \mathbb{K}$) is the Euler-Lagrange functional equation

$$g(ax + by) + g(bx - ay) = (a^2 + b^2)\big(g(x) + g(y)\big), \qquad x, y \in X, \tag{1.4}$$

investigated by J. M. Rassias [31, 32] (see also [26]), for $a = b = 1$, equation (1.4) becomes the quadratic equation

$$g(x + y) + g(x - y) = 2g(x) + 2g(y), \qquad x, y \in X.$$

The study of stability problems of functional equations was motivated by a question of S. M. Ulam asked in 1940. The first result giving a partial answer to this question is due to D. H. Hyers (see [18]). Subsequently, his result was extended and generalized in several ways.
The following theorem is the most classical result concerning the Hyers-Ulam stability.

Theorem 1.1. *Let X and Y be two normed spaces, Y be complete, $c \geq 0$ and $p \neq 1$ be a real number. Let $f : X \to Y$ be an operator such that*

$$\|f(x + y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in X \setminus \{0\}.$$

Then there exists a unique solution $T : X \to Y$ of (1.2) with

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|1 - 2^{p-1}|}, \qquad x \in X \setminus \{0\}.$$

It is due to T. Aoki [1] (for $0 < p < 1$; see also [33]), Z. Gajda [13] (for $p > 1$) and Th. M.Rassias [34] (for $p < 0$; see also [35, p. 326]). Moreover, an example is given in [13] from which it follows that analogous result for $p = 1$ is not true. For $p = 0$ it is the first result of stability proved by Hyers [18]. Now, it is known that for $p < 0$ we have the hyperstability result, that is f satisfying (1.1) must be additive (see [8]).

Another generalization of the result of Hyers was considered by J. M. Rassias who has proved the following theorem

Theorem 1.2. *([28–30]). Let X and Y be two normed spaces, Y be complete, $c \geq 0$ and r, s be a real numbers such that $q = r + s \neq 1$. Let $f : X \to Y$ be an operator such that*

$$\|f(x+y) - f(x) - f(y)\| \leq c\|x\|^r\|y\|^s, \qquad x, y \in X \setminus \{0\}.$$

Then there exists a unique solution $T : X \to Y$ of (1.2) with

$$\|f(x) - T(x)\| \leq \frac{c}{|2^q - 2|}\|x\|^q, \qquad x \in X \setminus \{0\}.$$

Moreover, an example given in [15] shows that analogous result for $q = 1$ is not true. The above mentioned stability involving a product of powers of norms is sometimes called Ulam-Găvruta-Rassias stability (see [3, 25, 36]). In [14], a generalization of the above theorems was obtained by P. Găvruta. In [37] another control function $\varphi(x, y) = c(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$ for $x, y \in X$ with $c \geq 0$ and $p > 0$ (the mixed product sum of powers of norms) was considered. F. Skof (see [38]) was the first who has solved the Ulam's problem on a restricted domain. The stability and hyperstability of the particular cases of the functional equation (1.1) also were studied by several mathematicians (cf., e.g., [2, 6, 10, 20, 21, 26, 27]).

In this paper we consider the functions $g: E \to Y$, where E is a nonempty subset of X satisfying the equation

$$(1.5) \quad g(ax + by) + g(cx + dy) = Ag(x) + Bg(y), \quad x, y, ax + by, cx + dy \in E,$$

and we show that for some natural particular forms of φ, under some assumptions, the conditional functional equation (1.5) is φ-hyperstable in the class of functions $g: E \to Y$, i.e. each $g: E \to Y$ satisfying the inequality

$$\|g(ax + by) + g(cx + dy) - Ag(x) - Bg(y)\| \leq \varphi(x, y),$$

for $x, y, ax + by, cx + dy \in E$, must fulfil the equation (1.5). In this study, we consider the following control functions:

$$\begin{array}{lll} \varphi(x, y) = & C\|x\|^r\|y\|^s, & \text{with} \quad C \geq 0, r + s \neq 0; \\ \varphi(x, y) = & C\|x\|^r\|y\|^s + D(\|x\|^{r+s} + \|y\|^{r+s}), & \text{with} \quad C, D \geq 0,\ r, s < 0. \end{array}$$

The term *hyperstability* has been used for the first time in [23], however it seems that the first hyperstability result was published in [4] and concerned the ring homomorphisms.

One of the method of the proof is based on a fixed point result that can be derived from [5] (Theorem 1). To present it we need the following three hypotheses:

(H1) E is a nonempty set, Y is a Banach space, $f_1, \ldots, f_k: E \to E$ and $L_1, \ldots, L_k: E \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T}\colon Y^E \to Y^E$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi, \mu \in Y^E, x \in E.$$

(H3) $\Lambda\colon {\mathbb{R}_+}^E \to {\mathbb{R}_+}^E$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^E, x \in E.$$

Now we are in a position to present the above mentioned fixed point theorem.

Theorem 1.3. *Let hypotheses* (H1)–(H3) *be valid and functions* $\varepsilon\colon E \to \mathbb{R}_+$ *and* $\varphi : E \to Y$ *fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \qquad x \in E,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \qquad x \in E.$$

Then there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \qquad x \in E.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in E.$$

2. **Hyperstability results**

The next theorems are the main results in this paper and concern φ-hyperstability of the equation (1.5).
First, we show that for $\varphi(x, y) = C\|x\|^r\|y\|^s$, where $x, y \in E$, $r, s \in \mathbb{R}$, $r+s \neq 0$ and $C \geq 0$, under some additional assumptions, the conditional functional equation (1.5) is φ-hyperstable in the class of functions $g\colon E \to Y$. They correspond in particular to some results in [2, 6, 8, 9, 27].

We start with the hypothesis

(H4) X, Y are the normed spaces, E is a nonempty subset of $X \setminus \{0\}$, $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K} \setminus \{0\}$, $C \geq 0$ and $r, s \in \mathbb{R}$.

Theorem 2.1. *Let hypothesis (H4) be valid,* $r + s < 0$ *and* E *be such that there exists a positive integer* n_0 *with*

$$nx, (a + bn)x, (c + dn)x \in E, \qquad x \in E,\ n \in \mathbb{N},\ n \geq n_0, \tag{2.1}$$

and $g\colon E \to Y$ *satisfies*

$$\begin{aligned} &\|g(ax + by) + g(cx + dy) - Ag(x) - Bg(y)\| \\ &\qquad \leq \quad C\|x\|^r\|y\|^s, \qquad x, y, ax + by, cx + dy \in E. \end{aligned} \tag{2.2}$$

Then g *satisfies the equation* (1.5).

Proof. First we notice that without loss of generality we can assume that Y is a Banach space, because otherwise we can replace it by its completion. From the fact $r+s<0$ we get that at least one of r and s must be negative, so we may assume that $s<0$.
Observe that there exists $m_0 \in \mathbb{N}$ such that

$$\frac{1}{|A|}|a+bm|^{r+s}+\frac{1}{|A|}|c+dm|^{r+s}+\Big|\frac{B}{A}\Big|m^{r+s}<1 \qquad \text{for } m \geq m_0. \tag{2.3}$$

Fix $m \geq \max\{n_0, m_0\}$ and replace y by mx in (2.2). Thus

$$\|g\big((a+bm)x\big)+g\big((c+dm)x\big)-Ag(x)-Bg(mx)\| \leq Cm^s\|x\|^{r+s}, \quad x \in E. \tag{2.4}$$

Write

$$\begin{aligned} \mathcal{T}\xi(x) &:= \frac{1}{A}\xi\big((a+bm)x\big)+\frac{1}{A}\xi\big((c+dm)x\big)-\frac{B}{A}\xi(mx), \ \ \xi \in Y^E, x \in E \\ \varepsilon(x) &:= \frac{C}{|A|}m^s\|x\|^{r+s}, \qquad x \in E, \end{aligned}$$

then (2.4) takes the form

$$\|\mathcal{T}g(x)-g(x)\| \leq \varepsilon(x), \qquad x \in E.$$

Define

$$\Lambda\eta(x) := \frac{1}{|A|}\eta\big((a+bm)x\big)+\frac{1}{|A|}\eta\big((c+dm)x\big)+\Big|\frac{B}{A}\Big|\eta(mx), \ \ \eta \in {\mathbb{R}_+}^E, x \in E.$$

Then it is easily seen that Λ has the form described in (H3) with $k=3$ and $f_1(x)=(a+bm)x,\ f_2(x)=(c+dm)x,\ f_3(x)=mx,\ L_1(x)=L_2(x)=\frac{1}{|A|}$, $L_3(x)=\big|\frac{B}{A}\big|$ for $x \in E$.

Moreover, for every $\xi, \mu \in Y^E$, $x \in E$

$$\begin{aligned} &\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x)\| \\ &\quad = \Big\|\frac{1}{A}\xi\big((a+bm)x\big)+\frac{1}{A}\xi\big((c+dm)x\big)-\frac{B}{A}\xi(mx) \\ &\qquad -\frac{1}{A}\mu\big((a+bm)x\big)-\frac{1}{A}\mu\big((c+dm)x\big)+\frac{B}{A}\mu(mx)\Big\| \\ &\quad \leq \frac{1}{|A|}\|(\xi-\mu)((a+bm)x)\|+\frac{1}{|A|}\|(\xi-\mu)((c+dm)x)\|+\Big|\frac{B}{A}\Big|\|(\xi-\mu)(mx)\| \\ &\quad = \sum_{j=1}^{3} L_j(x)\|(\xi-\mu)(f_j(x))\|, \end{aligned}$$

so (H2) is valid.

From (2.3) we have

$$\begin{aligned}\varepsilon^*(x) &:= \sum_{n=0}^{\infty}\Lambda^n\varepsilon(x)\\ &= \varepsilon(x)\sum_{n=0}^{\infty}\Big(\frac{1}{|A|}|a+bm|^{r+s}+\frac{1}{|A|}|c+dm|^{r+s}+\Big|\frac{B}{A}\Big|m^{r+s}\Big)^n\\ &= \frac{\varepsilon(x)}{1-\frac{1}{|A|}|a+bm|^{r+s}-\frac{1}{|A|}|c+dm|^{r+s}-\big|\frac{B}{A}\big|m^{r+s}}, \qquad x\in E.\end{aligned}$$

Hence, according to Theorem 1.3 there exists a unique solution $G\colon E\to Y$ of the equation

$$G(x)=\frac{1}{A}G\big((a+bm)x\big)+\frac{1}{A}G\big((c+dm)x\big)-\frac{B}{A}G(mx), \qquad x\in E$$

such that

$$\|g(x)-G(x)\|\le\frac{\varepsilon(x)}{1-\frac{1}{|A|}|a+mb|^{r+s}-\frac{1}{|A|}|c+dm|^{r+s}-\big|\frac{B}{A}\big|m^{r+s}}, \quad x\in E. \tag{2.5}$$

Moreover,

$$G(x):=\lim_{n\to\infty}(\mathcal{T}^n g)(x), \qquad x\in E.$$

Now, we show that

$$\begin{aligned}&\|\mathcal{T}^n g(ax+by)+\mathcal{T}^n g(cx+dy)-A\mathcal{T}^n g(x)-B\mathcal{T}^n g(y)\|\\ &\le C\Big(\frac{1}{|A|}|a+bm|^{r+s}+\frac{1}{|A|}|c+dm|^{r+s}+\Big|\frac{B}{A}\Big|m^{r+s}\Big)^n\|x\|^r\|y\|^s,\end{aligned} \tag{2.6}$$

for every $x,y,ax+by,cx+dy\in E$ and $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. If $n=0$, then (2.6) is simply (2.2). So, take $k\in\mathbb{N}_0$ and suppose that (2.6) holds for $n=k$. Then

$$\begin{aligned}&\|\mathcal{T}^{k+1}g(ax+by)+\mathcal{T}^{k+1}g(cx+dy)-A\mathcal{T}^{k+1}g(x)-B\mathcal{T}^{k+1}g(y)\|\\ &= \|\frac{1}{A}\mathcal{T}^k g((a+bm)(ax+by))+\frac{1}{A}\mathcal{T}^k g((c+dm)(ax+by))\\ &\quad-\frac{B}{A}\mathcal{T}^k g(m(ax+by))+\frac{1}{A}\mathcal{T}^k g((a+bm)(cx+dy))\\ &\quad+\frac{1}{A}\mathcal{T}^k g((c+dm)(cx+dy))-\frac{B}{A}\mathcal{T}^k g(m(cx+dy))\\ &\quad-\frac{1}{A}\mathcal{T}^k g((a+bm)x)-\frac{1}{A}\mathcal{T}^k g((c+dm)x)+\frac{B}{A}\mathcal{T}^k g(mx)\\ &\quad-\frac{1}{A}\mathcal{T}^k g((a+bm)y)-\frac{1}{A}\mathcal{T}^k g((c+dm)y)+\frac{B}{A}\mathcal{T}^k g(my)\|\\ &\le C\Big(\frac{1}{|A|}|a+bm|^{r+s}+\frac{1}{|A|}|c+dm|^{r+s}+\Big|\frac{B}{A}\Big|m^{r+s}\Big)^k\\ &\quad\Big(\frac{1}{|A|}\|(a+bm)x\|^r\|(a+bm)y\|^s+\frac{1}{|A|}\|(c+dm)x\|^r\|(c+dm)y\|^s\end{aligned}$$

$$
\begin{aligned}
&+\Big|\frac{B}{A}\Big|\|mx\|^r\|my\|^s\Big)\\
=\;& C\Big(\frac{1}{|A|}|a+bm|^{r+s}+\frac{1}{|A|}|c+dm|^{r+s}+\Big|\frac{B}{A}\Big|m^{r+s}\Big)^{k+1}\|x\|^r\|y\|^s,
\end{aligned}
$$

for every $x, y, ax+by, cx+dy \in E$. Thus, by induction we have shown that (2.6) holds for every $n \in \mathbb{N}_0$ and for every $x, y, ax+by, cx+dy \in E$. Letting in (2.6) $n \to \infty$, we obtain

$$G(ax+by)+G(cx+dy)=AG(x)+BG(y), \qquad x, y, ax+by, cx+dy \in E.$$

In this way, for every $m \geq max\{n_0, m_0\}$ there exists a function G_m satisfying the equation (1.5) and the inequality

$$\|g(x)-G_m(x)\| \leq \frac{\varepsilon(x)}{1-\frac{1}{|A|}|a+mb|^{r+s}-\frac{1}{|A|}|c+dm|^{r+s}-\big|\frac{B}{A}\big|m^{r+s}}, \quad x \in E.$$

Letting $m \to \infty$, It follows that g satisfies the equation (1.5). □

In similar way we can prove the following theorem in which we consider the case when $r+s>0$. Then obviously at least one of r and s must be positive and without loss of generality we can assume that $s>0$.

Theorem 2.2. *Let hypothesis (H4) be valid and $r+s>0$, $s>0$. If there exist two sequences $\{e_m\}_{m\in\mathbb{N}}$,$\{f_m\}_{m\in\mathbb{N}}$ of the elements of $\mathbb{F}$ such that $\{e_m\}_{m\in\mathbb{N}}$ is bounded, $\lim_{m\to\infty} f_m=0$ and there exists a positive integer n_0 with*

$$e_m x, f_m x, (ae_m+bf_m)x, (ce_m+df_m)x \in E, \;\; x \in E,\, m \in \mathbb{N},\, m \geq n_0 \tag{2.7}$$

such that one of the conditions is satisfied:

$$
\begin{aligned}
&(C1) && e_m \equiv 1 \quad \textit{and} \quad \lim_{m\to\infty}\gamma_m^1<1\\
& && \textit{where}\quad \gamma_m^1 := \tfrac{1}{|A|}\big(|a+f_m b|^{r+s}+|c+f_m d|^{r+s}+|B||f_m|^{r+s}\big),\\
&(C2) && ae_m+bf_m=1,\;\; ce_m+df_m=1 \quad \textit{and} \quad \lim_{m\to\infty}\gamma_m^2<1\\
& && \textit{where}\quad \gamma_m^2 := \tfrac{1}{2}\big(|A||e_m|^{r+s}+|B||f_m|^{r+s}\big),\\
&(C3) && ae_m+bf_m=1,\;\; ce_m+df_m\neq 1 \quad \textit{and} \quad \lim_{m\to\infty}\gamma_m^3<1\\
& && \textit{where}\quad \gamma_m^3 := |ce_m+df_m|^{r+s}+|A||e_m|^{r+s}+|B||f_m|^{r+s},\\
&(C4) && ae_m+bf_m\neq 1,\;\; ce_m+df_m=1 \quad \textit{and} \quad \lim_{m\to\infty}\gamma_m^4<1\\
& && \textit{where}\quad \gamma_m^4 := |ae_m+bf_m|^{r+s}+|A||e_m|^{r+s}+|B||f_m|^{r+s},
\end{aligned}
$$

and $g\colon E \to Y$ fulfills (2.2), then the function g satisfies the equation (1.5).

Proof. Replacing in (2.2) x by $e_m x$ and y by $f_m x$, where $m \in \mathbb{N}_{n_0} := \{m \in \mathbb{N} : m \geq n_0\}$, we get

$$
\begin{aligned}
&\|g((ae_m+bf_m)x)+g((ce_m+df_m)x)-Ag(e_m x)-Bg(f_m x)\|\\
&\quad\leq\; C|e_m|^r|f_m|^s\|x\|^{r+s}, \qquad x \in E.
\end{aligned}
\tag{2.8}
$$

Let the case (Ci) holds, where $i \in \{1,2,3,4\}$. For $x \in E$ we define

$$\mathcal{T}_m\xi(x) := k_1^i\xi((ae_m + bf_m)x) + k_2^i\xi((ce_m + df_m)x) - k_3^iA\xi(e_mx) - k_4^iB\xi(f_mx),$$
$$\varepsilon_m(x) := k_0^iC|e_m|^r|f_m|^s\|x\|^{r+s},$$
$$\Lambda_m\eta(x) := |k_1^i|\eta((ae_m + bf_m)x) + |k_2^i|\eta((ce_m + df_m)x) + |k_3^iA|\eta(e_mx) + |k_4^iB|\eta(f_mx),$$

where $k_1^1 = k_2^1 = k_4^1 = \frac{1}{A}$, $k_3^1 = 0$,

$k_1^2 = k_2^2 = 0$, $k_3^2 = k_4^2 = -\frac{1}{2}$,

$k_1^3 = 0$, $k_2^3 = k_3^3 = k_4^3 = -1$,

$k_1^4 = k_3^4 = k_4^4 = -1$, $k_2^4 = 0$,

$k_0^1 = \frac{1}{|A|}$, $k_0^2 = \frac{1}{2}$, $k_0^3 = k_0^4 = 1$.

As in Theorem 2.1 we observe that (2.8) takes form

$$\|\mathcal{T}_mg(x) - g(x)\| \le \varepsilon_m(x), \qquad x \in E \tag{2.9}$$

and Λ_m has the form described in (H3) and (H2) is valid for every $\xi, \mu \in Y^E$, $x \in E$.

Next we can find $m_0 \in \mathbb{N}$, such that $m_0 \ge n_0$ and $\gamma_m^i < 1$ for $m \in \mathbb{N}_{m_0}$. Therefore

$$\varepsilon_m^*(x) := \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x) = \frac{\varepsilon_m(x)}{1-\gamma_m^i},$$

for $m \ge m_0$, $x \in E$. Hence, according to Theorem 1.3, for each $m \in \mathbb{N}_{m_0}$ there exists a unique solution $G_m\colon E \to Y$ of the equation

$$G_m(x) = k_1^iG_m((ae_m+bf_m)x)+k_2^iG_m((ce_m+df_m)x)-k_3^iAG_m(e_mx)-k_4^iBG_m(f_mx)$$

such that

$$\|g(x) - G_m(x)\| \le \varepsilon_m^*(x), \qquad x \in E. \tag{2.10}$$

Moreover,

$$G_m(ax+by)+G_m(cx+dy) = AG_m(x)+BG_m(y), \qquad x, y, ax+by, cx+dy \in E.$$

In this way we obtain a sequence $(G_m)_{m\in\mathbb{N}_{m_0}}$ satisfying the equation (1.5) such that (2.10) holds. It follows, with $m \to \infty$, that g satisfies the equation (1.5), because

$$\lim_{m\to\infty}\varepsilon_m^*(x) = \|x\|^{r+s}\lim_{m\to\infty}|f_m|^s\,\frac{k_0^iC|e_m|^r}{1-\gamma_m^i} = 0.$$

□

From the Theorem 2.2 we deduce in particular the following corollaries.

Corollary 2.3. *Let hypothesis (H4) be valid and $r+s>0$, $s>0$. If*

$$|a|^{r+s}+|c|^{r+s}<|A|$$

and there exists a positive integer n_0 with

$$-\frac{a}{bm}x, a\big(1-\frac{1}{m}\big)x, \Big(c-\frac{ad}{bm}\Big)x \in E, \qquad x\in E,\ m\in\mathbb{N}_{n_0}$$

and $g\colon E\to Y$ fulfills (2.2), then g satisfies the equation (1.5).

Proof. Putting $f_m=-\frac{a}{bm}$ and using Theorem 2.2 (1) we have

$$\gamma_m^1 := \frac{1}{|A|}\Big(\Big|a\big(1-\frac{1}{m}\big)\Big|^{r+s}+\Big|c-\frac{ad}{bm}\Big|^{r+s}+|B|\Big|\frac{a}{bm}\Big|^{r+s}\Big),$$

hence

$$\lim_{m\to\infty}\gamma_m^1=\frac{1}{|A|}\big(|a|^{r+s}+|c|^{r+s}\big)<1,$$

so the function g satisfies the equation (1.5). □

Corollary 2.4. *Let hypothesis (H4) be valid and $r+s>0$, $s>0$. If*

$$a=c,\ \ b=d,\ \ \frac{|A|}{2|a|^{r+s}}<1$$

and there exists a positive integer n_0 with

$$\frac{1}{a}(1-\frac{1}{m})x, \frac{1}{bm}x\in E, \qquad x\in E,\ m\in\mathbb{N}_{n_0}$$

and $g\colon E\to Y$ fulfills (2.2), then g satisfies the equation (1.5).

Proof. Setting $e_m=\frac{1}{a}-\frac{1}{am}$, $f_m=\frac{1}{bm}$ and using Theorem 2.2 (2) we have

$$\gamma_m^2 := \frac{1}{2}\Big(|A|\Big|\frac{1}{a}\Big(1-\frac{1}{m}\Big)\Big|^{r+s}+|B|\Big|\frac{1}{bm}\Big|^{r+s}\Big)$$

and

$$\lim_{m\to\infty}\gamma_m^2=\frac{|A|}{2|a|^{r+s}}<1,$$

so the function g satisfies the equation (1.5). □

Corollary 2.5. *Let hypothesis (H4) be valid and $r+s>0$, $s>0$. If*

$$\Big|\frac{c}{a}\Big|^{r+s}+\frac{|A|}{|a|^{r+s}}<1$$

and there exists a positive integer n_0 with

$$\frac{1}{a}(1-\frac{1}{m})x, \frac{1}{bm}x, \Big(\frac{c}{a}+\frac{1}{m}\big(\frac{d}{b}-\frac{c}{a}\big)\Big)x\in E, \qquad x\in E,\ m\in\mathbb{N}_{n_0}$$

and $g\colon E\to Y$ fulfills (2.2), then g satisfies the equation (1.5).

Proof. Setting $e_m = \frac{1}{a} - \frac{1}{am}$, $f_m = \frac{1}{bm}$ and using Theorem 2.2 (3) we have

$$\gamma_3 := \frac{|A|}{|a|^{r+s}}\Big|1 - \frac{1}{m}\Big|^{r+s} + \frac{|B|}{|b|^{r+s}}\Big|\frac{1}{m}\Big|^{r+s} + \Big|\frac{c}{a} + \frac{1}{m}\Big(\frac{d}{b} - \frac{c}{a}\Big)\Big|^{r+s}$$

hence

$$\lim_{m\to\infty} \gamma_m^3 = \Big|\frac{c}{a}\Big|^{r+s} + \frac{|A|}{|a|^{r+s}} < 1,$$

so the function g satisfies the equation (1.5). □

Now, we are going to prove analogues of the Corollaries (2.3)-(2.5). Namely, we show, under some assumptions, that the equation (1.5) is φ-hyperstable in the class of functions $g\colon E \to Y$, where the set E containing 0 and $\varphi(x,y) = C\|x\|^r\|y\|^s$ with $C, r, s \geq 0$, $r+s > 0$.

Theorem 2.6. *Let X, Y be the normed spaces, $E \subset X$ be such that $0 \in E$ and $a, b, c, d \in \mathbb{F} \setminus \{0\}$, $A, B \in \mathbb{K} \setminus \{0\}$, $C \geq 0$, $r, s \geq 0$, $r+s > 0$. If $g\colon E \to Y$ fulfills (2.2) and one of the conditions is satisfied*

$$\begin{array}{llll}
(a) & \frac{1}{|A|}\big(|a|^{r+s} + |c|^{r+s}\big) < 1 & \textit{and} & ax, cx \in E \textit{ for } x \in E,\\
(b) & a = c,\quad b = d,\quad \frac{|A|}{2|a|^{r+s}} < 1 & \textit{and} & \frac{1}{a}x \in E \textit{ for } x \in E,\\
(c) & \big|\frac{c}{a}\big|^{r+s} + \frac{|A|}{|a|^{r+s}} < 1 & \textit{and} & \frac{c}{a}x, \frac{1}{a}x \in E \textit{ for } x \in E,\\
(d) & \frac{1}{|B|}\big(|b|^{r+s} + |d|^{r+s}\big) < 1 & \textit{and} & bx, dx \in E \textit{ for } x \in E,\\
(e) & a = c,\quad b = d,\quad \frac{|B|}{2|b|^{r+s}} < 1 & \textit{and} & \frac{1}{b}x \in E \textit{ for } x \in E,\\
(f) & \big|\frac{d}{b}\big|^{r+s} + \frac{|B|}{|b|^{r+s}} < 1 & \textit{and} & \frac{d}{b}x, \frac{1}{b}x \in E \textit{ for } x \in E,
\end{array}$$

then g satisfies the equation (1.5).

Proof. First we observe that from (2.2) with $x = y = 0$ we obtain

$$g(0)(2 - A - B) = 0.$$

Assume that the case (a) holds. Note that (2.2) with $y = 0$ gives

$$g(x) = \frac{1}{A}\big(g(ax) + g(cx) - Bg(0)\big), \qquad x \in E. \tag{2.11}$$

We prove that for every $x, y \in E$ such that $ax + by, cx + dy \in E$ and for every $n \in \mathbb{N}_0$

$$\|g(ax+by)+g(cx+dy)-Ag(x)-Bg(y)\| \leq C\Big(\frac{|a|^{r+s}+|c|^{r+s}}{|A|}\Big)^n\|x\|^r\|y\|^s. \tag{2.12}$$

Clearly, the case $n = 0$ is just (2.2). Next, fix $n \in \mathbb{N}_0$ and assume that (2.12) holds for every $x, y, ax + by, cx + dy \in E$. Then using (2.11) we have

$$\begin{aligned}
&\|g(ax + by) + g(cx + dy) - Ag(x) - Bg(y)\| \\
&=\Big\| \frac{1}{A}\big(g(aax + bay) + g(acx + bcy) + g(cax + day) + g(ccx + dcy) \\
&\quad - A(ax) - Ag(cx) - Bg(ay) - Bg(cy)\big)\Big\| \\
&\le \frac{1}{|A|}\Big(C\Big(\frac{|a|^{r+s}+|c|^{r+s}}{|A|}\Big)^n \|ax\|^r\|ay\|^s + C\Big(\frac{|a|^{r+s}+|c|^{r+s}}{|A|}\Big)^n \|cx\|^r\|cy\|^s\Big) \\
&= C\Big(\frac{|a|^{r+s}+|c|^{r+s}}{|A|}\Big)^{n+1}\|x\|^r\|y\|^s,
\end{aligned}$$

for every $x, y, ax + by, cx + dy \in E$.
Thus, by induction we have shown that (2.12) holds for every $n \in \mathbb{N}_0$ and every $x, y \in E$ such that $ax+by, cx+dy \in E$. Since $\frac{|a|^{r+s}+|c|^{r+s}}{|A|} < 1$, letting in (2.12) $n \to \infty$, we obtain that g satisfies the equation (1.5).

If the case (b) holds, then replacing in (2.2) x by $\frac{1}{a}x$ and y by 0 we get

$$g(x) = \frac{1}{2}\big(Ag(\frac{1}{a}x) + Bg(0)\big), \qquad x \in E. \tag{2.13}$$

We show that for every $x, y \in E$ such that $ax + by \in E$ and for every $n \in \mathbb{N}_0$

$$\big\|g(ax+by) - \frac{A}{2}g(x) - \frac{B}{2}g(y)\big\| \le \frac{C}{2}\Big(\frac{|A|}{2|a|^{r+s}}\Big)^n\|x\|^r\|y\|^s. \tag{2.14}$$

For $n = 0$ (2.14) reduces to (2.2). Assuming (2.14) true for fix $n \in \mathbb{N}_0$ and every $x, y, ax + by \in E$, we have by (2.13)

$$\begin{aligned}
&\big\|g(ax + by) - \frac{A}{2}g(x) - \frac{B}{2}g(y)\big\| \\
&= \Big\|\frac{A}{2}\Big(g\Big(\frac{1}{a}(ax + by)\Big) - \frac{A}{2}g\Big(\frac{1}{a}x\Big) - \frac{B}{2}g\Big(\frac{1}{a}y\Big)\Big)\Big\| \\
&\le \frac{|A|}{2}\frac{C}{2}\Big(\frac{|A|}{2|a|^{r+s}}\Big)^n\Big\|\frac{1}{a}x\Big\|^r\Big\|\frac{1}{a}y\Big\|^s \\
&= \frac{C}{2}\Big(\frac{|A|}{2|a|^{r+s}}\Big)^{n+1}\|x\|^r\|y\|^s,
\end{aligned}$$

for every $x, y, ax + by \in E$ and thus we obtain (2.12) for every $n \in \mathbb{N}_0$ and every $x, y \in E$ such that $ax + by \in E$, which completes the induction. Since $\frac{|A|}{2|a|^{r+s}} < 1$, letting in (2.13) $n \to \infty$, we get that g satisfies the equation (1.5).

Now, we assume that the condition (c) holds. Replacing x by $\frac{1}{a}x$ and y by 0 in the condition (2.2), we obtain

$$g(x) = Ag\Big(\frac{1}{a}x\Big) - g\Big(\frac{c}{a}x\Big) + Bg(0), \qquad x \in E.$$

Using the above, it is easy to prove by induction that for every $n \in \mathbb{N}_0$ and for every $x, y \in E$ such that $ax + by, cx + dy \in E$

$$\|g(ax+by)+g(cx+dy)-Ag(x)-Bg(y)\| \leq C\Big(\Big|\frac{c}{a}\Big|^{r+s} + \frac{|A|}{|a|^{r+s}}\Big)^n \|x\|^r\|y\|^s.$$

Passing to the limit as $n \to \infty$, we obtain that g satisfies the equation (1.5). The proofs of the cases (d),(e),(f) are analogous to proofs of the cases (a), (b), (c), respectively. □

We end this section with theorem which shows that under some assumptions, the conditional functional equation (1.5) is φ-hyperstable in the class of functions $g\colon E \to Y$, with the control function $\varphi(x, y) = C\|x\|^r\|y\|^s + D(\|x\|^{r+s} + \|y\|^{r+s})$, where $C, D \geq 0$ and $r, s < 0$.

Theorem 2.7. *Let hypothesis (H4) be valid and $D \geq 0$, $r < 0$, $s < 0$, ($a \neq -c$ or $b \neq -d$) and if $a = tc$ and $b = td$ with some $|t| \neq 1$ then $|a| < |c|$. If there exist a sequence $\{e_m\}_{m\in\mathbb{N}}$ of the elements of $\mathbb{F}$ such that $\lim_{m\to\infty} |e_m| = +\infty$, and a positive integer n_0 with*

$$e_m x, \frac{1 - ae_m}{b}x, (ce_m + d\frac{1 - ae_m}{b})x \in E, \qquad x \in E,\ m \in \mathbb{N},\ m \geq n_0$$

and $g\colon E \to Y$ fulfills the estimation

$$\begin{aligned} &\|g(ax + by) + g(cx + dy) - Ag(x) - Bg(y)\| \\ &\quad\leq C\|x\|^r\|y\|^s + D(\|x\|^{r+s} + \|y\|^{r+s}), \quad x, y, ax + by, cx + dy \in E, \end{aligned} \tag{2.15}$$

then the function g satisfies the equation (1.5).

Proof. Denote $f_m := \frac{1-ae_m}{b}$. Replacing in (2.15) x by $e_m x$ and y by $f_m x$, where $m \in \mathbb{N}_{n_0}$, we get

$$\begin{aligned} &\|g(x) + g((ce_m + df_m)x) - Ag(e_m x) - Bg(f_m x)\| \\ &\quad\leq (C|e_m|^r|f_m|^s + D|e_m|^{r+s} + D|f_m|^{r+s})\|x\|^{r+s}, \qquad x \in E. \end{aligned} \tag{2.16}$$

We have three possibilities

$$\begin{aligned} &(D1) \quad a = c \quad \text{and} \quad b = d, \\ &(D2) \quad a = tc \quad \text{and} \quad b = td, \qquad \text{with } |t| < 1, \\ &(D3) \qquad ad \neq bc. \end{aligned}$$

Let the case (Di) holds, where $i \in \{1, 2, 3\}$. For $x \in E$ we define

$$\begin{aligned} \mathcal{T}_m\xi(x) &:= k_1^i\xi((ce_m + df_m)x) - k_2^i A\xi(e_m x) - k_3^i B\xi(f_m x), \\ \varepsilon_m(x) &:= k_0^i(C|e_m|^r|f_m|^s + D|e_m|^{r+s} + D|f_m|^{r+s})\|x\|^{r+s}, \\ \Lambda_m\eta(x) &:= |k_1^i|\eta((ce_m + df_m)x) + |k_2^i A|\eta(e_m x) + |k_3^i B|\eta(f_m x), \end{aligned}$$

where

$k_1^1 = 0$, $k_2^1 = k_3^1 = -\frac{1}{2}$,

$k_1^2 = k_2^2 = k_3^2 = -1,$

$k_1^3 = k_2^3 = k_3^3 = -1,$

$k_0^1 = \frac{1}{2}$, $k_0^2 = k_0^3 = 1.$

As in Theorem 2.2 we observe that (2.16) takes form (2.9), and Λ_m has the form described in (H3) and (H2) is valid for every $\xi, \mu \in Y^E$, $x \in E$. Next, we can find $m_0 \in \mathbb{N}$, such that $m_0 \geq n_0$ and $\gamma_m^i < 1$ for $m \in \mathbb{N}_{m_0}$, where

$$\begin{aligned} \gamma_m^1 &:= \tfrac{1}{2}\big(|A||e_m|^{r+s} + |B||f_m|^{r+s}\big), \\ \gamma_m^2 &:= |\tfrac{1}{t}|^{r+s} + |A||e_m|^{r+s} + |B||f_m|^{r+s}, \\ \gamma_m^3 &:= |ce_m + df_m|^{r+s} + |A||e_m|^{r+s} + |B||f_m|^{r+s}, \end{aligned}$$

because $\lim_{m\to\infty} \gamma_m^i < 1$.
Therefore

$$\varepsilon_m^*(x) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) = \frac{\varepsilon_m(x)}{1-\gamma_m^i},$$

for $m \geq m_0$, $x \in E$. Hence, according to Theorem 1.3, for each $m \in \mathbb{N}_{m_0}$ there exists a unique function $G_m\colon E \to Y$ such that $\mathcal{T}_m G_m(x) = G_m(x)$ and (2.10) holds. It is easy to check that for each $m \in \mathbb{N}_{m_0}$ the function G_m satisfies (1.5). In this way we obtain a sequence $(G_m)_{m\in\mathbb{N}_{m_0}}$ satisfying the equation (1.5) such that (2.10) holds. It follows, with $m \to \infty$, that g satisfies the equation (1.5), because

$$\lim_{m\to\infty} \varepsilon_m^*(x) = \|x\|^{r+s} \lim_{m\to\infty} \frac{k_0^i(C|e_m|^r|f_m|^s + D|e_m|^{r+s} + D|f_m|^{r+s})}{1-\gamma_m^i} = 0.$$

□

Remark 2.8. We notice that from the above theorem, when $C = 0$, we obtain that under some assumptions, the equation (1.5) is φ-hyperstable in the class of functions $g\colon E \to Y$, with $\varphi(x,y) = D(\|x\|^p + \|y\|^p)$, where $D \geq 0$ and $p < 0$.

Now, we give an example, which shows that in Theorem 2.7 the assumption $a \neq -c$ or $b \neq -d$ is necessary.

Example 2.9. The functional equation

$$f(px - py) + f(py - px) = f(x) + f(y), \tag{2.17}$$

where $x, y \in \mathbb{R}$, $p \in \mathbb{R} \setminus \{0,1\}$ is not θ-hyperstable in the class of function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, with the control function

$$\theta(x,y) := |x|^{-l-1}|y|^{-l} + |x|^{-2l-1} + |y|^{-2l-1}, \qquad x, y \in \mathbb{R} \setminus \{0\},$$

where $l \in \mathbb{N}$, because the function

$$f(x) = \begin{cases} x^{-2l-1} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

satisfies the inequality

$$\begin{aligned} |f(px-py)+f(py-px)-f(x)-f(y)| &= |x^{-2l-1}+y^{-2l-1}| \\ &\le |x|^{-2l-1}+|y|^{-2l-1} \le \theta(x,y), \end{aligned}$$

for all $x, y \in \mathbb{R} \setminus \{0\}$ and f is not a solution of (2.17).

3. **Particular case**

Using the above results we can obtain results for the particular form of the equation (1.5), for example, we present the results for the linear equation (for Jensen equation see [2]).

Corollary 3.1. *Let X, Y be the normed spaces, E be a nonempty subset of X, $a, b \in \mathbb{F} \setminus \{0\}$, $\alpha, \beta \in \mathbb{K} \setminus \{0\}$, $C \ge 0$, $r, s \in \mathbb{R}$ and $g\colon E \to Y$ fulfills*

$$\|g(ax+by) - \alpha g(x) - \beta g(y)\| \le C\|x\|^r\|y\|^s, \qquad x, y, ax+by \in E.$$

Moreover assume that one of the following conditions is satisfied

(a) $0 \notin E$, $r+s<0$, *and there exists* $n_0 \in \mathbb{N}$ *with*

$$nx, (a+bn)x \in E, \qquad x \in E,\ n \in \mathbb{N}_{n_0}, \tag{3.1}$$

(b) $0 \notin E$, $r+s>0$, $s>0$, $\frac{|\alpha|}{|a|^{r+s}} < 1$ *and there exists* $n_0 \in \mathbb{N}$ *with*

$$-\frac{a}{bn}x, a\Big(1-\frac{1}{n}\Big)x \in E, \qquad x \in E,\ n \in \mathbb{N}_{n_0},$$

(c) $0 \notin E$, $r+s>0$, $s>0$, $\frac{|a|^{r+s}}{|\alpha|} < 1$ *and there exists* $n_0 \in \mathbb{N}$ *with*

$$\frac{1}{bn}x, a\Big(1-\frac{1}{n}\Big)x \in E, \qquad x \in E,\ n \in \mathbb{N}_{n_0},$$

(d) $0 \in E$, $ax \in E$ *for* $x \in E$, $r+s>0$, $r, s \ge 0$ *and* $\frac{|a|^{r+s}}{|\alpha|} < 1$,

(e) $0 \in E$, $\frac{1}{a}x \in E$ *for* $x \in E$ $r+s>0$, $r, s \ge 0$ *and* $\frac{|\alpha|}{|a|^{r+s}} < 1$.

Then g satisfies the equation

$$g(ax+by) = \alpha g(x) + \beta g(y), \qquad x, y, ax+by \in E.$$

The following example shows that the linear equation is not φ-hyperstable with the control function $\varphi(x,y) := C|x|^r|y|^s$, if $r+s>0$ and $\frac{|a|^{r+s}}{|\alpha|} = 1$.

Example 3.2. Let $X = Y = E = \mathbb{R}$, $r = s = \frac{1}{2}$, $\alpha = a > 0$, $\beta = b > 0$ and $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = |x|$, $x \in \mathbb{R}$. Then g satisfies

$$|g(ax+by) - ag(x) - bg(y)| \le 2\sqrt{ab}\sqrt{|x||y|}, \qquad x, y \in \mathbb{R}$$

but g is not linear.

Now, we give an example, which shows that in above corollary the additional assumptions on E are necessary.

Example 3.3. Let $X = Y = \mathbb{R}$, $E = [-1, 0) \cup (0, 1]$, $r = s = -1$, $\alpha = a > 1$, $\beta = b > 1$ and $g\colon E \to \mathbb{R}$ be defined by $g(x) = |x|$, $x \in E$. Then f satisfies

$$|g(ax + by) - ag(x) - bg(y)| \leq 2ab|x|^{-1}|y|^{-1}, \qquad x, y \in E,$$

but g is not is not linear on E. We notice that $0 \notin E$ and E does not satisfy the condition (3.1).

Remark 3.4. We notice that Corollary 3.1 generalizes Theorem 20 from [9], where it has proved that linear equation is φ-hyperstable in the class of function mapping a linear space into a linear space with $\varphi(x, y) := C\|x\|^r\|y\|^s$, but under the assumption that $C, r, s \in [0, +\infty)$ and $r + s > 0$.

4. Open problem

We end the paper with an open problem.

For the case $r + s = 0$, the method used in the proofs of the above Theorems cannot be applied, thus this is still an open problem. However, the case when $r = s = 0$ ($\varphi(x, y) = C$), was obviously investigated.

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