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POLYNOMIAL EVALUATION GROUPOIDS AND THEIR GROUPS

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Abstract. In this paper we show how certain metabelian groups can be found within polynomial evaluation groupoids.
Keywords: polynomial evaluation groupoid, group evaluation polynomial, group.
MSC(2010): Primary: 20D99; Secondary: 20F99.

1. Introduction

Let R be a commutative ring with an identity, and let $P(x, y)$ be a two variable polynomial with coefficients in R. When $a, b \in R$, we let $a * b = P(a, b)$ denotes the binary operation obtained by evaluating the polynomial $P(x, y)$ at the point (a, b). Then $(R, *)$ is a binary system that we call a polynomial evaluation groupoid. When G is a finite group of order n, the authors in the article [1] presented a proof that there exists a two variables polynomial $P(x, y)$ of degree at most $2n$ with coefficients in a field R such that the polynomial evaluation groupoid $(R, *)$ would have a subsystem isomorphic to the group G. However, the proof in [1] does not yield a method for finding polynomials $P(x, y)$ of lower degree than $2n$ and having group G in $(R, *)$. It was also demonstrated in [1] that the dihedral group D_p of order $2p$, where $p > 2$ is prime, could be found within the polynomial evaluation groupoid $(R, *)$ when R is the Galois field $GF(p^2)$ and $P(x, y) = x + y + xy + xy^p$.

2. Polynomial evaluation groupoid

In this article, R will denote a commutative ring with an identity 1, and $\mathbb{U}(R)$ will denote the unit group in R. Moreover, we always assume that R has finite characteristic $p > 0$. Let f be a *polynomial* with coefficients in R where

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$f : R \to \mathbb{U}(R) \cup \{0\}$. Obviously, such polynomials always exist. We then define the polynomial $P(x, y) \in R[x, y]$ in two variables by

$$P(x, y) = y + xf(y).$$

We use the binary operation $* : R \times R \to R$ defined by

$$a * b = P(a, b)$$

to construct a **polynomial evaluation groupoid** $(R, *)$.

Lemma 2.1. *(The Associativity Lemma) Let G be a subset of R where G contains at least one unit of R. Then $*$ is associative on the set G if and only if $f(y * z) = f(y)f(z)$ for all y and z in G.*

Proof. Suppose that $f(y * z) = f(y)f(z)$ for $x, y \in G$. If $x, y, z \in G$ we have

$$\begin{aligned} x * (y * z) &= (y * z) + xf(y * z) \\ &= z + yf(z) + xf(y)f(z) \\ &= z + \big(y + xf(y)\big)f(z) \\ &= z + (x * y)f(z) \\ &= (x * y) * z \end{aligned}$$

and it follows that $*$ is associative on G.
On the other hand, suppose that $*$ is associative on G. Then, if $x, y, z \in G$, we have

$$\begin{aligned} z + yf(z) + xf(y * z) &= (y * z) + xf(y * z) \\ &= x * (y * z) \\ &= (x * y) * z \\ &= z + (x * y)f(z) \\ &= z + (y + xf(y))f(z) \\ &= z + yf(z) + xf(y)f(z) \end{aligned}$$

After canceling similar terms in the above equation, we obtain

$$xf(y * z) = xf(y)f(z)$$

or equivalently,

$$x\big[f(y * z) - f(y)f(z)\big] = 0. \qquad (2.1)$$

Since G contains a unit element, we may select x in equation 2.1 to be a unit without affecting the values of $f(y*z)$, $f(y)$ and $f(z)$. Consequently, equation 2.1 must imply that $f(y * z) - f(y)f(z) = 0$ for all $y, z \in R$. □

Theorem 2.2. *Let R denote a commutative ring with identity having finite characteristic $p > 0$. Let f be a polynomial with coefficients in R where $f : R \to \mathbb{U}(R) \cup \{0\}$, and let $G = \{x \in R | f(x) \neq 0\}$. If $f(y * z) = f(y)f(z)$ for all $y, z \in G$, then $G = (G, *)$ is a group that is contained in the polynomial evaluation groupoid $(R, *)$.*

Proof. (Closure) Let $x, y \in G$. By definition, $f : R \to \mathbb{U}(R) \cup \{0\}$, so $f(x)$ and $f(y)$ must be units in R, and their product $f(x)f(y)$ must also be a unit in R. Consequently

$$f(x * y) = f(x)f(y) \neq 0.$$

Therefore $x * y \in G$.
(Associativity) The Associativity Lemma immediately implies that $x*(y*z) = (x * y) * z$ for all $x, y, z \in G$.
(Left Identity) The following shows that $0 \in G$ is a left identity for every $y \in G$.

$$0 * y = y + 0f(y) = y + 0 = y.$$

(Left Inverse) Let $y \in G$. Therefore, $f(y) \neq 0$ in the commutative ring R of characteristic p, and it follows that $f(y)^{-1} = \frac{1}{f(y)}$ exists since $f(y)$ is a unit in R. Let

$${y_L}^{-1} = \frac{p-1}{f(y)} y$$

therefore,

$$\begin{aligned} {y_L}^{-1} * y &= y + {y_L}^{-1} f(y) \\ &= y + \frac{p-1}{f(y)} y f(y) \\ &= y + (p-1)y \\ &= py \\ &= 0. \end{aligned}$$

Thus, all these prove that $G = (G, *)$ is a group contained in the polynomial evaluation groupoid $(R, *)$. □

In view of Theorem 2.2, we will make the following:

Definition 2.3. Let R be a commutative ring with identity having finite characteristic $p > 0$, and let f be a polynomial with coefficients in R. We will call f a *group evaluation polynomial* over R provided

$$f : R \to \mathbb{U}(R) \cup \{0\} \tag{2.2}$$

and

$$f(y * z) = f(y)f(z) \tag{2.3}$$

for every $y, z \in G = \{x \in R | f(x) \neq 0\}$.

When the group evaluation polynomial f has no roots in R, then $G = R$ and the polynomial evaluation groupoid $(R, *)$ will be a group.

We will make no distinction between isomorphic copies of groups or rings. If $G = (G, +)$ is a finite abelian group, the Fundamental Theorem of Finite Abelian Groups implies that G is the direct sum of cyclic groups of prime power orders; i.e.,

$$G = Z_{{p_1}^{n_1}} \oplus Z_{{p_2}^{n_2}} \oplus \cdots Z_{{p_k}^{n_k}}$$

where each summand $Z_{{p_i}^{n_i}}$ denotes the ring of integers modulo ${p_i}^{n_i}$. Consequently, the above direct sum is in fact a commutative ring with an identity having finite characteristic $p > 0$. We then take $R = Z_{{p_1}^{n_1}} \oplus Z_{{p_2}^{n_2}} \oplus \cdots Z_{{p_k}^{n_k}}$ and let f be the constant polynomial $f(y) = 1$ over R. It is obvious that f is a group evaluation polynomial over R, and $x * y = y + xf(y) = x + y$ for every $x, y \in R$. We have just shown the following result:

Theorem 2.4. *Every finite abelian group can be obtained as a polynomial evaluation groupoid.*

Lemma 2.5. *Let R be a finite field with p^{2n} elements, where p is a prime. The polynomial $f(y) = y^{p^n} + y + 1$ is a group evaluation polynomial over R.*

Proof. Since the field R has finite characteristic p, we observe that

$$\begin{aligned}\big(f(z)\big)^{p^n} &= \big(z^{p^n} + z + 1\big)^{p^n} \\ &= z^{p^{2n}} + z^{p^n} + 1 \\ &= z + z^{p^n} + 1 \\ &= f(z).\end{aligned}$$

Whenever $y, z \in R$, we have

$$\begin{aligned}f(y * z) &= f(z + yf(z)) \\ &= (z + yf(z))^{p^n} + (z + yf(z)) + 1 \\ &= z^{p^n} + y^{p^n}(f(z))^{p^n} + yf(z) + z + 1 \\ &= z^{p^n} + y^{p^n} f(z) + yf(z) + z + 1 \\ &= y^{p^n} f(z) + yf(z) + z^{p^n} + z + 1 \\ &= y^{p^n} f(z) + yf(z) + f(z) \\ &= \big(y^{p^n} + y + 1\big) f(z) \\ &= f(y)f(z).\end{aligned}$$

It now follows that f is a group evaluation polynomial on the field R with p^{2n} elements. □

Our first example will involve the finite field R with 16 elements, and we must first briefly discuss several basic facts from elementary abstract algebra. We know that $g(\omega) = \omega^4 + \omega + 1$ is an irreducible polynomial over the field Z_2 of integers modulo 2, and that we can construct the unique field R with 16 elements as

$$R = Z_2[\omega]/(\omega^4 + \omega + 1).$$

Consequently, we can write the elements of R as

$$R = \{a\omega^3 + b\omega^2 + c\omega + d\}$$

where $a, b, c, d \in Z_2$. Since the non-zero elements in any finite field form a cyclic group under multiplication, it is easy to observe that the units in our finite field R can also be written as powers of ω; that is,

$$U(R) = \{\omega\ ,\ \omega^2\ ,\ \omega^3\ ,\ \cdots\ ,\ \omega^{15} = 1\}.$$

The above powers bigger than 3 can, of course, be rewritten as
$\omega^4 = \omega + 1$, $\omega^5 = \omega^2 + \omega$, etc..

It is also well known that any finite field having p^n elements with p a prime will have one and only one subfield with p^m elements if and only if the integer m divides n. So, our field R with $16 = 2^4$ elements must have a unique subfield K with $4 = 2^2$ elements. Since the non-zero elements in K form a cyclic group of order 3 under multiplication, and since the only elements in $U(R)$ of order 3 are ω^5 and ω^{10}, it is immediately obvious that

$$K = \{\ 0\ ,\ \omega^5\ ,\ \omega^{10}\ ,\ \omega^{15} = 1\ \}.$$

Since $(K, +)$ is an abelian 2-group, it is elementary to show that

$$1 + \omega^5 + \omega^{10} = 0.$$

Example 2.6. Let $f(y) = y^4 + y + 1$ be a polynomial in $R[y]$, where R is the field with 16 elements. Then

(2.4) $\qquad f$ is a group evaluation polynomial over R,

and

(2.5) $\qquad G = (G, *)$ is isomorphic to the alternating group A_4.

In fact, after rewriting $f(y)$ as $f(y) = y^{2^2} + y + 1$, it follows immediately from Lemma 2.5 (by using $n = 1$) that f is a group evaluation polynomial over R.
It is straightforward to show that

$$\omega\ ,\ \omega^2\ ,\ \omega + 1\ ,\ \text{and}\ \omega^2 + 1$$

are roots of f in the field R. Since f is a polynomial of degree 4, the above four roots are all of the roots of f in R. Therefore, $G = \{g \in R \mid f(g) \neq 0\}$ must have precisely 12 elements. Theorem 2.2 implies that $G = (G, *)$ is a group of order 12.

Let η denote the restriction of f to the group $G = (G, *)$. It follows from Lemma 2.5 and the definition of a group evaluation polynomial that η is a homomorphism of the group $G = (G, *)$ onto $\overline{G}$, where $\overline{G}$ is the image of η in the unit group $U(R) = (U(R), \cdot)$. The set $N = \{g \in R \mid \eta(g) = 1\}$ is the kernel of the group homomorphism η, and is therefore a normal subgroup of G. It is straightforward to show that

$$N = \{\ 0\ ,\ \omega^5\ ,\ \omega^{10}\ ,\ \omega^{15} = 1\ \}$$

Moreover, the Fundamental Theorem of Homomorphisms implies that G/N is isomorphic to $\overline{G}$.

In order to distinguish between powers of g in $(G, *)$ and powers of g in $\big(U(R), \cdot\big)$ we will use the notation $g^{k*} = g * g * \cdots * g$ and the usual notation $g^k = gg \cdots g$ for powers in $\big(U(R), \cdot\big)$. The elements $g \in N$ with $g \neq 0$ all have order 2 since

$$g^{2*} = g * g = g + gf(g) = g + g1 = g + g = 0.$$

Next, let $g \in G$ where $g \notin N$. The coset $g * N \in G/N = \overline{G}$ must have order 3; and consequently,

$$\big(g * N\big)^{3*} = (g * g * g) * N = 0 * N$$

which implies that $g * g * g \in N$. Therefore, $1 = \eta(g * g * g) = \eta(g) \cdot \eta(g) \cdot \eta(g)$. So, $\eta(g)$ must be an element of order 3 in $\overline{G} \subseteq U(R)$ and we have

$$\eta(g) = \omega^5 \quad \text{or} \quad \eta(g) = \omega^{10}.$$

In either case,

$$1 + \eta(g) + \eta(g)^2 = 0$$

from the remark immediately preceding this example. We now have

$$\begin{aligned} g^{3*} = (g * g) * g &= g + (g * g)\eta(g) \\ &= g + \big(g + g\eta(g)\big)\eta(g) \\ &= g\big(1 + \eta(g) + \eta(g)^2\big) \\ &= g \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, every element in G that is not in N must have order 3. There are only three non-abelian groups of order 12 (see Hungerford [2], page 99), and these three groups (using standard group notation) are A_4,

$$D_6 = <\ a, b \mid a^6 = 1, b^2 = 1, \text{and } ba = a^{-1}b\ >$$

and

$$T = <\ a, b \mid a^6 = 1, b^2 = a^3, \text{and } ba = a^{-1}b\ > .$$

Since our group $G = (G, *)$ does not contain elements of order 6, we have shown that G must be the alternating group A_4.

Our next three examples will involve a commutative ring R with identity 1 having characteristic $p = 4$ that is constructed using the following standard techniques of elementary abstract algebra. Notice that the polynomial $g(\omega) = \omega^2 + 1$ is irreducible over the ring $Z_4 = \{\ 0,\ 1,\ 2,\ 3\ \}$ of integers modulo 4. Therefore, our ring R will be

$$R = Z_4[\omega]/\big(\omega^2 + 1\big) \tag{2.6}$$

and we can write the 16 elements in R as

$$R = \{\ a\omega + b \mid a, b \in Z_4\ \}.$$

Example 2.7. Let $f(y) = 1 + 2\omega y^2$ be a polynomial in $R[y]$, where $R = Z_4[\omega]/\big(\omega^2 + 1\big)$ is the commutative ring with identity having 16 elements described above. Then

$$f \quad \text{is a group evaluation polynomial over } R \tag{2.7}$$

and

$$G = (G, *) \quad \text{is isomorphic to the group } Q_8 \times C_2 \tag{2.8}$$

where Q_8 is the quaternion group of order 8. Notice that $f(y)^2 = \big(1+2\omega y^2\big)^2 = 1 + 4\omega y^2 + 4\omega^2 y^4 = 1$. This immediately implies that $f(y) \neq 0$ for all $y \in R$ and that the polynomial f has no roots in R. It also implies that $f(y)$ is a unit in $U(R)$; and if $f(y) \neq 1$, then $f(y)$ is a unit of order 2 in $U(R)$. Next, suppose that $y, z \in R$ and observe that

$$\begin{aligned} f(y * z) &= f\big(z + yf(z)\big) \\ &= 1 + 2\omega\big(z + yf(z)\big)^2 \\ &= 1 + 2\omega z^2 + 4\omega zyf(z) + 2\omega y^2 f(z)^2 \\ &= 1 + 2\omega z^2 + 2\omega y^2 \\ &= 1 + 2\omega z^2 + 2\omega y^2 + 4\omega^2 y^2 z^2 \\ &= \big(1 + 2\omega y^2\big)\big(1 + 2\omega z^2\big) \\ &= f(y)f(z) \end{aligned}$$

and it follows that f is a group evaluation polynomial over R.

Since f has no roots in R we have

$$G = \{\ g \in R \mid f(g) \neq 0\ \} = R$$

so Theorem 2.2 implies that $G = (G, *)$ is a group of order 16. We let $a = 1$, $b = 3\omega + 1$ and $c = 2\omega$. It is straightforward to show that

$$f(a) = 2\omega + 1\ ,\ f(b) = 1\ ,\ \text{and } f(c) = 1$$

and we calculate the powers

$$\begin{aligned} a^{2*} &= a * a = a + af(a) = 1 + 1(2\omega + 1) = 2\omega + 2 \\ a^{3*} &= a^{2*} * a = a + a^{2*}f(a) = 1 + (2\omega + 2)(2\omega + 1) = 2\omega + 3 \\ a^{4*} &= a^{3*} * a = a + a^{3*}f(a) = 1 + (2\omega + 3)(2\omega + 1) = 0 \end{aligned}$$

and likewise, $b^{2*} = 2\omega + 2 = a^{2*}$, $b^{3*} = \omega + 3$ and $b^{4*} = 0$. Moreover,

$$\begin{aligned} b * a &= a + bf(a) \\ &= 1 + (3\omega + 1)(2\omega + 1) \\ &= \omega \\ &= (3\omega + 1) + (2\omega + 3)f(3\omega + 1) \\ &= a^{3*} * b. \end{aligned}$$

Since the quaternion group Q_8 can be described, using standard notation, as

$$Q_8 =< a, b \mid a^4 = b^4 = 1, a^2 = b^2, \text{and } ba = a^{-1}b >$$

we have shown that the elements $a = 1$, and $b = 3\omega + 1$ generate Q_8 as a subgroup of $G = (G, *)$. Finally, observe that $c = 2\omega$ is an element of order 2 in G, and that c commutes with both a and b. We have now established that $G = Q_8 \times C_2$.

Example 2.8. Let $f(y) = 1 + 2y + 2y^2$ be a polynomial in $R[y]$, where $R = Z_4[\omega]/(\omega^2 + 1)$ is the commutative ring with identity having 16 elements described in (8) above. Then

(2.9) f is a group evaluation polynomial over R

and

(2.10) G is the group generated by a, b, c

where (using standard group notation) $a^4 = b^4$, $a^2 = b^2$, $c^2 = 1$, $ba = a^3bc$, $ca = ac$, $cb = bc$. After showing that $f(y)^2 = 1$, it follows that $f(y)$ is a unit in $U(R)$ and it also follows that $f(y) \neq 0$ for all $y \in R$. By calculations similar to those in Example 2.7, we can show that $f(y * z) = f(y)f(z)$ for all $y, z \in R$ and it will follow that $f(y)$ is a group evaluation polynomial. Theorem 2.2 now implies that $G = (G, *)$ is a group of order 16. Let $a = 1$, $b = \omega$, and $c = 2\omega$. We leave it to the reader as an exercise to show that

$$a^{2*} = 2\ ,\ a^{3*} = 3\ ,\ a^{4*} = 4 = 0,$$

$$b^{2*} = 2 = a^{2*}\ ,\ b^{3*} = \omega + 2\ ,\ b^{4*} = 0,$$

$$c * a = 2\omega + 1 = a * c\,, \quad c * b = 3\omega = a * c,$$

and

$$b * a = \omega + 1 = a^{3*} * b * c.$$

Example 2.9. Let $f(y) = 1 + 2y^2$ be a polynomial in $R[y]$, where $R = Z_4[\omega]/\big(\omega^2 + 1\big)$ is the commutative ring with identity having 16 elements described in (10). Then $G = (G, *)$ is isomorphic to the group $D_4 \times C_2$, where D_4 is the dihedral group of order 8. The elements $a = \omega + 1$ and $b = 1$ will generate D_4 and $c = 2$ will generate the cyclic group of order 2. We will leave all the details as an easy exercise for the reader.

The reader may have already suspected that only metabelian groups $(G, *)$ can be obtained when the binary operation $* : R \times R \to R$ is defined by

$$x * y = y + xf(y)$$

and f is a group evaluation polynomial. This result will be proven in the following:

Theorem 2.10. *Let R denote any commutative ring with identity having finite characteristic $p > 0$. Let f be a group evaluation polynomial over R, and let $G = \{g \mid f(g) \neq 0\}$. If the set G contains at least one unit in R, then the group $G = (G, *)$ is metabelian.*

Proof. We let η be the restriction of f to the group G. Since $\eta(x * y) = \eta(x)\eta(y)$, we know that η is a group homomorphism from $(G, *)$ into $\big(U(R), \cdot\big)$, and consequently,

$$N = ker(\eta) = \{g \in G \mid \eta(g) = 1\}$$

is a normal subgroup of G. Let $x, y \in N$. Clearly, $\eta(x) = \eta(y) = 1$ immediately implies that

$$\begin{aligned} x * y &= y + x\eta(y) \\ &= x + y\eta(x) \\ &= y * x. \end{aligned}$$

We have shown that N is an abelian subgroup of G. Finally, let $[a, b] = a^{-1} * b^{-1} * a * b$ denote the commutator of $a, b \in G$. Then

$$\begin{aligned} \eta\big([a, b]\big) &= \eta\big(a^{-1} * b^{-1} * a * b\big) \\ &= \eta\big(a^{-1}\big)\eta\big(b^{-1}\big)\eta\big(a\big)\eta\big(b\big) \\ &= \eta\big(a^{-1}\big)\eta\big(a\big)\eta\big(b^{-1}\big)\eta\big(b\big) \\ &= \eta\big(a^{-1} * a\big)\eta\big(b^{-1} * b\big) \\ &= \eta\big(0\big)\eta\big(0\big) \\ &= 1. \end{aligned}$$

Consequently, $[a, b] \in N$ and it follows that the commutator subgroup $[G, G] \subset N$ which establishes the theorem. □

The reader should observe that the key element in the proof of Theorem 2.10 is the fact that f restricted to $G = (G, *)$ must be a group homomorphism since the binary operation $*$ is associative, when defined by $a*b = P(a, b) = b+af(b)$, if and only if f is a homomorphism (see Lemma 2.1.).

3. **Concluding remarks**

At this point, several natural questions arise. Can all metabelian groups be found using polynomial evaluations of the form

$$P(x, y) = y + xf(x) \tag{3.1}$$

where f is a group polynomial evaluation over a finite commutative ring R with identity having characteristic $p > 0$? If not, then what types of metabelian groups can be found ?

Suppose that the polynomial $P(x, y)$ is defined by a more general expression; namely,

$$P(x, y) = xf(y) + yg(x) \tag{3.2}$$

where both $f(y)$ and $g(x)$ are polynomials with coefficients in R and having constant coefficient 1. If both f and g are group evaluation polynomials, then the argument within the proof of Theorem 2.10 when extended to the normal abelian subgroup $N = ker(f) \cap ker(g)$ will show that any group $(G, *)$ found within the polynomial groupoid $(R, *)$ must be metabelian. However, for $P(x, y)$ defined by (12), it is possible to obtain groups even though *neither* f or g is a group evaluation polynomial. In fact, we can produce the dihedral group D_4 by using $f(y) = 1 + (\omega + 1)y$ and $g(x) = 1 + (\omega + 1)x^2$ as polynomials in R, where $R = Z_4[\omega]/(\omega^2+1)$ is our ring used in the last three examples. Should the reader be interested in checking this fact, take $a = \omega + 2$ and $b = 2$, and it will be straightforward to show that $D_4 =< a, b \mid a^4 = b^2 = 1 \text{ and } ba = a^{-1}b >$. Moreover, neither polynomial f or g will be a homomorphism on $G = D_4$. We also leave it to the reader to show that $f(a * a) \neq f(a)f(a)$ and that $g(a * a) \neq g(a)g(a)$ when $a = \omega + 2$. This discovery makes it *plausible* that a group $(G, *)$ which is not metabelian could still be found within some sufficiently large type of polynomial groupoid $(R, *)$. Can one find a polynomial $P(x, y)$ having a suitable polynomial groupoid that contains the symmetric group S_4 ?

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