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STRONGLY NONCOSINGULAR MODULES

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Abstract. An R-module M is called *strongly noncosingular* if it has no nonzero Rad-small (cosingular) homomorphic image in the sense of Harada. It is proven that (1) an R-module M is strongly noncosingular if and only if M is coatomic and noncosingular; (2) a right perfect ring R is Artinian hereditary serial if and only if the class of injective modules coincides with the class of (strongly) noncosingular R-modules; (3) absolutely coneat modules are strongly noncosingular if and only if R is a right max ring and injective modules are strongly noncosingular; (4) a commutative ring R is semisimple if and only if the class of injective modules coincides with the class of strongly noncosingular R-modules.
Keywords: coclosed submodules, (non) cosingular modules, coatomic modules.
MSC(2010): Primary: 16D10; Secondary: 16D50, 16D80.

1. **Introduction**

All rings are associative with an identity element and all modules are unitary right R-modules. We use the notation $E(M)$, $\mathrm{Soc}(M)$, $\mathrm{Rad}(M)$ for the injective hull, socle, radical of an R-module M, respectively. We denote the radical of R by $J(R)$. We use $N \leq M$ to signify that N is a submodule of M.

Let M be an R-module and let N be a submodule of M. N is called a *small* submodule of M, denoted as $N \ll M$, if $N + K = M$ implies $K = M$ for any submodule K of M. A submodule K of M is called a *supplement* of N in M if K is minimal with respect to the property $M = K + N$, equivalently, $M = K + N$ and $KapN \ll K$. N is called an *essential* submodule of M, denoted by $N \trianglelefteq M$, if $NapL \neq 0$ for each nonzero submodule L of M. For an R-module M, the submodule $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal I of R $\}$ is called the *singular* submodule of M. An R-module M is said to be a *singular* (*nonsingular*) if $Z(M) = M$ ($Z(M) = 0$). Suppose that

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$0 \leq A \leq B \leq N$. Then, A is an essential submodule of B if $A/0 \trianglelefteq B/0$. Dually, we say that A is a *coessential* submodule of B in N (denoted by $A \hookrightarrow^{ce} B$ in N) if $B/A \ll N/A$. It is easy to see that $A \hookrightarrow^{ce} B$ in N if and only if $B + X = N$ implies $A + X = N$. An R-module M is said to be a *lifting* module if, for any submodule A of M, there exists a direct summand B of M such that B is a coessential submodule of A in M. A submodule A of N is said to be *coclosed* in N (denoted by $A \hookrightarrow^{cc} N$) if A has no proper coessential submodule in N. A module M is called *weakly injective* if for every extension N of M, M is coclosed in N. All supplement submodules are coclosed (see, for example, [7, 20.2]).

In [16], Leonard defines a module M to be *small* if it is a small submodule of some R-module and he shows that M is small if and only if M is small in its injective hull. We put $Z^*(M) = \{m \in M \mid mR$ is a small module $\}$. As the dual notion of singular (nonsingular), an R-module M is called *cosingular* if $Z^*(M) = M$ (see [11]). An R-module M is cosingular if and only if $M \leq \mathrm{Rad}(L)$ for some R-modules L. Since $\mathrm{Rad}(M)$ is the union of all small submodules of M, we see that $Z^*(E) = \mathrm{Rad}(E)$ for any injective module E, and $Z^*(M) = M ap\, \mathrm{Rad}(E(M))$. The radical $\mathrm{Rad}(M)$ of an R-module M is a submodule of $Z^*(M)$. For further properties of $Z^*()$, see [21]. For convenience in concepts, the cosingular R-modules are called *Rad-small* in this work.

Following [26], a module M is called *noncosingular* if for every nonzero module N and every nonzero homomorphism $f : M \to N$, $\mathrm{Im}\, f$ is not a small module. An R-module M is noncosingular if and only if every homomorphic image of M is weakly injective (see [34]). Recently, there has been a significant interest in noncosingular R-modules, see [12, 27, 29, 30, 34].

In this article, we introduce the concept of strongly noncosingular R-module. An R-module M is called *strongly noncosingular* if for every nonzero module N and every nonzero homomorphism $f : M \to N$, $\mathrm{Im}\, f$ is not a Rad-small module. Since small modules are Rad-small, strongly noncosingular R-modules are noncosingular, but the converse is not true in general (see Example 2.8). Our aim is to work on the concept of strongly noncosingular modules and investigate the rings and modules that can be characterized via these modules. In particular, section 2 deals with strongly noncosingular modules and its characterizations. We have also proved that an R-module M is strongly noncosingular if and only if M is coatomic and every simple homomorphic image of M is injective. We have showed that a right perfect ring R is Artinian hereditary serial if and only if the class of injective R-modules coincides with the class of (strongly) noncosingular R-modules. A right hereditary ring R is max ring if and only if absolutely coneat R-modules are strongly noncosingular.

Section 3 deals with the structure of strongly noncosingular R-modules on commutative rings. We have showed that strongly noncosingular R-modules are exactly the semisimple injective modules on commutative noetherian rings.

A commutative ring R is semisimple if and only if the class of injective modules coincides with the class of strongly noncosingular R-modules.

2. **Strongly Noncosingular Modules**

We introduce the concept of strongly noncosingular R-module as follows.

Definition 2.1. An R-module M is called *strongly noncosingular* if for every nonzero R-module N and every nonzero homomorphism $f : M \to N$, $\operatorname{Im} f$ is not a Rad-small submodule of N, i.e. M has no nonzero Rad-small homomorphic image.

Remark 2.2. (1) Simple injective R-modules are strongly noncosingular.
(2) Let R be a division ring (e.g. the rational numbers $\mathbb{Q}$). An R-module M is a vector space, so it is a semisimple injective R-module. Therefore, it is strongly noncosingular.
(3) Let R be a right hereditary ring. Finitely generated injective R-modules are strongly noncosingular. Let M be a finitely generated injective R-module. Suppose that $Z^*(M/N) = M/N$ for a submodule N of M. Since R is a right hereditary ring, M/N is injective and so $\operatorname{Rad}(M/N) = Z^*(M/N) = M/N$. Since M/N is finitely generated, $\operatorname{Rad}(M/N) \ll M/N$, a contradiction. Thus, finitely generated injective R-modules are strongly noncosingular.
(4) Strongly noncosingular R-modules are noncosingular. However, there exists a noncosingular R-module which is not strongly noncosingular (see Example 2.8).

Proposition 2.3. *The class of all strongly noncosingular R-modules is closed under homomorphic images, direct sums, direct summands, extensions and small covers.*

Proof. (1) Let M be a strongly noncosingular R-module and N a submodule of M. Suppose that M/N is not a strongly noncosingular R-module. Then, there is a nonzero homomorphism g from M/N to the R-modules T with $\operatorname{Im} g \leq \operatorname{Rad}(T)$. Then $\operatorname{Im}(g\pi) \leq \operatorname{Rad}(T)$, where π is the canonical epimorphism $M \to M/N$. Since M is strongly noncosingular, $\operatorname{Im}(g\pi) = 0$. Then $g = 0$, a contradiction.

(2) Assume that $(M_i)_{i \in I}$ is a class of strongly noncosingular R-modules. Let f be a homomorphism from $\bigoplus_{i \in I} M_i$ to the R-module N with $\operatorname{Im} f \leq \operatorname{Rad}(N)$. Then, $\operatorname{Im}(f\iota_i) \leq \operatorname{Rad}(N)$ for the inclusion maps $\iota_i : M_i \to \bigoplus_{i \in I} M_i$ for every $i \in I$. Since M_i is a strongly noncosingular R-module, $\operatorname{Im}(f\iota_i) = 0$ for every $i \in I$. Then $f = 0$, and $\bigoplus_{i \in I} M_i$ is strongly noncosingular.

(3) Let N be a direct summand of a module M and $p : M \to N$ the projection map. Let f be a homomorphism from N to the R-modules T with $\operatorname{Im} f \leq \operatorname{Rad}(T)$. Then, $\operatorname{Im}(fp) \leq \operatorname{Rad}(T)$ and, by the hypothesis, $fp(M) = 0$. Hence $f = 0$, and N is strongly noncosingular.

(4) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence, and suppose that A and C are strongly noncosingular R-modules. Assume that there is a homomorphism f from B to some R-module T with $\operatorname{Im} f \leq \operatorname{Rad}(T)$. Then, $\operatorname{Im}(f\alpha) \leq \operatorname{Rad}(T)$. Since A is a strongly noncosingular R-module, $\operatorname{Im}(f\alpha) = 0$. Then, there is a homomorphism g from C to T such that $f = g\beta$ by the factor theorem (see [1, Theorem 3.6]). Therefore $\operatorname{Im} g \leq \operatorname{Rad}(T)$ and since C is a strongly noncosingular R-module, $\operatorname{Im} g = 0$. Thus, $f = 0$ and B is strongly noncosingular.

(5) Let B be a strongly noncosingular R-module and let $f : A \to B$ be a small cover, i.e. f is an epimorphism and $\operatorname{Ker} f \ll A$. Suppose that A is not a strongly noncosingular R-module. Then, there is a submodule X of A such that A/X is Rad-small. $B/f(X)$ is Rad-small, since it is a homomorphic image of the Rad-small module A/X by [21, Lemma 2.6]. But $B/f(X)$ is strongly noncosingular by (1), hence $B/f(X) = 0$ and $B = f(X)$. Then, $f^{-1}(B) = X + \operatorname{Ker} f = A$, and $X = A$ since $\operatorname{Ker} f \ll A$. Hence A is strongly noncosingular. $\square$

Corollary 2.4. *Let M be a module, and U and V be submodules of M such that V is a supplement of U. Then V is strongly noncosingular if and only if M/U is strongly noncosingular.*

Proof. By the hypothesis, $M = U + V$, $UapV \ll V$ and $M/UongV/(UapV)$. Suppose that V is strongly noncosingular. Since strongly noncosingular R-modules are closed under homomorphic images by Proposition 2.3, $M/UongV/(UapV)$ is strongly noncosingular. Conversely, assume that $M/UongV/(UapV)$ is strongly noncosingular. By Proposition 2.3, strongly noncosingular R-modules are closed under small covers and using $UapV \ll V$, we obtain that V is strongly noncosingular. $\square$

Proposition 2.5. *Let M be a strongly noncosingular R-module. The following properties hold:*

(1) Every Rad-small submodule of M is small in M.
(2) Coclosed submodules of M are strongly noncosingular.
(3) $\operatorname{Rad} M \ll M$.
(4) $\operatorname{Rad} M = \operatorname{Rad} NapM$ *for every extension N of M.*
(5) $\operatorname{Rad}(M) = Z^*(M)$.

Proof. (1) Suppose that K is a Rad-small submodule of M and $K + L = M$ for a submodule L of M. Since $\frac{K}{KapL}$ is a homomorphic image of K, it is Rad-small by [21, Lemma 2.6]. But M is strongly noncosingular, and so $\frac{K}{KapL} = 0$ by Proposition 2.3. Hence $KapL = K$ and so $L = M$ and $K \ll M$.

(2) Let A be a coclosed submodule of M. Suppose that A/X is a Rad-small R-module for a submodule X of A. Since M is strongly noncosingular, M/X is also strongly noncosingular by Proposition 2.3. Then, by (1), $A/X \ll M/X$.

But A is a coclosed submodule of M, so $X = A$. This implies A is a strongly noncosingular R-module.

(3) and (4) follow from (1).

(5) $\operatorname{Rad} M \leq Z^*(M)$ is clear. Conversely, let $m \in Z^*(M)$, then mR is a small module and, by (1) it follows that $mR \ll M$. Thus, $m \in \operatorname{Rad} M$. □

An R-module M is called *coatomic* if for every submodule N of M, $\operatorname{Rad}(M/N) = M/N$ implies $M/N = 0$, equivalently every proper submodule of M is contained in a maximal submodule of M. Finitely generated and semisimple modules are coatomic. Coatomic modules appear in the theory of supplemented, semiperfect, and perfect modules (see [35]).

Theorem 2.6. *Let M be an R-module.Then the following statements are equivalent:*

(1) *M is strongly noncosingular.*
(2) *M is coatomic and every simple homomorphic image of M is injective.*
(3) *M is coatomic and noncosingular.*

Proof. Note that any simple module is either small or injective ([7, 8.2]).

(1) ⇒ (2) Let N be a proper submodule of M. Suppose N is not contained in a maximal submodule of M. Then $M/N = Rad(M/N)$, and this implies M/N is Rad-small. But M is strongly noncosingular, and so $M/N = 0$, a contradiction. By the given above, a simple homomorphic image of a strongly noncosingular R-module is injective.

(2) ⇒ (3) Let N be a proper submodule of M with M/N is a small module. If N is maximal submodule of M, then M/N is injective. Suppose that N is not a maximal submodule of M. By the assumption, M is coatomic, hence there exists a maximal submodule K of M which contains N. By [16, Theorem 2], small modules are closed under homomorphic images, hence $M/Kong\frac{M/N}{K/N}$ is small. But M/K is injective by the assumption, hence $M/K = 0$, a contradiction . Then, M has no small homomorphic images, i.e. M is noncosingular.

(3) ⇒ (1) Let N be a proper submodule of M with M/N is a Rad-small R-module. By the assumption, M is coatomic, and hence there exists a maximal submodule K of M which contains N. M/K is injective since M is noncosingular. Rad-small modules are closed under homomorphic image by [21, Lemma 2.6], so $M/Kong\frac{M/N}{K/N}$ is Rad-small. Then, $\operatorname{Rad}(M/K) = M/K$, and this contradicts with the fact that M is coatomic. Hence, M is strongly noncosingular. □

By Theorem 2.6, a strongly noncosingular R-module exists if and only if there exists a simple injective R-module.

Proposition 2.7. *Let R be a domain which is not a division ring. Then there does not exist a strongly noncosingular R-module.*

Proof. It is enough to show that there is no simple injective R-module. Assume that there exists a simple injective R-module, say S. Then S is divisible. Since S is simple, there exists a nonzero maximal ideal I of R such that $SongR/I$. Then $(R/I)r = 0$ for each $r \in I$, which contradicts with the divisibility of S. Hence, there is no simple injective R-module. $\square$

Example 2.8. We give an example for a noncosingular R-module that fails to be strongly noncosingular. Consider the ring $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, c \in \mathbb{Z}, b \in \mathbb{Q}\}$ and the R-module ${}_RM = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}$. The left R-module structure of M is completely determined by the left $\mathbb{Z}$-module structure of $\mathbb{Q}$. Then M is not coatomic since ${}_{\mathbb{Z}}\mathbb{Q}$ is not coatomic. But M is noncosingular since every nonzero homomorphic image of the $\mathbb{Z}$-module $\mathbb{Q}$ is nonsmall.

A ring R is called a right *max ring* if $\mathrm{Rad}(M) \neq M$ for every R-module M. Equivalently, R is a right max ring if and only if every nonzero R-module is coatomic. Any right perfect ring R is right max ring and the converse is true if $R/J(R)$ is a semisimple ring (see [1, Theorem 28.4]). Theorem 2.6 yields the following.

Corollary 2.9. *Let R be a right max ring. An R-module M is strongly noncosingular if and only if it is noncosingular.*

Recall that an R-module M is called *weakly injective* if, for every extension N of M, M is coclosed in N.

Proposition 2.10. *Strongly noncosingular R-modules are weakly injective.*

Proof. Let M be a strongly noncosingular R-module and $M \leq N$ be any extension of M. Let L be a proper submodule of M. Since M/L is not Rad-small, M/L cannot be a small submodule of N/L. Hence L is not a coessential submodule of M in N, this implies M is coclosed in N. So, M is weakly injective. $\square$

The converse of Proposition 2.10 is not true in general. In Example 2.8, the R-module ${}_RM$ is injective, so it is weakly injective, but not strongly noncosingular.

Proposition 2.3, Theorem 2.6 and Proposition 2.10 yield the following corollary.

Corollary 2.11. *Let M be a coatomic module. Then the following statements are equivalent:*

(1) M is strongly noncosingular.
(2) Every homomorphic image of M is weakly injective.
(3) Every finitely generated quotient of M is weakly injective.
(4) Every cyclic quotient of M is weakly injective.

(5) Every simple quotient of M is injective.

A ring R is called a right *V-ring* if each simple R-module is injective. This is equivalent to the condition that $Rad(M) = 0$ for any R-module M (see, [15, Theorem 3.75]). Clearly, every R-module is coatomic if R is a right V-ring.

Corollary 2.12. *Let R be a ring. The following statements are equivalent:*

(1) R_R is strongly noncosingular.
(2) R is a right V-ring.
(3) Every quotient of R is weakly injective.
(4) Every R-module is strongly noncosingular.

Proof. (1) $\Leftrightarrow$ (2) ¡follows from¿ Theorem 2.6, (1) $\Leftrightarrow$ (3) ¡follow¿ by Corollary 2.11, and (4) $\Rightarrow$ (1) is clear. For (1) $\Rightarrow$ (4), note that every module is an epimorphic image of a free module. Since R_R is a strongly noncosingular module, every free module is strongly noncosingular by Proposition 2.3. Again by Proposition 2.3, every R-module is strongly noncosingular. $\square$

In [9], a submodule N of an R-module M is called *coneat* in M if for every simple R-module S, any homomorphism $\varphi : N \to S$ can be extended to a homomorphism $\theta : M \to S$. In [8], an R-module M is called *absolutely coneat* if M is a coneat submodule of any module containing it. Absolutely coneat modules are also studied in [4]. Coclosed submodules are coneat by [4, Proposition 2.1]. Thus we may say that weakly injective modules are absolutely coneat. We have the following implications among the concepts:

$$\begin{array}{ccccccc} \text{strongly noncosingular} & \Longrightarrow & \text{noncosingular} & \Longrightarrow & \text{weakly injective} & \Longleftarrow & \text{injective} \\ & & & & \Downarrow & & \\ & & & & \text{absolutely coneat} & & \end{array}$$

Proposition 2.13. *Every injective module is strongly noncosingular if and only if every weakly injective module is strongly noncosingular.*

Proof. Let M be a weakly injective module. By the assumption, $E(M)$ is strongly noncosingular. Then, M is strongly noncosingular by Proposition 2.5. The converse follows from the fact that injective modules are weakly injective. $\square$

Proposition 2.14. *Assume that every injective module is strongly noncosingular. Then the following holds.*

(1) The class of absolutely coneat modules is closed under homomorphic images.
(2) A simple module S is either injective or $\mathrm{Hom}(E, S) = 0$ for each injective module E.

If R is commutative, the following statement also holds.

(3) *Every simple submodule S of a flat module F is flat, i.e.* $\mathrm{Soc}(F)$ *is flat.*

Proof. (1) Let L be an absolutely coneat module and N be a submodule of L. Consider the coneat exact sequence $0 \to L \to E(L) \to M \to 0$. We have the pushout diagram

$$\begin{tikzcd}
& 0 \arrow[r] & L \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{E}: & 0 \arrow[r] & L/N \arrow[r] & P \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Since coneat exact sequences are closed under pushout, $\mathbb{E}$ is coneat exact. On the other hand, γ is epimorphism, and so P is absolutely coneat by the assumption and Proposition 2.3. Therefore, L/N is absolutely coneat by [8, Theorem 3.2].

(2) This follows by Proposition 2.3 since a simple module S is either injective or small.

(3) Consider the exact sequence $0 \to S \hookrightarrow F \to M \to 0$. We have the exact sequence $0 \to \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(F, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z}) \to 0$. Note that if R is commutative and E is an injective cogenerator, then $\mathrm{Hom}(S, E) \cong S$ for each simple module S. Since $\mathrm{Hom}(F, \mathbb{Q}/\mathbb{Z})$ is injective by [15, Theorem 4.9], S is injective by (2). Then, S is a direct summand of the flat module F, and so it is flat. □

It is well known that a ring R is right hereditary if and only if every homomorphic image of an injective R-module is injective (see [15, Theorem 3.22]). Recall that every module is coatomic on max rings. Thus, by Theorem 2.6 and Proposition 2.14, we have the following.

Corollary 2.15. *Let R be a right max ring. The following are equivalent.*

(1) *Every simple quotient of an injective module is injective.*
(2) *Every injective module is strongly noncosingular.*
(3) *The class of absolutely coneat modules is closed under homomorphic images.*
(4) *Every absolutely coneat module is strongly noncosingular.*

Remark 2.16. (1) For a right small ring R i.e. $\mathrm{Rad}(E) = E$ for every injective right R-module E, we have $\mathrm{Hom}(E, S) = 0$ for each simple right R-module S. Therefore, if R is a right small right max ring, every injective module is strongly noncosingular.

(2) It is clear that injective modules are noncosingular on hereditary rings. Let R be a right hereditary max ring. Then, every injective module is strongly noncosingular.

Example 2.17. This example exhibits a V-ring that is not hereditary. Let $R = \prod_{i=1}^{\infty} F_i$ with $F_i = F$ is a field for all $i \geq 1$. R is a right V-ring that is not semisimple. Hence, every R-module is strongly noncosingular, so injective modules are strongly noncosingular, and R is right self injective by [15, Corollary 3.11.B]. Hence R is not right hereditary, otherwise it must be semisimple (see, [15, Theorem 7.52]).

Although we suspect that a ring whose injective modules are strongly noncosingular is a max ring, we have not yet been able to prove it.

Proposition 2.18. *Let R be a local ring. The following statements are equivalent.*

(1) R is a division ring.
(2) Every injective right R-module is strongly noncosingular.
(3) Every injective left R-module is strongly noncosingular.

Proof. (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (1), let E be a nonzero injective right R-module. Then E is coatomic by the hypothesis. So E has an injective simple factor module, say E/K. Since R is a local ring, every simple right R-module is isomorphic to $R/J(R)$. Hence $R/J(R)$ is injective. This implies R is a V-ring, and so $J(R) = 0$. Therefore R is a division ring. (1) $\Rightarrow$ (3) follows by left-right symmetry. □

Proposition 2.19. *Let R be a ring. The following statements hold:*

(1) If R is a right max ring, then absolutely coneat modules are weakly injective.
(2) If absolutely coneat modules are strongly noncosingular, then R is a right max ring.

Proof. (1) Let A be an absolutely coneat module and M be any extension of A. Suppose A is not a coclosed submodule of M. Then for some proper submodule B of A , $A/B \ll M/B$. Since R is a right max ring, A is coatomic. Thus, B is contained in a maximal submodule, say K, of A. Then, $A/K \ll M/K$, which contradicts with the fact that A is coneat in M. Hence, A is weakly injective.
(2) Let M be an R-module with $\mathrm{Rad}(M) = M$. Then, $\mathrm{Hom}(M, S) = 0$ for each simple module S. Hence M is absolutely coneat and, by the assumption, M is strongly noncosingular. By Proposition 2.5(3), $\mathrm{Rad}(M) \ll M$. So, $M = 0$ since $\mathrm{Rad}(M) = M$. Therefore, R is a right max ring. □

Corollary 2.20. *Let R be a ring. R is a right max ring and every injective module is strongly noncosingular if and only if every absolutely coneat module is strongly noncosingular.*

A ring R is called a right *H-ring* if every injective right R-module is lifting (see [19]). Let R be a right nonsingular ring. Then every nonsingular right

R-module is projective if and only if R is Artinian hereditary serial (see [6, Theorem 4.2]). Artinian hereditary serial rings are right (left) H-rings.

Theorem 2.21. *Let R be a right perfect ring. The following statements are equivalent:*

(1) The class of injective modules coincides with the class of (strongly) noncosingular R-modules.

(2) R is Artinian hereditary serial.

Proof. (1) $\Rightarrow$ (2) Since strongly noncosingular R-modules are closed under homomorphic images, every homomorphic image of an injective module is injective by the assumption. R is a right hereditary ring by [15, Theorem 3.22] and so it is right nonsingular by [15, Corollary 7.7]. Under the assumption, injective R-modules are closed under small covers by Proposition 2.3. Then R is a right H-ring by [19, Theorem I]. Therefore nonsingular right R-modules are projective by [19, Theorem II and Theorem 4.6]. And so R is hereditary Artinian serial by [6, Theorem 4.2].

(2) $\Rightarrow$ (1) By Remark 2.16(2), injective modules are strongly noncosingular. Let M be a strongly noncosingular R-module. Since an Artinian hereditary serial ring R is a right H-ring by [19], M has a decomposition $M = M_1 \oplus M_2$, where M_1 is injective and M_2 is small. By Proposition 2.3, M_1 and M_2 are strongly noncosingular. But M_2 is a small module, and so $M_2 = 0$. Therefore, $M_1 = M$ is injective. □

A ring R is called right *Kasch* if every simple right R-module S can be embedded in R_R (see [15, 8.26]).

Lemma 2.22. *Let R be a right Kasch ring. An R-module M is strongly noncosingular if and only if M is semisimple and every simple submodule of M is injective.*

Proof. Suppose M is not semisimple, i.e. $\mathrm{Soc}(M) \neq M$. Since M is strongly noncosingular, M is coatomic by Theorem 2.6. Then, the proper submodule $\mathrm{Soc}(M)$ is contained in a maximal submodule of M, say K. Since M is strongly noncosingular, M/K is injective by Theorem 2.6. By the hypothesis, M/K embeds in R. But M/K is injective, and so it is a direct summand of R. Hence M/K is projective. So, $M = K \bigoplus S$ for some submodule S of M. Since K is maximal in M, $S \cong M/K$ is a simple module. Then, $S \leq \mathrm{Soc}(M)$, and hence $S \leq \mathrm{Soc}(M)apS \leq KapS = 0$, a contradiction. Thus, we must have $M = \mathrm{Soc}(M)$. Therefore, M is semisimple. Since M is semisimple, every simple submodule N of M is isomorphic to a simple homomorphic image of M. Then, N is injective by Theorem 2.6. The converse follows from Theorem 2.6, since semisimple modules are coatomic. □

A ring R is said to be *semilocal*, if $R/J(R)$ is a semisimple ring (see [15, §20]). Any right or left Artinian ring, any serial ring, and any semiperfect ring is semilocal. A ring R is semilocal if and only if every product of simple modules is semisimple (see [1, 15.17]).

Lemma 2.23. *Let R be a semilocal ring. An R-module M is strongly noncosingular and every maximal submodule of M is a direct summand if and only if M is semisimple injective.*

Proof. Suppose on the contrary that $\mathrm{Soc}(M) \neq M$. Since M is stongly noncosingular, M is coatomic and the proper submodule $\mathrm{Soc}(M)$ is contained in a maximal submodule of M, say K. By the assumption, every maximal submodule of M is a direct summand of M. So, $M = K \bigoplus S$ for some submodule S of M. Since K is maximal in M, M/K is a simple module. Thus, $S \cong M/K$ is simple and $S \leq \mathrm{Soc}(M)$, so $S \leq \mathrm{Soc}(M) apS \leq K apS = 0$, a contradiction. Thus, we must have $M = \mathrm{Soc}(M)$. Therefore, M is semisimple, and $M = \bigoplus_{\lambda \in \Lambda} S_\lambda$ for some index set Λ and simple submodules S_λ of M. Then $M \leq N := \prod_{\lambda \in \Lambda} S_\lambda$. Since R is semilocal, the right side N is also a semisimple R-module. Every simple summand $(S_\lambda, \lambda \in \Lambda)$ of M is injective since M is strongly noncosingular. Thus, $N = \prod_{\lambda \in \Lambda} S_\lambda$ is injective. Then, the direct summand M of N is injective. So, M is semisimple injective. The converse is clear. $\square$

Lemma 2.22 and Lemma 2.23 yield the following.

Corollary 2.24. *Let R be a semilocal right Kasch ring. An R-module M is strongly noncosingular if and only if M is semisimple injective.*

3. Strongly Noncosingular Modules Over Commutative Rings

In this section¡,¿ we investigate the strongly noncosingular R-modules over commutative rings. For a ring R, let $Z(R) = \{r \in R \mid rs = sr, \text{for each}\, s \in R\}$ be the center of R. Recall that for any ring R and R-module M, $J(R).M \leq \mathrm{Rad}(M)$, see [1, Corollary 15.18.].

Proposition 3.1. *Let R be a ring and M a strongly noncosingular right R-module. Then, $Z^*(R_R) apZ(R) \leq \mathrm{Ann}_R(M)$.*

Proof. Let $r \in Z^*(R_R) apZ(R)$. Since $r \in Z(R)$, the map $f : M \to M$, defined by $f(m) = mr$ for each $m \in M$, is an R-homomorphism. Note that $MZ^*(R_R) \leq Z^*(M)$ by [21, Lemma 3.8]. Then, $r \in Z^*(R_R)$ implies that $\mathrm{Im}(f) = Mr \leq \mathrm{Rad}(E(M))$. Therefore, $f = 0$, and so $Mr = 0$ by the hypothesis. Hence $r \in \mathrm{Ann}_R(M)$. $\square$

Corollary 3.2. *Let R be a ring and M a strongly noncosingular right R-module. Then, $J(R) apZ(R) \leq \mathrm{Ann}_R(M)$.*

Corollary 3.3. *Let R be a commutative ring and M a strongly noncosingular R-module. Then, $Z^*(R).M = J(R).M = 0$.*

Corollary 3.4. *Let R be a commutative semilocal ring and M an R-module. M is strongly noncosingular if and only if M is a semisimple injective module.*

Proof. If R is semilocal then, $\operatorname{Rad}(M) = J(R).M = 0$. So, M is an $R/J(R)$-module. Therefore M is a semisimple $R/J(R)$-module, and so M is semisimple as an R-module. The injectivity of M follows from Lemma 2.23. The converse is clear. □

Proposition 3.5. *Let R be a commutative ring and M a strongly noncosingular R-module. Then $\operatorname{Ann}(M/\operatorname{Rad}(M)) = \operatorname{Ann}(M)$.*

Proof. Let $r \in \operatorname{Ann}(M/\operatorname{Rad}(M))$. Then $rM \leq \operatorname{Rad}(M)$ and so from the proof of Proposition 3.1, we get $rM = 0$. Therefore, $r \in \operatorname{Ann}(M)$ and $\operatorname{Ann}(M/\operatorname{Rad}(M)) \leq \operatorname{Ann}(M)$. On the other hand, we always have $\operatorname{Ann}(M) \leq \operatorname{Ann}(M/\operatorname{Rad}(M))$. This completes the proof. □

Lemma 3.6. *[1, Exercises 15.(5)] Let R be a commutative ring and Ω the set of all maximal ideals of R. Then $\operatorname{Rad}(M) = \bigcap_{P\in\Omega} PM$ for each R-module M.*

Proposition 3.7. *Let R be a commutative ring and M a nonzero R-module with a unique maximal submodule. Then M is strongly noncosingular if and only if M is simple injective.*

Proof. We first claim that M is a simple R-module. By the hypothesis $\operatorname{Rad}(M)$ is a maximal submodule of M, i.e. $M/\operatorname{Rad}(M)$ is simple. Then $M/\operatorname{Rad}(M) \cong$

R/P for some maximal ideal P of R. Since M is strongly noncosingular, $\operatorname{Ann}(M) = \operatorname{Ann}(M/\operatorname{Rad}(M)) = P$ by Proposition 3.5. Then $P.M = 0$, and so $\operatorname{Rad}(M) = 0$ by Lemma 3.6. Therefore, M is a simple R-module and, by the hypothesis, M is injective. The converse is clear. □

Lemma 3.8. *[31, Lemma 2.6] Let R be a commutative ring and M a simple R-module. Then, M is flat if and only if M is injective.*

Lemma 3.9. *Let R be a commutative Noetherian ring and M an R-module. Then, M is strongly noncosingular if and only if M is semisimple injective.*

Proof. Suppose on the contrary that $\operatorname{Soc}(M) \neq M$. Since M is strongly noncosingular, the proper submodule $\operatorname{Soc}(M)$ is contained in a maximal submodule K of M such that M/K is injective. So, M/K is flat by the above lemma. Since M/K is finitely generated, it is finitely presented by [15, Proposition 4.29]. Therefore, M/K is projective by [15, Theorem 4.30]. Then K is a direct summand of M such that $M = K \bigoplus S$ for some simple submodule S of M. But $S \leq \operatorname{Soc}(M) apS \leq (KapS) = 0$, a contradiction. Hence M is semisimple. Then $M = \bigoplus_{i\in I} S_i$, where each S_i is a simple module for all $i \in I$. Since M is strongly noncosingular, all simple summands of M is strongly noncosingular by Proposition 2.3, and so they are injective by Theorem 2.6. Then M is injective since direct sums of injective modules are injective by [1, Proposition 18.13]. The converse is clear. □

¡In General¿ Lemma 3.9 is not true in the noncommutative case.

Example 3.10. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ be upper triangular matrices over a field F. R is a right hereditary Artinian ring and $\mathrm{Soc}(R_R) = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$. Let $A = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$. Then $R_R = A \oplus B$, A is injective and B is simple. Since R is a right hereditary Artinian ring, A is strongly noncosingular by Remark 2.16. However A is not a semisimple R-module otherwise $Soc(R_R) = R_R$, a contradiction .

Theorem 3.11. *Let R be a commutative ring. Then the following statements are equivalent:*

(1) *The class of injective modules coincides with the class of strongly noncosingular R-modules.*
(2) *R is semisimple.*

Proof. (1) $\Rightarrow$ (2) Since strongly noncosingular R-modules are closed under direct sums by Proposition 2.3, direct sums of injective modules are injective by the assumption. Then R is a Noetherian ring by [1, Proposition 18.13]. By (1) and by Lemma 3.9, every injective module is semisimple, and so every module is semisimple. Then R is semisimple. (2) $\Rightarrow$ (1) ¡follows¿ by Lemma 3.9. □

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