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ON CERTAIN SEMIGROUPS OF TRANSFORMATIONS THAT PRESERVE DOUBLE DIRECTION EQUIVALENCE

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Abstract. Let T_X be the full transformation semigroups on the set X. For an equivalence E on X, let $T_{E^*}(X) = \{\alpha \in T_X : \forall (x, y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}$ It is known that $T_{E^*}(X)$ is a subsemigroup of T_X. In this paper, we discuss the Green's *-relations, certain *-ideal and certain Rees quotient semigroup for $T_{E^*}(X)$.
Keywords: Transformation semigroups, Equivalence, Green's *-relations, *-Ideal, Rees quotient semigroup.
MSC(2010): Primary: 20M20; Secondary: 54H15.

1. Introduction

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroup S are generalization of the familiar Green's relations $\mathcal{L}$ and $\mathcal{R}$. Two elements a and b in S are said to be $\mathcal{L}^*$-related if and only if they are $\mathcal{L}$-related in some oversemigroup of S. The relation $\mathcal{R}^*$ can be defined dually. The join of the equivalence relations $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection is denoted by $\mathcal{H}^*$. A semigroup S is called abundant if any $\mathcal{L}^*$-class and any $\mathcal{R}^*$-class contains an idempotent of S. It is known that a regular semigroup is abundant but the converse is not true. For example, Umar [6] showed that the semigroup of order-decreasing finite full transformations is abundant but not regular.

The $\mathcal{L}^*$-class containing the element a of the semigroup S will be denoted by L^*_a. The corresponding notation will be used for the classes of the other relations. A left (right) *-ideal of a semigroup S is defined to be a left(right) ideal I of S such that $L^*_a \subseteq I$ ($R^*_a \subseteq I$) for all $a \in I$. A subset I of S is a *-ideal of S if it is both a left *-ideal and a right *-ideal. A principal *-ideal $J^*(a)$ generated by the element a of S is the intersection of all *-ideals of S to

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which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that $(a,b) \in \mathcal{J}^*$ if and only if $J^*(a) = J^*(b)$.

In the theory of abundant semigroups, the relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$, and $\mathcal{D}^*$ together with the relation $\mathcal{J}^*$ play a role which is to some extent analogous to that of Green's relations in the theory of regular semigroups.

Let T_X be the full transformation semigroups on a set X and E be an equivalence on X. Denote

$$T_{E^*}(X) = \{\alpha \in T_X : \forall (x,y) \in E \Leftrightarrow (x\alpha, y\alpha) \in E\}.$$

Then $T_{E^*}(X)$ is a subsemigroup of $T_E(X)$, it's Green's relations and regularity are investigated in [1].

2. **Preliminaries**

We denote composition of two mappings by juxtaposition and adopt a right mapping convention: $\alpha\beta$ denotes the mapping obtained by performing first α and then β.

Denote by X/E the quotient set. The symbol $\pi(\alpha)$ denotes the decomposition of X induced by the map α, namely

$$\pi(\alpha) = \{x\alpha^{-1} : x \in X\alpha\}.$$

Then $\pi(\alpha) = X/ker(\alpha)$, where $ker(\alpha) = \{(x,y) \in X \times X : x\alpha = y\alpha\}$.

Denote by I the identical equivalence on X, i.e.:

$$I = \{(x,x) : x \in X\}.$$

Lemma 2.1. *Let $\alpha, \beta \in T_X$. Then the following statements hold:*
(1) $(\alpha,\beta) \in \mathcal{L}$ if and only if $X\alpha = X\beta$.
(2) $(\alpha,\beta) \in \mathcal{R}$ if and only if $\pi(\alpha) = \pi(\beta)$.
(3) $(\alpha,\beta) \in \mathcal{D}$ if and only if $|X\alpha| = |X\beta|$.
(4) $\mathcal{D} = \mathcal{J}$.

Lemma 2.2. *[2] Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*
(1) $(a,b) \in \mathcal{L}^*$.
(2) For all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

Dually, we have:

Lemma 2.3. *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*
(1) $(a,b) \in \mathcal{R}^*$.
(2) For all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$.

3. Green's *-relations

In this section, we focus our attention on Green's *-relations for the semigroup $T_{E^*}(X)$, beginning with $\mathcal{L}^*$.

Theorem 3.1. *Let $\alpha, \beta \in T_{E^*}(X)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ if and only if $X\alpha = X\beta$.*

Proof. If $X\alpha = X\beta$, then by Lemma 2.1, $(\alpha, \beta) \in \mathcal{L}$ in T_X. Hence $(\alpha, \beta) \in \mathcal{L}^*$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^*$, by Lemma 2.2, for all $\delta, \gamma \in T_{E^*}(X)$, $\alpha\delta = \alpha\gamma$ if and only if $\beta\delta = \beta\gamma$. If $X\alpha \neq X\beta$, without loss of generality, we may assume that $X\beta \setminus X\alpha \neq \emptyset$. Then there exists $a \in X\beta \setminus X\alpha$ and $b\beta = a$ for some $b \in X$. There are two cases to consider: (Denote by 1_X the identity mapping on X)

Case 1. $a \in A \in X/E$ and $A \bigcap X\alpha \neq \emptyset$.

Define $\delta : X \to X$ by:

$$a\delta = c,\ x\delta = x(x \neq a), \text{ where } c \in A \text{ and } c \neq a.$$

It is easy to verify that $\delta \in T_{E^*}(X)$ and $\alpha\delta = \alpha \cdot 1_X$. However, $b\beta\delta = a\delta = c \neq a = b\beta = b(\beta \cdot 1_X)$. This contradicts with $\beta\delta = \beta \cdot 1_X$.

Case 2. $a \in A \in X/E$ and $A \bigcap X\alpha = \emptyset$.

Define $\gamma : X \to X$ by:

$$\text{for } x \in A,\ x\gamma = a; \text{ otherwise, } x\gamma = x\alpha.$$

It is easy to verify that $\gamma \in T_{E^*}(X)$ and $\alpha^2 = \alpha\gamma$. However, $b\beta\alpha = a\alpha \neq a = a\gamma = b\beta\gamma$. This contradicts with $\beta\alpha = \beta\gamma$.

Consequently, we have $X\alpha = X\beta$. □

Next we consider the relation $\mathcal{R}^*$.

Theorem 3.2. *Let $\alpha, \beta \in T_{E^*}(X)$. Then $(\alpha, \beta) \in \mathcal{R}^*$ if and only if $\pi(\alpha) = \pi(\beta)$.*

Proof. If $\pi(\alpha) = \pi(\beta)$, then by Lemma 2.1, $(\alpha, \beta) \in \mathcal{R}$ in T_X. Hence $(\alpha, \beta) \in \mathcal{R}^*$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{R}^*$, by Lemma 2.3, for all $\delta, \gamma \in T_{E^*}(X)$, $\delta\alpha = \gamma\alpha$ if and only if $\delta\beta = \gamma\beta$. Define $\delta : X \to X$ by:

$$x\delta = a, \text{ where } x \in y\alpha^{-1},\ y \in X\alpha,\ a \text{ is a fixed element and } a \in y\alpha^{-1}.$$

It is easy to verify that $\delta \in T_{E^*}(X)$, $\pi(\delta) = \pi(\alpha)$ and $\delta\alpha = 1_X \cdot \alpha$. Then $\delta\beta = 1_X \cdot \beta = \beta$ which implies that $\pi(\delta)$ refines $\pi(\beta)$. That is to say, $\pi(\alpha)$ refines $\pi(\beta)$. Dually, $\pi(\beta)$ refines $\pi(\alpha)$. Consequently, we have $\pi(\alpha) = \pi(\beta)$. □

As an immediate consequence of the previous theorems we get the following theorem:

Theorem 3.3. *Let $\alpha, \beta \in T_{E^*}(X)$. Then $(\alpha, \beta) \in \mathcal{H}^*$ if and only if $X\alpha = X\beta$ and $\pi(\alpha) = \pi(\beta)$.*

Now we consider Green's *-relation $\mathcal{D}^*$. Let Y, Z be two subsets of X and φ be a mapping from Y into Z. φ is said to be E-preserving if for any $x, y \in Y, (x, y) \in E$ implies $(x\varphi, y\varphi) \in E$. φ is said to be E^*-preserving if for any $x, y \in Y$, $(x, y) \in E$ if and only if $(x\varphi, y\varphi) \in E$.

Theorem 3.4. *Let $\alpha, \beta \in T_{E^*}(X)$. Then $(\alpha, \beta) \in \mathcal{D}^*$ if and only if there exists an E^*-preserving bijection $\rho : X\alpha \to X\beta$.*

Proof. We define the relation $\mathcal{K}$ on $T_{E^*}(X)$ by the rule:

$(\alpha, \beta) \in \mathcal{K}$ if and only if there exists an E^*-preserving bijection:
$\rho : X\alpha \to X\beta.$

Suppose that $(\alpha, \beta) \in \mathcal{L}^*$, then $X\alpha = X\beta$. Clearly $(\alpha, \beta) \in \mathcal{K}$ and so $\mathcal{L}^* \subseteq \mathcal{K}$. Next, we suppose that $(\alpha, \beta) \in \mathcal{R}^*$, then $\pi(\alpha) = \pi(\beta)$. Clearly $|X\alpha| = |X\beta|$. Define $\rho : X\alpha \to X\beta$ by:

$$x\rho = x\alpha^{-1}\beta.$$

It is easy to verify that $\rho : X\alpha \to X\beta$ is an E^*-preserving bijection. Further, $(\alpha, \beta) \in \mathcal{K}$ and so $\mathcal{R}^* \subseteq \mathcal{K}$. Therefore $\mathcal{D}^* \subseteq \mathcal{K}$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{K}$, then there exists an E^*-preserving bijection $\rho : X\alpha \to X\beta$. Define $\gamma : X \to X$ by:

$$x\gamma = a\rho, \text{ where } x \in a\alpha^{-1} \text{ and } a \in X\alpha.$$

It is easy to verify that $\gamma \in T_{E^*}(X)$, $\pi(\gamma) = \pi(\alpha)$ and $X\gamma = X\beta$. So that $(\alpha, \gamma) \in \mathcal{R}^*$ and $(\gamma, \beta) \in \mathcal{L}^*$. Thus $(\alpha, \beta) \in \mathcal{D}^*$ and so $\mathcal{K} \subseteq \mathcal{D}^*$.

Consequently, we have $\mathcal{D}^* = \mathcal{K}$. □

The discussion of the last relation $\mathcal{J}^*$ is more complicated than the others. We first observe a necessary condition for two elements of $T_{E^*}(X)$ to be $\mathcal{J}^*$-related.

Theorem 3.5. *Let $\alpha, \beta \in T_{E^*}(X)$, $(\alpha, \beta) \in \mathcal{J}^*$ then $|X\alpha| = |X\beta|$.*

Proof. Suppose that $(\alpha, \beta) \in \mathcal{J}^*$, then $J^*(\alpha) = J^*(\beta)$. Let

$$I(X, \beta) = \{\gamma \in T_{E^*}(X) : |X\gamma| \le |X\beta|\}.$$

It is easy to verify that $I(X, \beta)$ is a *-ideal of $T_{E^*}(X)$ to which β belongs. Since $\alpha \in J^*(\alpha) = J^*(\beta) \subseteq I(X, \beta)$, then $|X\alpha| \le |X\beta|$. Dually, we obtain the same result for β. Hence $|X\alpha| = |X\beta|$. □

Next we characterize Green's *-relation $\mathcal{D}^*$ and $\mathcal{J}^*$ when X is a finite set.

Theorem 3.6. *Let X be a finite set, then on the semigroup $T_{E^*}(X)$, $\mathcal{D}^* = \mathcal{J}^*$.*

Proof. Suppose that $(\alpha,\beta)\in\mathcal{J}^*$, then $J^*(\alpha)=J^*(\beta)$. Let

$$I(X,\beta)=\{\gamma\in T_{E^*}(X):|X\gamma|<|X\beta|\}$$
$$\bigcup\{\gamma\in T_{E^*}(X):\text{there exists an }E^*\text{-preserving bijection }\rho:X\gamma\to X\beta\}.$$

It is easy to verify that $I(X,\beta)$ is a *-ideal of $T_{E^*}(X)$ to which β belongs. Since $\alpha\in J^*(\alpha)=J^*(\beta)\subseteq I(X,\beta)$, then $|X\alpha|<|X\beta|$, or there exists an E^*-preserving bijection $\rho:X\alpha\to X\beta$.

Dually, we obtain the same results for β. Hence there exists an E^*-preserving bijection $\rho:X\alpha\to X\beta$. By Theorem 3.4, so that $(\alpha,\beta)\in\mathcal{D}^*$. Further, $\mathcal{J}^*\subseteq\mathcal{D}^*$. It is well known that $\mathcal{D}^*\subseteq\mathcal{J}^*$. Consequently, we have $\mathcal{D}^*=\mathcal{J}^*$. □

4. **Abundant semigroups**

In this section we investigate some conditions under which the monoid $T_{E^*}(X)$ is abundant.

Theorem 4.1. *For each $\alpha\in T_{E^*}(X)$, there exists an idempotent $e\in T_{E^*}(X)$ such that $\pi(e)=\pi(\alpha)$. Consequently, each $\mathcal{R}^*$-class of $T_{E^*}(X)$ contains an idempotent.*

Proof. Define $e:X\to X$ by:

$$xe=a\in x\alpha\alpha^{-1},\text{ where }a\text{ is a fixed element.}$$

It's easy to verify that $e\in T_{E^*}(X)$, $e^2=e$ and $\pi(e)=\pi(\alpha)$. By Theorem 3.2, we have $(e,\alpha)\in\mathcal{R}^*$. □

However, the conclusion is not true for $\mathcal{L}^*$-classes. In other words, there may be no idempotent in $\mathcal{L}^*$-classes.

Example 4.2. Let $X=\{1,2,3,\cdots\}$ and $E=I$. Let

$$\alpha=\begin{pmatrix}1&2&3&\cdots\\2&3&4&\cdots\end{pmatrix}$$

It is easy to verify that there is not idempotent $e\in T_{E^*}(X)$ such that $Xe=X\alpha$.

Theorem 4.3. *Let $\alpha\in T_{E^*}(X)$, L^*_α contains an idempotent if and only if $A\bigcap X\alpha\neq\emptyset$ for all $A\in X/E$.*

Proof. If $A\bigcap X\alpha\neq\emptyset$ for all $A\in X/E$, then we define $e:X\to X$ by:

$$xe=\begin{cases}x & if\ x\in X\alpha\\ a & if\ x\in A\setminus X\alpha. \quad where\ a\in A\bigcap X\alpha\end{cases}.$$

It is easy to verify that $e\in T_{E^*}(X)$, $e^2=e$ and $Xe=X\alpha$. By Theorem 3.1, we have $(e,\alpha)\in\mathcal{L}^*$.

Conversely, suppose that $(e,\alpha)\in\mathcal{L}^*$ and $e^2=e$, then for all $A\in X/E$, $Ae\subseteq A$. Hence $A\bigcap Xe\neq\emptyset$. So by Theorem 3.1, $A\bigcap X\alpha\neq\emptyset$. □

Theorem 4.4. *$T_{E^*}(X)$ is abundant if and only if $|X/E|$ is finite.*

Proof. If $|X/E|$ is infinite, without loss of generality, we may assume that $X/E = \{A_1, A_2, A_3, \cdots\}$. Define $\alpha : X \to X$ by:

$$\text{for } x \in A_i,\ x\alpha = a_{i+1}, \text{ where } a_{i+1} \in A_{i+1}, i = 1, 2, 3, \cdots.$$

It is clear that $\alpha \in T_{E^*}(X)$ and $A_1 \bigcap X\alpha = \emptyset$. So by Theorem 4.3, there is no idempotent in L^*_α. Hence $T_{E^*}(X)$ is not abundant which contradicts with $T_{E^*}(X)$ is abundant. Consequently, $|X/E|$ is finite.

Conversely, suppose that $|X/E|$ is finite. It is clear that $A \bigcap X\alpha \neq \emptyset$ for any $\alpha \in T_{E^*}(X)$, $A \in X/E$. So by Theorem 4.1 (Theorem 4.3), any $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) contain an idempotent. Thus $T_{E^*}(X)$ is abundant. □

The following lemma has been proved in [1].

Lemma 4.5. *$T_{E^*}(X)$ is regular if and only if $|X/E|$ is finite.*

As an immediate consequence of the previous theorems we the following theorem.

Theorem 4.6. *$T_{E^*}(X)$ is abundant if and only if $T_{E^*}(X)$ is regular.*

5. *-Ideal

Throughout this section X/E is an infinite set. Let $\alpha \in T_{E^*}(X)$, we put

$$Z(\alpha) = \{A \in X/E : A \bigcap X\alpha = \emptyset\}.$$

For a given nonnegative integer ξ, let

$$K^*(X, \xi) = \{\alpha \in T_{E^*}(X) : \xi \leq |Z(\alpha)| < +\infty\}.$$

One can easily prove the following theorem.

Theorem 5.1. *(1) If $\xi = 0$, then $K^*(X, \xi)$ is a *-ideal of $T_{E^*}(X)$.*
*(2) If $\xi > 0$, then $K^*(X, \xi)$ is a left *-ideal of $T_{E^*}(X)$.*

However, if $\xi > 0$, $K^*(X, \xi)$ is not a right *-ideal of $T_{E^*}(X)$. In other words, there may exist $\alpha \in K^*(X, \xi)$ such that $R^*_\alpha \nsubseteq K^*(X, \xi)$.

Example 5.2. Let $X = \{1, 2, 3, \cdots\}$ and $E = I$. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots \\ 2 & 3 & 4 & \cdots \end{pmatrix}.$$

It is easy to verify that $\alpha \in K^*(X, 1)$ and $(1_X, \alpha) \in \mathcal{R}^*$. But $1_X \notin K^*(X, 1)$.

Lemma 5.3. *Let $\alpha, \beta \in T_{E^*}(X)$, then $|Z(\alpha\beta)| = |Z(\alpha)| + |Z(\beta)|$.*

Proof. Suppose that

$$Z(\alpha) = \{A_1, \cdots, A_k\} \text{ and } Z(\beta) = \{B_1, \cdots, B_l\}.$$

By $\beta \in T_{E^*}(X)$, so that for $i = 1, \cdots, k$, there exists $C_i \in X/E$ such that $A_i\beta \subseteq C_i$. Thus

$$Z(\alpha\beta) = \{C_1, \cdots, C_k, B_1, \cdots, B_l\}.$$

Consequently, we have $|Z(\alpha\beta)| = |Z(\alpha)| + |Z(\beta)|$. □

By Lemma 5.3, we can obtain following theorem.

Theorem 5.4. *If $\xi > 0$, then all the Green's relations are trivial in $K^*(X, \xi)$.*

Next, we study the Green's *-relations for $K^*(X, \xi)(\xi > 0)$. Denote by E_α the restriction of the equivalence E on $X\alpha$.

$$E_\alpha = \{(x, y) \in E : x, y \in X\alpha\}.$$

Theorem 5.5. *Let $\xi > 0$ and $\alpha, \beta \in K^*(X, \xi)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ if and only if $X\alpha = X\beta$.*

Proof. If $X\alpha = X\beta$, then by Lemma 2.1, $(\alpha, \beta) \in \mathcal{L}$ in T_X. Hence $(\alpha, \beta) \in \mathcal{L}^*$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^*$, by Lemma 2.2, for all $\delta, \gamma \in K^*(X, \xi)$, $\alpha\delta = \alpha\gamma$ if and only if $\beta\delta = \beta\gamma$. If $X\alpha \neq X\beta$, without loss of generality, we may assume that $X\beta \setminus X\alpha \neq \emptyset$. Then there exists $a \in X\beta \setminus X\alpha$ and $b\beta = a$ for some $b \in X$. There are two cases to consider:

Case 1. $a \in A \in X/E$ and $A \bigcap X\alpha \neq \emptyset$. Without loss of generality, we may assume that there exists $c \in A \bigcap X\alpha$. By $\alpha \in T_{E^*}(X)$, $|X/E| = |X\alpha/E_\alpha|$. Further, there exists an E^*-preserving mapping:

$$\rho : X \setminus A \to X\alpha \setminus A.$$

Define $\delta : X \to X$ by:

$$x\delta = \begin{cases} x & if\ x \in A \\ x\rho & else \end{cases}.$$

Define $\gamma : X \to X$ by:

$$x\gamma = \begin{cases} c & if\ x = a \\ x & if\ x \in A \setminus \{a\} \\ x\rho & else \end{cases}.$$

It is easy to verify that $\delta, \gamma \in K^*(X, \xi)$ and $\alpha\delta = \alpha\gamma$. However,

$$b\beta\delta = a\delta = a \neq c = a\gamma = b\beta\gamma.$$

This contradicts with $\beta\delta = \beta\gamma$.

Case 2. $a \in A \in X/E$ and $A \bigcap X\alpha = \emptyset$.

The proof is identical to that of Case 2 of Theorem 3.1.

Consequently, we have $X\alpha = X\beta$. □

Theorem 5.6. *Let $\xi > 0$ and $\alpha, \beta \in K^*(X, \xi)$. Then $(\alpha, \beta) \in \mathcal{R}^*$ if and only if $\pi(\alpha) = \pi(\beta)$.*

Proof. If $\pi(\alpha) = \pi(\beta)$, then by Lemma 2.1, $(\alpha, \beta) \in \mathcal{R}$ in T_X. Hence $(\alpha, \beta) \in \mathcal{R}^*$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{R}^*$, by Lemma 2.3, for all $\delta, \gamma \in K^*(X, \xi)$, $\delta\alpha = \gamma\alpha$ if and only if $\delta\beta = \gamma\beta$. If $\pi(\alpha) \neq \pi(\beta)$, without loss of generality, we may assume that there exist $x_1, x_2 \in A \in X/E$ such that

$$x_1 \neq x_2,\ x_1\alpha\alpha^{-1} = x_2\alpha\alpha^{-1} \text{ and } x_1\beta\beta^{-1} \neq x_2\beta\beta^{-1}.$$

There are two cases to consider:

Case 1. $A \bigcap X\alpha \neq \emptyset$.

Since $\alpha \in T_{E^*}(X)$, then $|X/E| = |X\alpha/E_\alpha|$. Further, there exists an E^*-preserving mapping:

$$\rho : X \setminus A \to X\alpha \setminus A.$$

Define $\delta : X \to X$ by:

$$x\delta = \begin{cases} x_1 & if\ x \in A \\ x\rho & else \end{cases}.$$

Define $\gamma : X \to X$ by:

$$x\gamma = \begin{cases} x_2 & if\ x \in A \\ x\rho & else \end{cases}.$$

It is easy to verify that $\delta, \gamma \in K^*(X, \xi)$ and $\delta\alpha = \gamma\alpha$. However,

$$A\delta\beta = x_1\beta \neq x_2\beta = A\gamma\beta.$$

This contradicts with $\delta\beta = \gamma\beta$.

Case 2. $A \bigcap X\alpha = \emptyset$.

Define $\delta : X \to X$ by:

$$x\delta = \begin{cases} x_1 & if\ x \in A \\ x\alpha & else \end{cases}.$$

Define $\gamma : X \to X$ by:

$$x\gamma = \begin{cases} x_2 & if\ x \in A \\ x\alpha & else \end{cases}.$$

It is easy to verify that $\delta, \gamma \in K^*(X, \xi)$ and $\delta\alpha = \gamma\alpha$. However,

$$A\delta\beta = x_1\beta \neq x_2\beta = A\gamma\beta.$$

This contradicts with $\delta\beta = \gamma\beta$.

Consequently, we have $\pi(\alpha) = \pi(\beta)$. □

As an immediate consequence of the previous theorems we have the following result.

Theorem 5.7. *Let $\xi > 0$ and $\alpha, \beta \in K^*(X, \xi)$. Then $(\alpha, \beta) \in \mathcal{H}^*$ if and only if $X\alpha = X\beta$ and $\pi(\alpha) = \pi(\beta)$.*

Theorem 5.8. *Let $\xi > 0$ and $\alpha, \beta \in K^*(X, \xi)$. Then $(\alpha, \beta) \in \mathcal{D}^*$ if and only if there exists an E^*-preserving bijection $\rho : X\alpha \to X\beta$.*

Proof. The proof is identical to that of Theorem 3.4. □

Theorem 5.9. *Let $\xi > 0$, $\alpha, \beta \in K^*(X, \xi)$ and $(\alpha, \beta) \in \mathcal{J}^*$. Then $|X\alpha| = |X\beta|$.*

Proof. The proof is identical to that of Theorem 3.5. □

6. **Rees quotient semigroup**

Throughout this section X/E is an infinite set. For a given nonnegative integer ξ, let

$$P^*(X, \xi) = K^*(X, \xi)/K^*(X, \xi + 1)$$

be the Rees quotient semigroup whose non-zero element α may be thought of as the element of $T_{E^*}(X)$ with $|Z(\alpha)| = \xi$. The product of two elements α, β of $P^*(X, \xi)$, is 0 whenever their product in $T_{E^*}(X)$ is of $|Z(\alpha\beta)| \geq \xi + 1$.

We begin our investigation on the properties of $P^*(X, \xi)$ by first characterizing the Green's relations.

Theorem 6.1. *Let $(\alpha, \beta) \in P^*(X, 0)$. Then*

(1) $(\alpha, \beta) \in \mathcal{L}$ if and only if $X\alpha = X\beta$.

(2) $(\alpha, \beta) \in \mathcal{R}$ if and only if $\pi(\alpha) = \pi(\beta)$.

(3) $(\alpha, \beta) \in \mathcal{H}$ if and only if $X\alpha = X\beta$ and $\pi(\alpha) = \pi(\beta)$.

(4) $(\alpha, \beta) \in \mathcal{D}$ if and only if there exists $\delta \in T_{E^*}(X)$ such that $\delta|_{X\alpha} : X\alpha \to X\beta$ is a bijection.

(5) $(\alpha, \beta) \in \mathcal{J}$ if and only if $|X\alpha| = |X\beta|$ and there exist $\rho, \tau \in T_{E^*}(X)$, for any $A \in X/E$, we have $A\alpha = B\beta\rho$ and $A\beta = C\alpha\tau$ for some $B, C \in X/E$.

Proof. The proof is identical to that of $T_{E^*}(X)$ in [1]. □

By Lemma 5.3, we can obtain the following theorem.

Theorem 6.2. *If $\xi > 0$, then all the Green's relations in $P^*(X, \xi)$ are trivial.*

Next, we study the Green's *-relations for $P^*(X, \xi)$.

Theorem 6.3. *Let $(\alpha, \beta) \in P^*(X, \xi)$. Then the following statements are hold.*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ if and only if $X\alpha = X\beta$.

(2) $(\alpha, \beta) \in \mathcal{R}^*$ if and only if $\pi(\alpha) = \pi(\beta)$.

(3) $(\alpha, \beta) \in \mathcal{H}^*$ if and only if $X\alpha = X\beta$ and $\pi(\alpha) = \pi(\beta)$.

(4) $(\alpha, \beta) \in \mathcal{D}^*$ if and only if there exists an E^*-preserving bijection $\rho : X\alpha \to X\beta$.

(5) If $(\alpha, \beta) \in \mathcal{J}^*$, then $|X\alpha| = |X\beta|$.

Proof. If $\xi = 0$, the proof is similar to that of $T_{E^*}(X)$. If $\xi > 0$, the proof is similar to that of $K^*(X, \xi)$. □

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