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THE STRUCTURE OF LINEAR PRESERVERS OF GS-MAJORIZATION

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ABSTRACT. Let $\mathbf{M}_{n,\mathbf{m}}$ be the set of all $n \times m$ matrices with entries in \mathbb{F} , where \mathbb{F} is the field of real or complex numbers. In this paper we introduce the relation gs-majorization on $\mathbf{M}_{n,\mathbf{m}}$. We study some properties of this relation on $\mathbf{M}_{n,\mathbf{m}}$. We also characterize all linear operators that preserve (or strongly preserve) gs-majorization on $\mathbf{M}_{n,\mathbf{m}}$.

1. Introduction

A matrix $R \in \mathbf{M_n}$ where all its row sums are equal to one is said to be a g-row stochastic matrix and a g-column stochastic matrix is the transpose of a g-row stochastic matrix. A matrix $D \in \mathbf{M_n}$ with the property that D and D^t are g-row stochastic matrices is said to be g-doubly stochastic. The set of all g-doubly stochastic matrices is a convex set in $\mathbf{M_n}$. For more information see [5].

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The notion of majorization plays an important role in mathematics, statistics and economics. We begin with the definition of matrix majorization. Let A and B be $n \times m$ real matrices.

(1) If there exists an $n \times n$ doubly stochastic matrix D such that A = DB, then A is said to be multivariate majorized by B, and this is denoted by $A \prec_m B$.

(2) If for every $x \in \mathbb{R}^n$, Ax is vector majorized by Bx, then A is said to be directionally majorized by B, and this is denoted by $A \prec_d B$.

The definitions of multivariate and directional majorization are motivated by the theorem of Hardy-Littlewood and Polya for vector majorization which says that for $x, y \in \mathbb{R}^n, x \prec y$ if and only if there exists an $n \times n$ doubly stochastic matrix D such that x = Dy. For more information see [2], [3], [6] and [9].

Now we introduce the notion of gs-majorization on $\mathbf{M}_{n,\mathbf{m}}$ as follows.

Definition 1.1. Let $A, B \in \mathbf{M}_{n,m}$. The matrix B is said to be gsmajorized by A if there exists an $n \times n$ g-doubly stochastic matrix Dsuch that B = DA. This is denoted by $A \succ_{qs} B$.

Let ~ be a relation on $\mathbf{M}_{\mathbf{n},\mathbf{m}}$. A linear operator $T: \mathbf{M}_{\mathbf{n},\mathbf{m}} \longrightarrow \mathbf{M}_{\mathbf{n},\mathbf{m}}$ is said to be a linear preserver (or strong linear preserver) of ~ if $T(x) \sim T(y)$ whenever $x \sim y$ (or $T(x) \sim T(y)$ if and only if $x \sim y$).

Li and Poon proved the following result regarding the linear preservers of multivariate majorization.

Proposition 1.2. [7, Theorem 2] Let $T : \mathbf{M}_{n,\mathbf{m}} \to \mathbf{M}_{n,\mathbf{m}}$ be a linear operator, then T preserves directional majorization if and only if T preserves multivariate majorization if and only if one of the following holds:

- (a) There exist $A_1, ..., A_m \in \mathbf{M}_{n,m}$, such that $T(X) = \sum_{j=1}^m tr(x_j)A_j$, where x_j is j^{th} column of X.
- (b) There exist $R, S \in \mathbf{M_{n,m}}$ and permutation $P \in \mathbf{M_n}$ such that T(X) = PXR + JXS.

The main result of this paper is to characterize all linear preservers of gs-majorization on $\mathbf{M}_{n,\mathbf{m}}$. Let T be a linear preserver of gs-majorization on $\mathbf{M}_{n,\mathbf{m}}$, then one of the following holds:

(i) $T(X) = \sum_{j=1}^{m} \operatorname{tr}(x_j) A_j$, where $A_1, \dots, A_m \in \mathbf{M}_{\mathbf{n},\mathbf{m}}$ and x_j is j^{th} column of X, or

(ii)
$$T(X) = [A_1 X a_1] \cdots [A_m X a_m] + J X S,$$

where $S \in \mathbf{M}_{\mathbf{m}}, a_1, \cdots, a_m \in \mathbb{F}^m$ and $A_1, \cdots, A_m \in \mathbf{GD}_{\mathbf{n}}$ are invertible.

The following elementary properties of gs-majorization on $\mathbf{M}_{n,m}$ are used throughout this paper.

Let $X, Y \in \mathbf{M_{n,m}}, A, B \in \mathbf{GD_n}, C \in \mathbf{M_m}$ and $\alpha, \beta \in \mathbb{F}$ such that A, B and C are invertible and $\alpha \neq 0$. Then the following conditions are equivalent:

(1)
$$X \succ_{qs} Y$$
,

- (2) $AX \succ_{qs} BY$,
- (3) $\alpha X + \beta J_{n,m} \succ_{gs} \alpha Y + \beta J_{n,m},$
- (4) $XC \succ_{gs} YC$,

where $J_{n,m}$ is the $n \times m$ matrix with all entries equal to one.

Throughout this paper, $\mathbf{GR_n}$, $\mathbf{GC_n}$ and $\mathbf{GD_n}$ are the sets of g-row stochastic, g-column stochastic and g-doubly stochastic matrices, respectively. Also J is the $n \times n$ matrix with all entries equal to one.

2. Linear preservers of gs-majorization on \mathbb{F}^n

In this section we will characterize all linear operators that preserve (or strongly preserve) gs-majorization on \mathbb{F}^n . The following proposition gives an equivalent condition for gs-majorization on \mathbb{F}^n . Let $e = (1, ..., 1)^t \in \mathbb{F}^n$.

Proposition 2.1. Let x and y be two distinct vectors in \mathbb{F}^n . Then, $x \succ_{gs} y$ if and only if $x \notin \operatorname{span}\{e\}$ and $\operatorname{tr}(x) = \operatorname{tr}(y)$.

Lemma 2.2. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T preserves the subspace $\{x \in \mathbb{F}^n : Jx = 0\}$ if and only if there exists $A \in \text{span}(\mathbf{GC}_n)$ such that T(x) = Ax, for each $x \in \mathbb{F}^n$.

Proof. Let $A \in \mathbf{M_n}$ be the matrix representation of T with respect to the standard basis of \mathbb{F}^n . Let $A \in \text{span}(\mathbf{GC_n})$, it is easy to show that T preserves the subspace $\{x \in \mathbb{F}^n : Jx = 0\}$.

Conversely, let T preserve the subspace $\{x \in \mathbb{F}^n : Jx = 0\}$, then $J(T(e_i - e_j)) = 0$, for all $1 \leq i, j \leq n$, so $J(A(e_i - e_j)) = 0$, thus $\sum_{k=1}^n a_{ki} = \sum_{k=1}^n a_{kj}$. Therefore, $A \in \text{span}(\mathbf{GC_n})$.

By Lemma 2.2, we may state the following Proposition.

Proposition 2.3. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator that preserves gs-majorization. Then there exists $A \in span(\mathbf{GC_n})$ such that $T(x) = Ax, \forall x \in \mathbb{F}^n$.

We obtain a result similar to Ando's Theorem in [1], for gs-majorization.

Theorem 2.4. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T preserves gs-majorization if and only if one of the following holds :

(a) T(x) = tr(x)a, for some $a \in \mathbb{F}^n$.

(b) $T(x) = \alpha Dx + \beta Jx$, for some $\alpha, \beta \in \mathbb{F}$ and invertible matrix $D \in \mathbf{GD}_{\mathbf{n}}$.

Proof. Let $A \in \mathbf{M_n}$ be the matrix representation of T with respect to the standard basis of \mathbb{F}^n . If (a) or (b) holds, it is clear that T preserves gs-majorization. Conversely, let T preserve gs-majorization. We consider two parts:

Part (i): Let there exist $b \in (\mathbb{F}^n \setminus \text{span}\{e\})$, such that T(b) = se, for some $s \in \mathbb{F}$. We consider two cases;

Case 1; Let tr(b)=0, then tr(Ab)=0, by Proposition 2.3. Hence

$$J(Ab) = 0 \Rightarrow J(se) = 0 \Rightarrow se = 0 \Rightarrow s = 0 \Rightarrow T(b) = 0 \Rightarrow Ab = 0.$$

By Proposition 2.1, $b \succ_{gs} (e_i - e_j)$, for $1 \leq i, j \leq n$, so $0 = Ab \succ_{gs} A(e_i - e_j)$ and hence $Ae_i = Ae_j$, for $1 \leq i, j \leq n$. Then, $A = [a| \cdots |a]$, for some $a \in \mathbb{F}^n$. Thus, $T(x) = \operatorname{tr}(x)a, \forall x \in \mathbb{F}^n$.

Case 2; Let $\operatorname{tr}(b) = \delta \neq 0$. Consider the basis{ $\delta e_1, \dots, \delta e_n$ } for \mathbb{F}^n . By Proposition 2.1, $b \succ_{gs} \delta e_i$, for all $1 \leq i \leq n$. Then, $T(b) \succ_{gs} \delta T(e_i)$ and hence $se \succ_{gs} \delta T(e_i)$. Therefore, $T(e_i) = \frac{s}{\delta}e$, for all $1 \leq i \leq n$, so $T(x) = \operatorname{tr}(x)(\frac{s}{\delta}e), \forall x \in \mathbb{F}^n$.

Part(ii): Let $x \notin \text{span}\{e\}$ imply that $T(x) \notin \text{span}\{e\}$. We consider two cases;

Case 1; Let T be invertible. Then there exists $b \in \mathbb{F}^n$ such that T(b) = e, so by hypothesis b = se, for some $s \in \mathbb{F}$. Thus $Ae = \frac{1}{s}e$ and hence $A \in \text{span}(\mathbf{GR_n})$. Also $A \in \text{span}(\mathbf{GC_n})$, by Proposition 2.3. Therefore, $A \in \text{span}(\mathbf{GD_n})$. Put D = sA, $\alpha = \frac{1}{s}$ and $\beta = 0$. So, $T(x) = \alpha Dx + \beta Jx$.

Case 2; Let T be singular. By hypothesis, $\operatorname{Ker}(A) = \operatorname{span}\{e\}$, then $\frac{A+J}{n} \in \mathbf{GR_n}$. It is clear that, $\frac{A+J}{n}$ preserves gs-majorization, therefore, by Proposition 2.3, $\frac{A+J}{n} \in \mathbf{GD_n}$. We will show that A + J is invertible.

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If x is in ker(A + J) then Ax = -Jx lies in the span $\{e\}$, and hence so does x by the hypothesis of part(ii), thus x = re for some $r \in \mathbb{F}$. Then, (A + J)(re) = 0 implies that r = 0 and hence x = 0. Therefore, A + J is invertible. Define $D = \frac{A+J}{n}$, $\alpha = n$ and $\beta = -1$. Then $T(x) = \alpha Dx + \beta Jx$. \Box Let $T : \mathbf{M}_{n,\mathbf{m}} \to \mathbf{M}_{n,\mathbf{m}}$ be a linear operator that strongly preserves gsmajorization. It is easy to show that T is invertible.

Corollary 2.5. Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T strongly preserves gs-majorization if and only if $T(x) = \alpha Dx$, for some nonzero scalar $\alpha \in \mathbb{F}$ and invertible matrix $D \in \mathbf{GD}_n$.

3. Linear preservers of gs-majorization on $M_{n,m}$

In this section we characterize all linear operators that preserve (strongly preserve) gs-majorization on $\mathbf{M}_{n,\mathbf{m}}$.

Lemma 3.1. Let $A \in \mathbf{GD_n}$ be invertible. Then the following conditions are equivalent:

(a) $A = \alpha I + \beta J$, for some $\alpha, \beta \in \mathbb{F}$, (b) $(x + Ay) \succ_{gs} (Dx + ADy)$, for all $D \in \mathbf{GD_n}$ and for all $x, y \in \mathbb{F}^n$

Proof. $a \to b$) If $A = \alpha I + \beta J$, it is easy to show that $(x + Ay) \succ_{gs} (Dx + ADy)$ for all $D \in \mathbf{GD}_n$ and $x, y \in \mathbb{F}^n$.

 $b \to a$) The matrix A is invertible. Thus, for every $1 \le i \le n$ there exists $y_i \in \mathbb{F}^n$ such that $Ay_i = e - e_i$. It is trivial that $tr(y_i) = n - 1$. By hypothesis $(e_i + Ay_i) \succ_{gs} (De_i + ADy_i), \forall D \in \mathbf{GD_n}$, and hence $e \succ_{gs} (De_i + ADy_i), \forall D \in \mathbf{GD_n}$. Thus,

$$(De_i + ADy_i) = e, \ \forall D \in \mathbf{GD_n}.$$
(3.1)

It is clear that $[J - (n-1)A] \in \mathbf{GD}_{\mathbf{n}}$ and hence $D[J - (n-1)A] \in \mathbf{GD}_{\mathbf{n}}$, $\forall D \in \mathbf{GD}_{\mathbf{n}}$. Therefore, by (3.1),

$$D[J - (n - 1)A]e_i + AD[J - (n - 1)A]y_i = e$$

$$\Rightarrow (DA - AD)e_i = 0, \text{ for all } 1 \le i \le n$$

$$\Rightarrow AD = DA, \forall D \in \mathbf{GD_n}.$$
(3.2)

Put $D = P_{ij}$ in (3.2), where P_{ij} is the permutation that interchanges the i^{th} and j^{th} rows of the identity matrix. Then, $P_{ij}A = AP_{ij}$, so $A = \alpha I + \beta J$, for some $\alpha, \beta \in \mathbb{F}$.

Lemma 3.2. Let $T_1, T_2: \mathbb{F}^n \to \mathbb{F}^n$ satisfy $T_1(x) = \alpha Ax + \beta Jx$ and $T_2(x) = \operatorname{tr}(x)a$, for some $\alpha, \beta \in \mathbb{F}, \alpha \neq 0$, invertible matrix $A \in \mathbf{GD_n}$ and $a \in (\mathbb{F}^n \setminus \operatorname{span}\{e\})$. Then there exist a g-doubly stochastic matrix D and a vector $x \in \mathbb{F}^n$ such that $T_1(x) + T_2(x)$.

Proof. Assume that, if possible, $T_1(x) + T_2(x) \succ_{gs} T_1(Dx) + T_2(Dx)$, $\forall D \in \mathbf{GD}_{\mathbf{n}}, \forall x \in \mathbb{F}^n$. Then by elementary properties of gs-majorization, $\alpha Ax + \operatorname{tr}(x)a \succ_{gs} \alpha ADx + \operatorname{tr}(x)a, \forall D \in \mathbf{GD}_{\mathbf{n}}, \forall x \in \mathbb{F}^n$. Put $b = \frac{1}{\alpha}a$. Then

$$Ax + \operatorname{tr}(x)b \succ_{as} ADx + \operatorname{tr}(x)b, \quad \forall \ D \in \mathbf{GD}_{\mathbf{n}}, \ \forall x \in \mathbb{F}^{n}.$$
 (3.3)

The matrix A is invertible thus there exists $x_0 \in \mathbb{F}^n$ such that $Ax_0 = (e - b)$. Put $x = x_0 - (\frac{n-1-tr(b)}{n})e$ in (3.3). Then $ADx_0 = (e - b)$, $\forall D \in \mathbf{GD_n}$, so $b \in \mathrm{span}\{e\}$ and hence $a \in \mathrm{span}\{e\}$, which is a contradiction.

Now, we state the main theorem of this section.

Theorem 3.3. Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator that preserves gs-majorization. Then one the following holds:

(i) There exist $A_1, \dots, A_m \in \mathbf{M}_{\mathbf{n},\mathbf{m}}$ such that $T(X) = \sum_{j=1}^{m} \operatorname{tr}(x_j) A_j$, where $X = [x_1| \cdots |x_m]$.

(ii) There exist $S \in \mathbf{M}_{\mathbf{m}}$, $a_1, \dots, a_m \in \mathbb{F}^m$ and invertible matrices $A_1, \dots, A_m \in \mathbf{GD}_{\mathbf{n}}$, such that $T(X) = [A_1Xa_1| \dots |A_mXa_m] + JXS$.

Proof. Define the embedding $E^j:\mathbb{F}^n \to \mathbf{M}_{\mathbf{n},\mathbf{m}}$ by $E^j(x) = xe_j{}^t$ and projection $E_i:\mathbf{M}_{\mathbf{n},\mathbf{m}} \to \mathbb{F}^n$ by $E_i(A) = Ae_i$ for $1 \leq i,j \leq m$. Put $T_i{}^j = E_iTE^j$. Then,

$$T(X) = T[x_1|\cdots|x_m] = \left[\sum_{j=1}^m T_1^j(x_j)|\cdots|\sum_{j=1}^m T_m^j(x_j)\right].$$
 (3.4)

It is easy to show that $T_i^{j}:\mathbb{F}^n \to \mathbb{F}^n$ preserves gs-majorizaton. Then each T_i^{j} is of the form (a) or (b) in Theorem 2.4. Now, we consider two cases:

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Case 1; Let $T_i^{j}(x) = tr(x)a_i^{j}$, for some $a_i^{j} \in \mathbb{F}^n$, $\forall 1 \leq i, j \leq m$. Define $A_j = [a_1^{j}|\cdots|a_m^{j}]$. By (3.4), it is clear that $T(X) = \sum_{j=1}^{m} tr(x_j)A_j$. Hence the condition (i) holds.

Case 2; Let there exist $1 \leq p, q \leq m$, such that $T_p{}^q(x) = \gamma_p{}^q B_p{}^q x + \delta_p^q Jx$, for some $\gamma_p^q, \delta_p^q \in \mathbb{F}, \gamma_p^q \neq 0$, and invertible matrix $B_p{}^q \in \mathbf{GD_n}$.

Step 1. We show that for all $1 \leq j \leq m$, $T_p^j(x) = \alpha_p{}^j A_p x + \beta_p{}^j J x$, for some $\alpha_p^j, \beta_p{}^j \in \mathbb{F}$, and invertible matrix $A_p \in \mathbf{GD}_n$.

For every $x, y \in \mathbb{F}^n$, define $B_{x,y} = E^j(x) + E^q(y) \in \mathbf{M}_{n,m}$. Then for every $D \in \mathbf{GD}_n$,

$$B_{x,y} \succ_{gs} DB_{x,y} \Rightarrow T(B_{x,y}) \succ_{gs} T(DB_{x,y})$$

$$\Rightarrow [T_1^j(x) + T_1^q(y)| \cdots |T_m^j(x) + T_m^q(y)]$$

$$\succ_{gs} [T_1^j(Dx) + T_1^q(Dy)| \cdots |T_m^j(Dx) + T_m^q(Dy)]$$

$$\Rightarrow T_p^j(x) + T_p^q(y) \succ_{gs} T_p^j(Dx) + T_p^q(Dy),$$

$$\forall D \in \mathbf{GD_n}. \qquad (3.5)$$

Then by Lemma 3.2, $T_p^j(x) = \gamma_p^j B_p^j x + \delta_p^j J x$, for some $\gamma_p^j, \delta_p^j \in \mathbb{F}$, and invertible matrix $B_p^j \in \mathbf{GD_n}$. Put $T_p^j(x) = \gamma_p^j B_p^j x + \delta_p^j J x$ as in (3.5). Thus

$$\begin{split} \gamma_{p}^{j}B_{p}^{j}x + \delta_{p}^{j}Jx + \gamma_{p}^{q}B_{p}^{q}y + \delta_{p}^{q}Jy \\ \succ_{gs} \gamma_{p}^{j}B_{p}^{j}Dx + \delta_{p}^{j}JDx + \gamma_{p}^{q}B_{p}^{q}Dy + \delta_{p}^{q}JDy \\ \Rightarrow \gamma_{p}^{j}B_{p}^{j}x + \gamma_{p}^{q}B_{p}^{q}y \succ_{gs} \gamma_{p}^{j}B_{p}^{j}Dx + \gamma_{p}^{q}B_{p}^{q}Dy \\ \Rightarrow (\gamma_{p}^{q}B_{p}^{q})^{-1}\gamma_{p}^{j}B_{p}^{j}x + y \succ_{gs} (\gamma_{p}^{q}B_{p}^{q})^{-1}\gamma_{p}^{j}B_{p}^{j}Dx + Dy \\ \forall D \in \mathbf{GD_{n}} \quad . \end{split}$$
(3.6)

If $\gamma_p^j = 0$, we choose $B_p^j = 0$. Let $\gamma_p^j \neq 0$ then, by Lemma 3.1, $B_p^j = r_p^j B_p^q + s_p^j J$, for some $r_p^j, s_p^j \in \mathbb{F}$. Define $A_p = B_p^q$, then $T_p^j(x) = \alpha_p^j A_p x + \beta_p^j J x$, for some α_p^j , $\beta_p^j \in \mathbb{F}$.

Step 2. Now, we show that, for all $1 \leq i, j \leq m, T_i^j$ are of the form (b) of Theorem 2.4. By Step 1, $T_p^j(x) = \alpha_p^j A_p x + \beta_p^j J x$. For every $x \in \mathbb{F}^n$, define $B_x = E^j(x)$. Then for all $D \in \mathbf{GD}_n$,

$$B_{x} \succ_{gs} DB_{x} \Rightarrow T(B_{x}) \succ_{gs} T(DB_{x})$$

$$\Rightarrow [T_{1}^{j}(x)|\cdots|T_{m}^{j}(x)] \succ_{gs} [T_{1}^{j}(Dx)|\cdots|T_{m}^{j}(Dx)]$$

$$\Rightarrow T_{p}^{j}(x) + T_{i}^{j}(x) \succ_{gs} T_{p}^{j}(Dx) + T_{i}^{j}(Dx), \forall D \in \mathbf{GD_{n}}.$$

Thus by Lemma 3.2, $T_i^j(x) = \gamma_i^j B_i^j x + \delta_i^j J x$, for some $\gamma_i^j, \delta_i^j \in \mathbb{F}$ and invertible matrix $B_i^j \in \mathbf{GD_n}$. Now again by Step 1, $T_i^j(x) = \alpha_i^j A_i x + \beta_i^j J x$, for some $\alpha_i^j, \beta_i^j \in \mathbb{F}$ and invertible matrix $A_i \in \mathbf{GD_n}$. For every $1 \le i \le m$, define

$$a_{i} = \begin{pmatrix} \alpha_{i}^{1} \\ \vdots \\ \alpha_{i}^{m} \end{pmatrix} \quad and \quad S = \begin{pmatrix} \beta_{1}^{1} & \cdots & \beta_{m}^{1} \\ \vdots & \vdots & \vdots \\ \beta_{1}^{m} & \cdots & \beta_{m}^{m} \end{pmatrix}.$$

Then,

$$T(X) = T[x_1|\cdots|x_m] = \left[\sum_{j=1}^m T_1^j(x_j)|\cdots|\sum_{j=1}^m T_m^j(x_j)\right]$$

= $\left[A_1 \sum_{j=1}^m \alpha_1^j x_j|\cdots|A_m \sum_{j=1}^m \alpha_m^j x_j] + J\left[\sum_{j=1}^m \beta_1^j x_j|\cdots|\sum_{j=1}^m \beta_m^j x_j\right]$
= $\left[A_1 X a_1|\cdots|A_m X a_m\right] + JXS$.

Hence the condition (ii) holds.

Corollary 3.4. Let T satisfy the condition (ii) of Theorem 3.3 and let $\operatorname{rank}[a_1|\cdots|a_m] \geq 2$. Then T(X) = AXR + JXS, for some $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$.

Proof. Without loss of generality, let $\{a_1, a_2\}$ be a linearly independent set. Let $X \in \mathbf{M}_{n,m}$, $D \in \mathbf{GD}_n$ be arbitrary. Then

$$\begin{aligned} X \succ_{gs} DX \Rightarrow T(X) \succ_{gs} T(DX) \\ \Rightarrow [A_1 X a_1 | \dots | A_m X a_m] \succ_{gs} [A_1 D X a_1 | \dots | A_m D X a_m] \\ \Rightarrow A_1 X a_1 + A_2 X a_2 \succ_{gs} A_1 D X a_1 + A_2 D X a_2 \\ \Rightarrow X a_1 + (A_1^{-1} A_2) X a_2 \succ_{gs} D X a_1 + (A_1^{-1} A_2) D X a_2 \quad . \quad (3.7) \end{aligned}$$

Since $\{a_1, a_2\}$ is linearly independent, for every $x, y \in \mathbb{F}^n$, there exists $B_{x,y} \in \mathbf{M}_{n,\mathbf{m}}$ such that $B_{x,y} a_1 = x$ and $B_{x,y} a_2 = y$. Put $X = B_{x,y}$ as in (3.7). Thus,

$$B_{x,y} a_1 + (A_1^{-1} A_2) B_{x,y} a_2 \succ_{gs} DB_{x,y} a_2 + (A_1^{-1} A_2) DB_{x,y} a_2 \Rightarrow$$
$$x + (A_1^{-1} A_2) y \succ_{gs} Dx + (A_1^{-1} A_2) Dy, \forall D \in \mathbf{GD_n}.$$

Then by Lemma 3.1, $A_1^{-1}A_2 = \alpha I + \beta J$ and hence $A_2 = \alpha A_1 + \beta J$, for some $\alpha, \beta \in \mathbb{F}, \alpha \neq 0$.

For every $i \geq 3$, if $a_i = 0$ we can choose $A_i = A_1$. If $a_i \neq 0$ then $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is linearly independent. Then by the same argument as above, $A_i = \gamma_i A_1 + \delta_i J$, for some $\gamma_i, \delta_i \in \mathbb{F}, \gamma_i \neq 0$, or $A_i = \lambda_i A_2 + \mu_i J$, for some $\lambda_i, \mu_i \in \mathbb{F}, \lambda_i \neq 0$.

Define $A = A_1$. Then for every $i \ge 2$, $A_i = \alpha_i A + \beta_i J$, for some $\alpha_i, \beta_i \in \mathbb{F}$ and hence

$$T(X) = [AXa_1 \mid AX(r_2a_2) \mid \cdots \mid AX(r_ma_m)] + JXS = AXR + JXS,$$

where, $R = [a_1|r_2a_2|\cdots|r_ma_m]$, for some $r_2, \cdots, r_m \in \mathbb{F}$ and S is the same as in Theorem 3.3.

The following example shows that if $rank[a_1|\cdots|a_m] = 1$, the above corollary does not hold when $\mathbb{F} = \mathbb{R}$.

Example 3.5. Let $T : \mathbf{M}_{3,2} \to \mathbf{M}_{3,2}$ be defined by $T(X) = [Xe_1 | PXe_1]$ where $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We show that T preserves gs-majorization, and T is not of the form T(X) = AXR + JXS in Corollary 3.4. Let $X = [x \mid x'], Y = [y \mid y'] \in \mathbf{M}_{3,2}$ and $X \succ_{gs} Y$. Now we consider two

cases:

Case 1; Let $x \in \text{span}\{e\}$, then y = x, therefore, T(X) = T(Y).

Case 2; Let $x \notin \operatorname{span}\{e\}$. Put $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, then $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$.

 $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$.Set

$$R = \left(\begin{array}{ccc} x_1 - x_3 & x_2 - x_3 \\ x_3 - x_2 & x_1 - x_2 \end{array}\right).$$

It is clear that R is invertible. Define,

$$D := \begin{pmatrix} r_1 & r_2 & 1 - (r_1 + r_2) \\ s_1 & s_2 & 1 - (s_1 + s_2) \\ 1 - (r_1 + s_1) & 1 - (r_2 + s_2) & r_1 + r_2 + s_1 + s_2 - 1 \end{pmatrix},$$

where

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = R^{-1} \begin{pmatrix} y_1 - x_3 \\ y_3 - x_2 \end{pmatrix} \quad and \quad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = R^{-1} \begin{pmatrix} y_2 - x_3 \\ y_1 - x_2 \end{pmatrix}.$$

It is easy to check that $D \in \mathbf{GD}_3$, Dx = y and DPx = Py. Therefore, T preserves gs-majorization.

Finally we show that T is not of the form T(X) = AXR + JXS as in Corollary 3.4. Assume that, if possible, T(X) = AXR + JXS, for some $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$. Then $AXR + JXS = [Xe_1 \mid PXe_1], \forall X \in \mathbf{M_{3,2}}$, so

$$[AXR_1 + JXS_1 \mid AXR_2 + JXS_2] = [Xe_1 \mid PXe_1] \quad \forall X \in \mathbf{M}_{3,2},$$

where R_i and S_i are i^{th} column of R and S respectively. Let $x \in \mathbb{F}^3$ be arbitrary. Put $X = [x \mid 0]$ in the above equation, then $[r_{11}Ax + s_{11}Jx \mid r_{12}Ax + s_{12}Jx] = [x \mid Px], \forall x \in \mathbb{F}^3$. Therefore, $P = \alpha I + \beta J$, for some $\alpha, \beta \in \mathbb{F}$, which is a contradiction.

Lemma 3.6. Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator, such that, T(X) = DXR + JXS for some $R, S \in \mathbf{M}_m$ and invertible matrix $D \in \mathbf{GD}_n$. Then T is invertible if and only if R and (R+nS) are invertible.

Proof. Without loss of generality, we can assume that D = I. Let A be the matrix representation of T with respect to the standard basis of $\mathbf{M}_{n,\mathbf{m}}$. Then, it is easy to show that A is similar to the following block matrix,

$$\left(\begin{array}{cccc} R+nS & S & \cdots & S\\ 0 & R & \cdots & S\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & R\end{array}\right).$$

Therefore, T is invertible if and only if R and (R+nS) are invertible. \Box

Theorem 3.7. Let $T : \mathbf{M}_{n,\mathbf{m}} \to \mathbf{M}_{n,\mathbf{m}}$ be a linear operator. Then T strongly preserves gs-majorization if and only if T(X) = AXR + JXS for some $R, S \in \mathbf{M}_{\mathbf{m}}$ and invertible matrix $A \in \mathbf{GD}_{\mathbf{n}}$, such that, R and R + nS are invertible.

Proof. If m=1, the result holds by Corollary 2.5. So let $m \ge 2$. Let T(X) = AXR + JXS, such that R and (R + nS) are invertible. Let $X \succ_{gs} Y$. It is easy to show that $T(X) \succ_{gs} T(Y)$. Now, let

$$T(X) \succ_{gs} T(Y)$$
. Then, $DT(X) = T(Y)$, for some $D \in \mathbf{GD_n}$. Thus,
 $DT(X) = T(Y) \Rightarrow DAXR + JXS = AYR + JYS$
 $\Rightarrow J[DAXR + JXS] = J[AYR + JYS]$
 $\Rightarrow (JX)(R + nS) = (JY)(R + nS)$
 $\Rightarrow JX = JY.$

Then, $(A^{-1}DA)X = Y$. Therefore, $X \succ_{gs} Y$.

Conversely, let T strongly preserve gs-majorization. Then T is invertible and by Theorem 3.3, $T(X) = [A_1Xa_1|...|A_mXa_m] + JXS'$, for some $S' \in M_m$, $a_1, \dots, a_m \in \mathbb{F}^m$ and invertible matrices $A_1, \dots, A_m \in \mathbf{GD_n}$. We show that $\operatorname{rank}[a_1|...|a_m] \geq 2$. Assume that, if possible, $\{a_1, \dots, a_m\} \subseteq \operatorname{span}\{a\}$, for some $a \in \mathbb{F}^n$. Since $m \geq 2$, we choose $0 \neq b \in (\operatorname{span}\{a\})^{\perp}$. Define, $X_0 \in \mathbf{M_{n,m}}$ such that the first and the second rows are b^t and $-b^t$, respectively, and the other rows are zero. It is clear that $X_0 \neq 0$ and $T(X_0) = 0$, which is a contradiction. Then by Corollary 3.4, there exist $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$ such that T(X) = AXR + JXS. Hence by Lemma 3.6, R and R + nS are invertible.

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