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# THE STRUCTURE OF LINEAR PRESERVERS OF GS-MAJORIZATION

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ABSTRACT. Let $\mathbf{M_{n,m}}$ be the set of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. In this paper we introduce the relation gs-majorization on $\mathbf{M_{n,m}}$. We study some properties of this relation on $\mathbf{M_{n,m}}$. We also characterize all linear operators that preserve (or strongly preserve) gs-majorization on $\mathbf{M_{n,m}}$.

## 1. Introduction

A matrix $R \in \mathbf{M_n}$ where all its row sums are equal to one is said to be a g-row stochastic matrix and a g-column stochastic matrix is the transpose of a g-row stochastic matrix. A matrix $D \in \mathbf{M_n}$ with the property that $D$ and $D^t$ are g-row stochastic matrices is said to be g-doubly stochastic. The set of all g-doubly stochastic matrices is a convex set in $\mathbf{M_n}$. For more information see [5].

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The notion of majorization plays an important role in mathematics, statistics and economics. We begin with the definition of matrix majorization. Let $A$ and $B$ be $n \times m$ real matrices.

(1) If there exists an $n \times n$ doubly stochastic matrix $D$ such that $A = DB$, then $A$ is said to be multivariate majorized by $B$, and this is denoted by $A \prec_m B$.

(2) If for every $x \in \mathbb{R}^n$, $Ax$ is vector majorized by $Bx$, then $A$ is said to be directionally majorized by $B$, and this is denoted by $A \prec_d B$.

The definitions of multivariate and directional majorization are motivated by the theorem of Hardy-Littlewood and Polya for vector majorization which says that for $x, y \in \mathbb{R}^n$, $x \prec y$ if and only if there exists an $n \times n$ doubly stochastic matrix $D$ such that $x = Dy$. For more information see [2], [3], [6] and [9].

Now we introduce the notion of gs-majorization on $\mathbf{M_{n,m}}$ as follows.

**Definition 1.1.** Let $A, B \in \mathbf{M_{n,m}}$. The matrix $B$ is said to be gs-majorized by $A$ if there exists an $n \times n$ $g$-doubly stochastic matrix $D$ such that $B = DA$. This is denoted by $A \succ_{gs} B$.

Let $\sim$ be a relation on $\mathbf{M_{n,m}}$ . A linear operator $T : \mathbf{M_{n,m}} \longrightarrow \mathbf{M_{n,m}}$ is said to be a linear preserver (or strong linear preserver) of $\sim$ if $T(x) \sim T(y)$ whenever $x \sim y$ (or $T(x) \sim T(y)$ if and only if $x \sim y$).

Li and Poon proved the following result regarding the linear preservers of multivariate majorization.

**Proposition 1.2.** [7, Theorem 2] *Let $T : \mathbf{M_{n,m}} \rightarrow \mathbf{M_{n,m}}$ be a linear operator, then $T$ preserves directional majorization if and only if $T$ preserves multivariate majorization if and only if one of the following holds:*

(a) *There exist $A_1, ..., A_m \in \mathbf{M_{n,m}}$, such that $T(X) = \sum_{j=1}{}^m tr(x_j)A_j$, where $x_j$ is $j^{th}$ column of $X$.*

(b) *There exist $R, S \in \mathbf{M_{n,m}}$ and permutation $P \in \mathbf{M_n}$ such that $T(X) = PXR + JXS$.*

The main result of this paper is to characterize all linear preservers of gs-majorization on $\mathbf{M_{n,m}}$. Let $T$ be a linear preserver of gs-majorization on $\mathbf{M_{n,m}}$, then one of the following holds:

(i) $T(X) = \sum_{j=1}^m \mathrm{tr}(x_j)A_j$,

where $A_1, \cdots, A_m \in \mathbf{M_{n,m}}$ and $x_j$ is $j^{th}$ column of $X$, or

(ii) $T(X) = [A_1Xa_1|\cdots|A_mXa_m] + JXS$,

where $S \in \mathbf{M_m}$, $a_1, \cdots, a_m \in \mathbb{F}^m$ and $A_1, \cdots, A_m \in \mathbf{GD_n}$ are invertible.

The following elementary properties of gs-majorization on $\mathbf{M_{n,m}}$ are used throughout this paper.

Let $X, Y \in \mathbf{M_{n,m}}$, $A, B \in \mathbf{GD_n}$, $C \in \mathbf{M_m}$ and $\alpha, \beta \in \mathbb{F}$ such that $A$, $B$ and $C$ are invertible and $\alpha \neq 0$. Then the following conditions are equivalent:

(1) $X \succ_{gs} Y$,

(2) $AX \succ_{gs} BY$,

(3) $\alpha X + \beta J_{n,m} \succ_{gs} \alpha Y + \beta J_{n,m}$,

(4) $XC \succ_{gs} YC$,

where $J_{n,m}$ is the $n \times m$ matrix with all entries equal to one.

Throughout this paper, $\mathbf{GR_n}$, $\mathbf{GC_n}$ and $\mathbf{GD_n}$ are the sets of $g$-row stochastic, $g$-column stochastic and $g$-doubly stochastic matrices, respectively. Also $J$ is the $n \times n$ matrix with all entries equal to one.

## 2. Linear preservers of gs-majorization on $\mathbb{F}^n$

In this section we will characterize all linear operators that preserve (or strongly preserve) gs-majorization on $\mathbb{F}^n$. The following proposition gives an equivalent condition for gs-majorization on $\mathbb{F}^n$. Let $e = (1, ..., 1)^t \in \mathbb{F}^n$.

**Proposition 2.1.** *Let $x$ and $y$ be two distinct vectors in $\mathbb{F}^n$. Then, $x \succ_{gs} y$ if and only if $x \notin \operatorname{span}\{e\}$ and $\operatorname{tr}(x) = \operatorname{tr}(y)$.*

**Lemma 2.2.** *Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then $T$ preserves the subspace $\{x \in \mathbb{F}^n : Jx = 0\}$ if and only if there exists $A \in \operatorname{span}(\mathbf{GC}_n)$ such that $T(x) = Ax$, for each $x \in \mathbb{F}^n$.*

**Proof.** Let $A \in \mathbf{M_n}$ be the matrix representation of $T$ with respect to the standard basis of $\mathbb{F}^n$. Let $A \in \operatorname{span}(\mathbf{GC_n})$, it is easy to show that $T$ preserves the subspace $\{x \in \mathbb{F}^n : Jx = 0\}$.

Conversely, let $T$ preserve the subspace $\{x \in \mathbb{F}^n : Jx = 0\}$, then $J(T(e_i - e_j)) = 0$, for all $1 \leq i, j \leq n$, so $J(A(e_i - e_j)) = 0$, thus $\sum_{k=1}^n a_{ki} = \sum_{k=1}^n a_{kj}$. Therefore, $A \in \operatorname{span}(\mathbf{GC_n})$. □

By Lemma 2.2, we may state the following Proposition.

**Proposition 2.3.** *Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator that preserves gs-majorization. Then there exists $A \in span(\mathbf{GC_n})$ such that $T(x) = Ax, \forall x \in \mathbb{F}^n$.*

We obtain a result similar to Ando's Theorem in [1], for gs-majorization.

**Theorem 2.4.** *Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then $T$ preserves gs-majorization if and only if one of the following holds :*
(a) *$T(x) = \mathrm{tr}(x)a$, for some $a \in \mathbb{F}^n$.*
(b) *$T(x) = \alpha Dx + \beta Jx$, for some $\alpha, \beta \in \mathbb{F}$ and invertible matrix $D \in \mathbf{GD_n}$.*

**Proof.** Let $A \in \mathbf{M_n}$ be the matrix representation of $T$ with respect to the standard basis of $\mathbb{F}^n$. If (a) or (b) holds, it is clear that $T$ preserves gs-majorization. Conversely, let $T$ preserve gs-majorization. We consider two parts:

Part (i): Let there exist $b \in (\mathbb{F}^n \setminus \mathrm{span}\{e\})$, such that $T(b) = se$, for some $s \in \mathbb{F}$. We consider two cases;

Case 1; Let tr($b$)=0, then tr($Ab$)=0, by Proposition 2.3. Hence

$$J(Ab) = 0 \Rightarrow J(se) = 0 \Rightarrow se = 0 \Rightarrow s = 0 \Rightarrow T(b) = 0 \quad \Rightarrow Ab = 0.$$

By Proposition 2.1, $b \succ_{gs} (e_i - e_j)$, for $1 \leq i, j \leq n$, so $0 = Ab \succ_{gs} A(e_i - e_j)$ and hence $Ae_i = Ae_j$, for $1 \leq i, j \leq n$. Then, $A = [a|\cdots|a]$, for some $a \in \mathbb{F}^n$. Thus, $T(x) = \mathrm{tr}(x)a$, $\forall x \in \mathbb{F}^n$.

Case 2; Let $\mathrm{tr}(b) = \delta \neq 0$. Consider the basis$\{\delta e_1, \cdots, \delta e_n\}$ for $\mathbb{F}^n$. By Proposition 2.1, $b \succ_{gs} \delta e_i$, for all $1 \leq i \leq n$. Then, $T(b) \succ_{gs} \delta T(e_i)$ and hence $se \succ_{gs} \delta T(e_i)$ . Therefore, $T(e_i) = \frac{s}{\delta}e$, for all $1 \leq i \leq n$, so $T(x) = \mathrm{tr}(x)(\frac{s}{\delta}e)$, $\forall x \in \mathbb{F}^n$.

Part(ii): Let $x \notin \mathrm{span}\{e\}$ imply that $T(x) \notin \mathrm{span}\{e\}$. We consider two cases;

Case 1; Let $T$ be invertible. Then there exists $b \in \mathbb{F}^n$ such that $T(b) = e$, so by hypothesis $b = se$, for some $s \in \mathbb{F}$. Thus $Ae = \frac{1}{s}e$ and hence $A \in \mathrm{span}(\mathbf{GR_n})$. Also $A \in \mathrm{span}(\mathbf{GC_n})$, by Proposition 2.3. Therefore, $A \in \mathrm{span}(\mathbf{GD_n})$. Put $D = sA$, $\alpha = \frac{1}{s}$ and $\beta = 0$. So, $T(x) = \alpha Dx + \beta Jx$.

Case 2; Let $T$ be singular. By hypothesis, Ker($A$)= span$\{e\}$, then $\frac{A+J}{n} \in \mathbf{GR_n}$. It is clear that, $\frac{A+J}{n}$ preserves gs-majorization, therefore, by Proposition 2.3, $\frac{A+J}{n} \in \mathbf{GD_n}$. We will show that $A + J$ is invertible.

If $x$ is in $\ker(A+J)$ then $Ax=-Jx$ lies in the span$\{e\}$, and hence so does $x$ by the hypothesis of part(ii), thus $x=re$ for some $r\in\mathbb{F}$. Then, $(A+J)(re)=0$ implies that $r=0$ and hence $x=0$. Therefore, $A+J$ is invertible. Define $D=\frac{A+J}{n}, \alpha=n$ and $\beta=-1$. Then $T(x)=\alpha Dx+\beta Jx$ . □

Let $T:\mathbf{M_{n,m}}\rightarrow\mathbf{M_{n,m}}$ be a linear operator that strongly preserves gs-majorization. It is easy to show that $T$ is invertible.

**Corollary 2.5.** *Let $T:\mathbb{F}^n\rightarrow\mathbb{F}^n$ be a linear operator. Then $T$ strongly preserves gs-majorization if and only if $T(x)=\alpha Dx$, for some nonzero scalar $\alpha\in\mathbb{F}$ and invertible matrix $D\in\mathbf{GD_n}$.*

## 3. Linear preservers of gs-majorization on $\mathbf{M_{n,m}}$

In this section we characterize all linear operators that preserve (strongly preserve) gs-majorization on $\mathbf{M_{n,m}}$.

**Lemma 3.1.** *Let $A\in\mathbf{GD_n}$ be invertible. Then the following conditions are equivalent:*

(a) $A=\alpha I+\beta J$, *for some* $\alpha,\beta\in\mathbb{F}$,

(b) $(x+Ay)\succ_{gs}(Dx+ADy)$, *for all* $D\in\mathbf{GD_n}$ *and for all* $x,y\in\mathbb{F}^n$ .

**Proof.** $a\rightarrow b)$ If $A=\alpha I+\beta J$, it is easy to show that $(x+Ay)\succ_{gs}(Dx+ADy)$ for all $D\in\mathbf{GD_n}$ and $x,y\in\mathbb{F}^n$.

$b\rightarrow a)$ The matrix A is invertible. Thus, for every $1\le i\le n$ there exists $y_i\in\mathbb{F}^n$ such that $Ay_i=e-e_i$ . It is trivial that $tr(y_i)=n-1$. By hypothesis $(e_i+Ay_i)\succ_{gs}(De_i+ADy_i)$, $\forall D\in\mathbf{GD_n}$, and hence $e\succ_{gs}(De_i+ADy_i)$, $\forall D\in\mathbf{GD_n}$. Thus,

$$(De_i+ADy_i)=e,\ \forall D\in\mathbf{GD_n}. \tag{3.1}$$

It is clear that $[J-(n-1)A]\in\mathbf{GD_n}$ and hence $D[J-(n-1)A]\in\mathbf{GD_n}$, $\forall D\in\mathbf{GD_n}$. Therefore, by (3.1),

$$\begin{aligned} D[J-(n-1)A]e_i+AD[J-(n-1)A]y_i=e \\ \Rightarrow(DA-AD)e_i=0,\ for\ all\ 1\le i\le n \\ \Rightarrow AD=DA,\forall\ D\in\mathbf{GD_n}. \end{aligned} \tag{3.2}$$

Put $D = P_{ij}$ in (3.2), where $P_{ij}$ is the permutation that interchanges the $i^{th}$ and $j^{th}$ rows of the identity matrix. Then, $P_{ij}A = AP_{ij}$, so $A = \alpha I + \beta J$, for some $\alpha, \beta \in \mathbb{F}$. □

**Lemma 3.2.** *Let $T_1, T_2$: $\mathbb{F}^n \to \mathbb{F}^n$ satisfy $T_1(x) = \alpha Ax + \beta Jx$ and $T_2(x) = \mathrm{tr}(x)a$, for some $\alpha, \beta \in \mathbb{F}$, $\alpha \neq 0$, invertible matrix $A \in \mathbf{GD_n}$ and $a \in (\mathbb{F}^n \setminus \mathrm{span}\{e\})$. Then there exist a g-doubly stochastic matrix $D$ and a vector $x \in \mathbb{F}^n$ such that $T_1(x) + T_2(x)$.*

**Proof.** Assume that, if possible, $T_1(x) + T_2(x) \succ_{gs} T_1(Dx) + T_2(Dx)$, $\forall\, D \in \mathbf{GD_n}$, $\forall\, x \in \mathbb{F}^n$. Then by elementary properties of gs-majorization,
$\alpha Ax + \mathrm{tr}(x)a \succ_{gs} \alpha ADx + \mathrm{tr}(x)a, \forall\ D \in \mathbf{GD_n}\ , \forall x \in \mathbb{F}^n$.
Put $b = \frac{1}{\alpha}a$. Then

$$Ax + \mathrm{tr}(x)b \succ_{gs} ADx + \mathrm{tr}(x)b, \quad \forall\ D \in \mathbf{GD_n}\ ,\ \forall x \in \mathbb{F}^n. \tag{3.3}$$

The matrix $A$ is invertible thus there exists $x_0 \in \mathbb{F}^n$ such that $Ax_0 = (e - b)$. Put $x = x_0 - (\frac{n-1-tr(b)}{n})e$ in (3.3). Then $ADx_0 = (e - b)$, $\forall\ D \in \mathbf{GD_n}$, so $b \in \mathrm{span}\{e\}$ and hence $a \in \mathrm{span}\{e\}$, which is a contradiction. □

Now, we state the main theorem of this section.

**Theorem 3.3.** *Let $T : \mathbf{M_{n,m}} \to \mathbf{M_{n,m}}$ be a linear operator that preserves gs-majorization. Then one the following holds:*

*(i) There exist $A_1, \cdots, A_m \in \mathbf{M_{n,m}}$ such that $T(X) = \sum_{j=1}{}^m \mathrm{tr}(x_j)A_j$, where $X = [x_1|\cdots|x_m]$.*

*(ii) There exist $S \in \mathbf{M_m}$, $a_1, \cdots, a_m \in \mathbb{F}^m$ and invertible matrices $A_1, \cdots, A_m \in \mathbf{GD_n}$, such that $T(X) = [A_1Xa_1|\cdots|A_mXa_m] + JXS$.*

**Proof.** Define the embedding $E^j$:$\mathbb{F}^n \to \mathbf{M_{n,m}}$ by $E^j(x) = x{e_j}^t$ and projection $E_i$ :$\mathbf{M_{n,m}} \to \mathbb{F}^n$ by $E_i(A) = Ae_i$ for $1 \leq i, j \leq m$. Put ${T_i}^j = E_iTE^j$. Then,

$$T(X) = T[x_1|\cdots|x_m] = [\sum_{j=1}^{m} T_1^j(x_j)|\cdots|\sum_{j=1}^{m} T_m^j(x_j)]. \tag{3.4}$$

It is easy to show that ${T_i}^j$:$\mathbb{F}^n \to \mathbb{F}^n$ preserves gs-majorizaton. Then each ${T_i}^j$ is of the form (a) or (b) in Theorem 2.4. Now, we consider two cases:

Case 1; Let $T_i{}^j(x) = tr(x)a_i{}^j$, for some $a_i{}^j \in \mathbb{F}^n$, $\forall\ 1 \leq i,j \leq m$. Define $A_j = [a_1{}^j|\cdots|a_m{}^j]$. By (3.4), it is clear that $T(X) = \sum_{j=1}{}^m tr(x_j)A_j$ . Hence the condition (i) holds.

Case 2; Let there exist $1 \leq p,q \leq m$, such that $T_p{}^q(x) = \gamma_p{}^q B_p{}^q x + \delta_p^q Jx$, for some $\gamma_p^q, \delta_p^q \in \mathbb{F}$, $\gamma_p^q \neq 0$, and invertible matrix $B_p{}^q \in \mathbf{GD_n}$.

Step 1. We show that for all $1 \leq j \leq m$, $T_p^j(x) = \alpha_p{}^j A_p x + \beta_p{}^j Jx$, for some $\alpha_p^j, \beta_p{}^j \in \mathbb{F}$, and invertible matrix $A_p \in \mathbf{GD_n}$.

For every $x, y \in \mathbb{F}^n$, define $B_{x,y} = E^j(x) + E^q(y) \in \mathbf{M_{n,m}}$. Then for every $D \in \mathbf{GD_n}$,

$$\begin{aligned} B_{x,y} \succ_{gs} DB_{x,y} &\Rightarrow T(B_{x,y}) \succ_{gs} T(DB_{x,y}) \\ &\Rightarrow [T_1^j(x) + T_1^q(y)|\cdots|T_m^j(x) + T_m^q(y)] \\ &\quad\ \succ_{gs} [T_1^j(Dx) + T_1^q(Dy)|\cdots|T_m^j(Dx) + T_m^q(Dy)] \\ &\Rightarrow T_p^j(x) + T_p^q(y) \succ_{gs} T_p^j(Dx) + T_p^q(Dy), \\ &\quad\ \forall D \in \mathbf{GD_n}\ . \end{aligned} \tag{3.5}$$

Then by Lemma 3.2, $T_p^j(x) = \gamma_p^j B_p^j x + \delta_p^j Jx$, for some $\gamma_p^j, \delta_p^j \in \mathbb{F}$, and invertible matrix $B_p^j \in \mathbf{GD_n}$. Put $T_p^j(x) = \gamma_p^j B_p^j x + \delta_p^j Jx$ as in (3.5). Thus

$$\begin{aligned} &\gamma_p^j B_p^j x + \delta_p^j Jx + \gamma_p^q B_p^q y + \delta_p^q Jy \\ &\succ_{gs} \gamma_p^j B_p^j Dx + \delta_p^j JDx + \gamma_p^q B_p^q Dy + \delta_p^q JDy \\ &\Rightarrow \gamma_p^j B_p^j x + \gamma_p^q B_p^q y \succ_{gs} \gamma_p^j B_p^j Dx + \gamma_p^q B_p^q Dy \\ &\Rightarrow (\gamma_p^q B_p^q)^{-1}\gamma_p^j B_p^j x + y \succ_{gs} (\gamma_p^q B_p^q)^{-1}\gamma_p^j B_p^j Dx + Dy \\ &\quad\ \forall D \in \mathbf{GD_n}\ . \end{aligned} \tag{3.6}$$

If $\gamma_p^j = 0$, we choose $B_p^j = 0$. Let $\gamma_p^j \neq 0$ then, by Lemma 3.1, $B_p^j = r_p^j B_p^q + s_p^j J$, for some $r_p^j, s_p^j \in \mathbb{F}$. Define $A_p = B_p^q$ , then $T_p^j(x) = \alpha_p^j A_p x + \beta_p^j Jx$, for some $\alpha_p^j$ , $\beta_p^j \in \mathbb{F}$.

Step 2. Now, we show that, for all $1 \leq i,j \leq m$, $T_i^j$ are of the form (b) of Theorem 2.4. By Step 1, $T_p^j(x) = \alpha_p^j A_p x + \beta_p^j Jx$. For every $x \in \mathbb{F}^n$, define $B_x = E^j(x)$. Then for all $D \in \mathbf{GD_n}$,

$$\begin{aligned} B_x \succ_{gs} DB_x &\Rightarrow T(B_x) \succ_{gs} T(DB_x) \\ &\Rightarrow [T_1^j(x)|\cdots|T_m^j(x)] \succ_{gs} [T_1^j(Dx)|\cdots|T_m^j(Dx)] \\ &\Rightarrow T_p^j(x) + T_i^j(x) \succ_{gs} T_p^j(Dx) + T_i^j(Dx), \forall D \in \mathbf{GD_n}. \end{aligned}$$

Thus by Lemma 3.2, $T_i^j(x) = \gamma_i^j B_i^j x + \delta_i^j Jx$, for some $\gamma_i^j, \delta_i^j \in \mathbb{F}$ and invertible matrix $B_i^j \in \mathbf{GD_n}$. Now again by Step 1, $T_i^j(x) = \alpha_i^j A_i x + \beta_i^j Jx$, for some $\alpha_i^j, \beta_i^j \in \mathbb{F}$ and invertible matrix $A_i \in \mathbf{GD_n}$. For every $1 \le i \le m$, define

$$a_i = \begin{pmatrix} \alpha_i^1 \\ \vdots \\ \alpha_i^m \end{pmatrix} \quad and \quad S = \begin{pmatrix} \beta_1^1 & \cdots & \beta_m^1 \\ \vdots & \vdots & \vdots \\ \beta_1^m & \cdots & \beta_m^m \end{pmatrix}.$$

Then,

$$\begin{aligned} T(X) &= T[x_1|\cdots|x_m] = [\sum_{j=1}^m T_1^j(x_j)|\cdots|\sum_{j=1}^m T_m^j(x_j)] \\ &= [A_1 \sum_{j=1}^m \alpha_1^j x_j|\cdots|A_m \sum_{j=1}^m \alpha_m^j x_j] + J[\sum_{j=1}^m \beta_1^j x_j|\cdots|\sum_{j=1}^m \beta_m^j x_j] \\ &= [A_1 X a_1|\cdots|A_m X a_m] + JXS \quad . \end{aligned}$$

Hence the condition (ii) holds. □

**Corollary 3.4.** *Let $T$ satisfy the condition (ii) of Theorem* 3.3 *and let* $\text{rank}[a_1|\cdots|a_m] \ge 2$*. Then $T(X) = AXR + JXS$, for some $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$.*

**Proof.** Without loss of generality, let $\{a_1, a_2\}$ be a linearly independent set. Let $X \in \mathbf{M_{n,m}}$, $D \in \mathbf{GD_n}$ be arbitrary. Then

$$\begin{aligned} & X \succ_{gs} DX \Rightarrow T(X) \succ_{gs} T(DX) \\ & \Rightarrow [A_1 X a_1|...|A_m X a_m] \succ_{gs} [A_1 DX a_1|...|A_m DX a_m] \\ & \Rightarrow A_1 X a_1 + A_2 X a_2 \succ_{gs} A_1 DX a_1 + A_2 DX a_2 \\ & \Rightarrow X a_1 + (A_1^{-1} A_2) X a_2 \succ_{gs} DX a_1 + (A_1^{-1} A_2) DX a_2 \quad . \quad (3.7) \end{aligned}$$

Since $\{a_1, a_2\}$ is linearly independent, for every $x, y \in \mathbb{F}^n$, there exists $B_{x,y} \in \mathbf{M_{n,m}}$ such that $B_{x,y}\ a_1 = x$ and $B_{x,y}\ a_2 = y$. Put $X = B_{x,y}$ as in (3.7). Thus,

$$\begin{aligned} B_{x,y}\ a_1 + (A_1^{-1}\ A_2)\ B_{x,y}\ a_2 &\succ_{gs} DB_{x,y}\ a_2 + (A_1^{-1}\ A_2)\ DB_{x,y}\ a_2 \Rightarrow \\ x + (A_1^{-1} A_2)\ y &\succ_{gs} Dx + (A_1^{-1} A_2) Dy\ , \forall D \in \mathbf{GD_n}\ . \end{aligned}$$

Then by Lemma 3.1, $A_1^{-1} A_2 = \alpha I + \beta J$ and hence $A_2 = \alpha A_1 + \beta J$, for some $\alpha, \beta \in \mathbb{F}$, $\alpha \ne 0$.

For every $i \geq 3$, if $a_i = 0$ we can choose $A_i = A_1$. If $a_i \neq 0$ then $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is linearly independent. Then by the same argument as above, $A_i = \gamma_i A_1 + \delta_i J$, for some $\gamma_i, \delta_i \in \mathbb{F}$, $\gamma_i \neq 0$, or $A_i = \lambda_i A_2 + \mu_i J$, for some $\lambda_i, \mu_i \in \mathbb{F}$, $\lambda_i \neq 0$.

Define $A = A_1$. Then for every $i \geq 2$, $A_i = \alpha_i A + \beta_i J$, for some $\alpha_i, \beta_i \in \mathbb{F}$ and hence

$$T(X) = [AXa_1 \mid AX(r_2 a_2)|\cdots|AX(r_m a_m)] + JXS = AXR + JXS,$$

where, $R = [a_1|r_2 a_2|\cdots|r_m a_m]$, for some $r_2, \cdots, r_m \in \mathbb{F}$ and $S$ is the same as in Theorem 3.3. □

The following example shows that if $\operatorname{rank}[a_1|\cdots|a_m] = 1$ , the above corollary does not hold when $\mathbb{F} = \mathbb{R}$.

**Example 3.5.** Let $T : \mathbf{M_{3,2}} \to \mathbf{M_{3,2}}$ be defined by $T(X) = [Xe_1 \mid PXe_1]$ where $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We show that $T$ preserves gs-majorization, and $T$ is not of the form $T(X) = AXR + JXS$ in Corollary 3.4. Let $X = [x \mid x']$, $Y = [y \mid y'] \in \mathbf{M_{3,2}}$ and $X \succ_{gs} Y$. Now we consider two cases:

Case 1; Let $x \in \operatorname{span}\{e\}$, then $y = x$, therefore, $T(X) = T(Y)$.

Case 2; Let $x \notin \operatorname{span}\{e\}$. Put $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, then $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$ .Set

$$R = \begin{pmatrix} x_1 - x_3 & x_2 - x_3 \\ x_3 - x_2 & x_1 - x_2 \end{pmatrix}.$$

It is clear that $R$ is invertible. Define,

$$D := \begin{pmatrix} r_1 & r_2 & 1 - (r_1 + r_2) \\ s_1 & s_2 & 1 - (s_1 + s_2) \\ 1 - (r_1 + s_1) & 1 - (r_2 + s_2) & r_1 + r_2 + s_1 + s_2 - 1 \end{pmatrix},$$

where

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = R^{-1} \begin{pmatrix} y_1 - x_3 \\ y_3 - x_2 \end{pmatrix} \quad and \quad \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = R^{-1} \begin{pmatrix} y_2 - x_3 \\ y_1 - x_2 \end{pmatrix}.$$

It is easy to check that $D \in \mathbf{GD_3}$, $Dx = y$ and $DPx = Py$. Therefore, $T$ preserves gs-majorization.

Finally we show that $T$ is not of the form $T(X) = AXR + JXS$ as in Corollary 3.4 . Assume that, if possible, $T(X) = AXR + JXS$, for some $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$. Then $AXR + JXS = [Xe_1 \mid PXe_1]$, $\forall X \in \mathbf{M_{3,2}}$, so

$$[AXR_1 + JXS_1 \mid AXR_2 + JXS_2] = [Xe_1 \mid PXe_1] \quad \forall X \in \mathbf{M_{3,2}},$$

where $R_i$ and $S_i$ are $i^{th}$ column of R and S respectively. Let $x \in \mathbb{F}^3$ be arbitrary. Put $X = [x \mid 0]$ in the above equation, then $[r_{11}Ax + s_{11}Jx \mid r_{12}Ax + s_{12}Jx]= [x \mid Px]$, $\forall x \in \mathbb{F}^3$. Therefore, $P = \alpha I + \beta J$, for some $\alpha, \beta \in \mathbb{F}$, which is a contradiction.

**Lemma 3.6.** *Let $T : \mathbf{M_{n,m}} \rightarrow \mathbf{M_{n,m}}$ be a linear operator, such that, $T(X) = DXR + JXS$ for some $R, S \in \mathbf{M_m}$ and invertible matrix $D \in \mathbf{GD_n}$. Then $T$ is invertible if and only if $R$ and $(R+nS)$ are invertible.*

**Proof.** Without loss of generality, we can assume that $D = I$. Let $A$ be the matrix representation of $T$ with respect to the standard basis of $\mathbf{M_{n,m}}$. Then, it is easy to show that $A$ is similar to the following block matrix,

$$\begin{pmatrix} R+nS & S & \cdots & S \\ 0 & R & \cdots & S \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R \end{pmatrix}.$$

Therefore, $T$ is invertible if and only if $R$ and $(R+nS)$ are invertible. $\square$

**Theorem 3.7.** *Let $T : \mathbf{M_{n,m}} \rightarrow \mathbf{M_{n,m}}$ be a linear operator. Then $T$ strongly preserves gs-majorization if and only if $T(X) = AXR + JXS$ for some $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$, such that, $R$ and $R + nS$ are invertible .*

**Proof.** If $m$=1, the result holds by Corollary 2.5. So let $m \geq 2$. Let $T(X) = AXR + JXS$, such that $R$ and $(R + nS)$ are invertible. Let $X \succ_{gs} Y$. It is easy to show that $T(X) \succ_{gs} T(Y)$. Now, let

$T(X) \succ_{gs} T(Y)$. Then, $DT(X) = T(Y)$, for some $D \in \mathbf{GD_n}$. Thus,

$$\begin{aligned} DT(X) = T(Y) &\Rightarrow DAXR + JXS = AYR + JYS \\ &\Rightarrow J[DAXR + JXS] = J[AYR + JYS] \\ &\Rightarrow (JX)(R + nS) = (JY)(R + nS) \\ &\Rightarrow JX = JY. \end{aligned}$$

Then, $(A^{-1}DA)X = Y$. Therefore, $X \succ_{gs} Y$.

Conversely, let $T$ strongly preserve gs-majorization. Then $T$ is invertible and by Theorem 3.3, $T(X) = [A_1Xa_1|...|A_mXa_m] + JXS'$, for some $S' \in M_m$, $a_1, \cdots, a_m \in \mathbb{F}^m$ and invertible matrices $A_1, ..., A_m \in \mathbf{GD_n}$. We show that $\text{rank}[a_1|...|a_m] \geq 2$. Assume that, if possible, $\{a_1, ..., a_m\} \subseteq \text{span}\{a\}$, for some $a \in \mathbb{F}^n$. Since $m \geq 2$, we choose $0 \neq b \in (\text{span}\{a\})^{\perp}$. Define, $X_0 \in \mathbf{M_{n,m}}$ such that the first and the second rows are $b^t$ and $-b^t$, respectively, and the other rows are zero. It is clear that $X_0 \neq 0$ and $T(X_0) = 0$, which is a contradiction. Then by Corollary 3.4, there exist $R, S \in \mathbf{M_m}$ and invertible matrix $A \in \mathbf{GD_n}$ such that $T(X) = AXR + JXS$. Hence by Lemma 3.6, $R$ and $R + nS$ are invertible. □

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