# THE STRUCTURE OF LINEAR PRESERVERS OF GS-MAJORIZATION 

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Communicated by Heydar Radjavi

Dedicated to Professor Mehdi Radjabalipour for his outstanding contributions to mathematics


#### Abstract

Let $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$ be the set of all $n \times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. In this paper we introduce the relation gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$. We study some properties of this relation on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$. We also characterize all linear operators that preserve (or strongly preserve) gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$.


## 1. Introduction

A matrix $R \in \mathbf{M}_{\mathbf{n}}$ where all its row sums are equal to one is said to be a g-row stochastic matrix and a g-column stochastic matrix is the transpose of a g-row stochastic matrix. A matrix $D \in \mathbf{M}_{\mathbf{n}}$ with the property that $D$ and $D^{t}$ are g-row stochastic matrices is said to be g-doubly stochastic. The set of all g-doubly stochastic matrices is a convex set in $\mathbf{M}_{\mathbf{n}}$. For more information see [5].

[^0]The notion of majorization plays an important role in mathematics, statistics and economics. We begin with the definition of matrix majorization. Let $A$ and $B$ be $n \times m$ real matrices.
(1) If there exists an $n \times n$ doubly stochastic matrix $D$ such that $A=D B$, then $A$ is said to be multivariate majorized by $B$, and this is denoted by $A \prec_{m} B$.
(2) If for every $x \in \mathbb{R}^{n}, A x$ is vector majorized by $B x$, then $A$ is said to be directionally majorized by $B$, and this is denoted by $A \prec_{d} B$.

The definitions of multivariate and directional majorization are motivated by the theorem of Hardy-Littlewood and Polya for vector majorization which says that for $x, y \in \mathbb{R}^{n}, x \prec y$ if and only if there exists an $n \times n$ doubly stochastic matrix $D$ such that $x=D y$. For more information see [2], [3], [6] and [9].

Now we introduce the notion of gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$ as follows.
Definition 1.1. Let $A, B \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$. The matrix $B$ is said to be gsmajorized by $A$ if there exists an $n \times n g$-doubly stochastic matrix $D$ such that $B=D A$. This is denoted by $A \succ_{g s} B$.

Let $\sim$ be a relation on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$. A linear operator $T: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \longrightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ is said to be a linear preserver (or strong linear preserver) of $\sim$ if $T(x) \sim$ $T(y)$ whenever $x \sim y$ (or $T(x) \sim T(y)$ if and only if $x \sim y$ ).

Li and Poon proved the following result regarding the linear preservers of multivariate majorization.

Proposition 1.2. [7, Theorem 2] Let $T: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \rightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ be a linear operator, then $T$ preserves directional majorization if and only if $T$ preserves multivariate majorization if and only if one of the following holds:
(a) There exist $A_{1}, \ldots, A_{m} \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$, such that $T(X)=\sum_{j=1}{ }^{m} \operatorname{tr}\left(x_{j}\right) A_{j}$, where $x_{j}$ is $j^{\text {th }}$ column of $X$.
(b) There exist $R, S \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ and permutation $P \in \mathbf{M}_{\mathbf{n}}$ such that $T(X)=P X R+J X S$.

The main result of this paper is to characterize all linear preservers of gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$. Let $T$ be a linear preserver of gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$, then one of the following holds:
(i) $T(X)=\sum_{j=1}^{m} \operatorname{tr}\left(x_{j}\right) A_{j}$,
where $A_{1}, \cdots, A_{m} \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ and $x_{j}$ is $j^{\text {th }}$ column of $X$, or
(ii) $T(X)=\left[A_{1} X a_{1}|\cdots| A_{m} X a_{m}\right]+J X S$,
where $S \in \mathbf{M}_{\mathbf{m}}, a_{1}, \cdots, a_{m} \in \mathbb{F}^{m}$ and $A_{1}, \cdots, A_{m} \in \mathbf{G D}_{\mathbf{n}}$ are invertible.

The following elementary properties of gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$ are used throughout this paper.

Let $X, Y \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}, A, B \in \mathbf{G D}_{\mathbf{n}}, C \in \mathbf{M}_{\mathbf{m}}$ and $\alpha, \beta \in \mathbb{F}$ such that $A, B$ and $C$ are invertible and $\alpha \neq 0$. Then the following conditions are equivalent:
(1) $X \succ_{g s} Y$,
(2) $A X \succ_{g s} B Y$,
(3) $\alpha X+\beta J_{n, m} \succ_{g s} \alpha Y+\beta J_{n, m}$,
(4) $X C \succ_{g s} Y C$,
where $J_{n, m}$ is the $n \times m$ matrix with all entries equal to one.
Throughout this paper, $\mathbf{G R}_{\mathbf{n}}, \mathbf{G C}_{\mathbf{n}}$ and $\mathbf{G D}_{\mathbf{n}}$ are the sets of $g$-row stochastic, $g$-column stochastic and $g$-doubly stochastic matrices, respectively. Also $J$ is the $n \times n$ matrix with all entries equal to one.

## 2. Linear preservers of gs-majorization on $\mathbb{F}^{n}$

In this section we will characterize all linear operators that preserve (or strongly preserve) gs-majorization on $\mathbb{F}^{n}$. The following proposition gives an equivalent condition for gs-majorization on $\mathbb{F}^{n}$. Let $e=(1, \ldots, 1)^{t} \in \mathbb{F}^{n}$.

Proposition 2.1. Let $x$ and $y$ be two distinct vectors in $\mathbb{F}^{n}$. Then, $x \succ_{g s} y$ if and only if $x \notin \operatorname{span}\{e\}$ and $\operatorname{tr}(x)=\operatorname{tr}(y)$.

Lemma 2.2. Let $T: \mathbb{F}^{n} \rightarrow \mathbb{F}^{n}$ be a linear operator. Then $T$ preserves the subspace $\left\{x \in \mathbb{F}^{n}: J x=0\right\}$ if and only if there exists $A \in \operatorname{span}\left(\mathbf{G} \mathbf{C}_{n}\right)$ such that $T(x)=A x$, for each $x \in \mathbb{F}^{n}$.

Proof. Let $A \in \mathbf{M}_{\mathbf{n}}$ be the matrix representation of $T$ with respect to the standard basis of $\mathbb{F}^{n}$. Let $A \in \operatorname{span}\left(\mathbf{G} \mathbf{C}_{\mathbf{n}}\right)$, it is easy to show that $T$ preserves the subspace $\left\{x \in \mathbb{F}^{n}: J x=0\right\}$.

Conversely, let $T$ preserve the subspace $\left\{x \in \mathbb{F}^{n}: J x=0\right\}$, then $J\left(T\left(e_{i}-e_{j}\right)\right)=0$, for all $1 \leq i, j \leq n$, so $J\left(A\left(e_{i}-e_{j}\right)\right)=0$, thus $\sum_{k=1}^{n} a_{k i}=\sum_{k=1}^{n} a_{k j}$. Therefore, $A \in \operatorname{span}\left(\mathbf{G C}_{\mathbf{n}}\right)$.

By Lemma 2.2, we may state the following Proposition.
Proposition 2.3. Let $T: \mathbb{F}^{n} \rightarrow \mathbb{F}^{n}$ be a linear operator that preserves gs-majorization. Then there exists $A \in \operatorname{span}\left(\mathbf{G C}_{\mathbf{n}}\right)$ such that $T(x)=$ $A x, \forall x \in \mathbb{F}^{n}$.

We obtain a result similar to Ando's Theorem in [1], for gs-majorization.
Theorem 2.4. Let $T: \mathbb{F}^{n} \rightarrow \mathbb{F}^{n}$ be a linear operator. Then $T$ preserves gs-majorization if and only if one of the following holds :
(a) $T(x)=\operatorname{tr}(x) a$, for some $a \in \mathbb{F}^{n}$.
(b) $T(x)=\alpha D x+\beta J x$, for some $\alpha, \beta \in \mathbb{F}$ and invertible matrix $D \in \mathbf{G D}_{\mathbf{n}}$.

Proof. Let $A \in \mathbf{M}_{\mathbf{n}}$ be the matrix representation of $T$ with respect to the standard basis of $\mathbb{F}^{n}$. If (a) or (b) holds, it is clear that $T$ preserves gs-majorization. Conversely, let $T$ preserve gs-majorization. We consider two parts:

Part (i): Let there exist $b \in\left(\mathbb{F}^{n} \backslash \operatorname{span}\{e\}\right)$, such that $T(b)=s e$, for some $s \in \mathbb{F}$. We consider two cases;

Case 1; Let $\operatorname{tr}(b)=0$, then $\operatorname{tr}(A b)=0$, by Proposition 2.3. Hence

$$
J(A b)=0 \Rightarrow J(s e)=0 \Rightarrow s e=0 \Rightarrow s=0 \Rightarrow T(b)=0 \Rightarrow A b=0
$$

By Proposition 2.1, $b \succ_{g s}\left(e_{i}-e_{j}\right)$, for $1 \leq i, j \leq n$, so $0=A b \succ_{g s}$ $A\left(e_{i}-e_{j}\right)$ and hence $A e_{i}=A e_{j}$, for $1 \leq i, j \leq n$. Then, $A=[a|\cdots| a]$, for some $a \in \mathbb{F}^{n}$. Thus, $T(x)=\operatorname{tr}(x) a, \forall x \in \mathbb{F}^{n}$.

Case 2; Let $\operatorname{tr}(b)=\delta \neq 0$. Consider the basis $\left\{\delta e_{1}, \cdots, \delta e_{n}\right\}$ for $\mathbb{F}^{n}$. By Proposition 2.1, $b \succ_{g s} \delta e_{i}$, for all $1 \leq i \leq n$. Then, $T(b) \succ_{g s} \delta T\left(e_{i}\right)$ and hence se $\succ_{g s} \delta T\left(e_{i}\right)$. Therefore, $T\left(e_{i}\right)=\frac{s}{\delta} e$, for all $1 \leq i \leq n$, so $T(x)=\operatorname{tr}(x)\left(\frac{s}{\delta} e\right), \forall x \in \mathbb{F}^{n}$.

Part(ii): Let $x \notin \operatorname{span}\{e\}$ imply that $T(x) \notin \operatorname{span}\{e\}$. We consider two cases;

Case 1; Let $T$ be invertible. Then there exists $b \in \mathbb{F}^{n}$ such that $T(b)=e$, so by hypothesis $b=s e$, for some $s \in \mathbb{F}$. Thus $A e=\frac{1}{s} e$ and hence $A \in \operatorname{span}\left(\mathbf{G R}_{\mathbf{n}}\right)$. Also $A \in \operatorname{span}\left(\mathbf{G C}_{\mathbf{n}}\right)$, by Proposition 2.3. Therefore, $A \in \operatorname{span}\left(\mathbf{G D}_{\mathbf{n}}\right)$. Put $D=s A, \alpha=\frac{1}{s}$ and $\beta=0$. So, $T(x)=\alpha D x+\beta J x$.

Case 2; Let $T$ be singular. By hypothesis, $\operatorname{Ker}(A)=\operatorname{span}\{e\}$, then $\frac{A+J}{n} \in \mathbf{G R}_{\mathbf{n}}$. It is clear that, $\frac{A+J}{n}$ preserves gs-majorization, therefore, by Proposition $2.3, \frac{A+J}{n} \in \mathbf{G D}_{\mathbf{n}}$. We will show that $A+J$ is invertible.

If $x$ is in $\operatorname{ker}(A+J)$ then $A x=-J x$ lies in the $\operatorname{span}\{e\}$, and hence so does $x$ by the hypothesis of part(ii), thus $x=r e$ for some $r \in \mathbb{F}$. Then, $(A+J)(r e)=0$ implies that $r=0$ and hence $x=0$. Therefore, $A+J$ is invertible. Define $D=\frac{A+J}{n}, \alpha=n$ and $\beta=-1$. Then $T(x)=\alpha D x+\beta J x$.
Let $T: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \rightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ be a linear operator that strongly preserves gsmajorization. It is easy to show that $T$ is invertible.

Corollary 2.5. Let $T: \mathbb{F}^{n} \rightarrow \mathbb{F}^{n}$ be a linear operator. Then $T$ strongly preserves gs-majorization if and only if $T(x)=\alpha D x$, for some nonzero scalar $\alpha \in \mathbb{F}$ and invertible matrix $D \in \mathbf{G D}_{\mathbf{n}}$.

## 3. Linear preservers of gs-majorization on $\mathrm{M}_{\mathrm{n}, \mathrm{m}}$

In this section we characterize all linear operators that preserve (strongly preserve) gs-majorization on $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$.

Lemma 3.1. Let $A \in \mathbf{G D}_{\mathbf{n}}$ be invertible. Then the following conditions are equivalent:
(a) $A=\alpha I+\beta J$, for some $\alpha, \beta \in \mathbb{F}$,
(b) $(x+A y) \succ_{g s}(D x+A D y)$, for all $D \in \mathbf{G D}_{\mathbf{n}}$ and for all $x, y \in \mathbb{F}^{n}$

Proof. $a \rightarrow b$ ) If $A=\alpha I+\beta J$, it is easy to show that $(x+A y) \succ_{g s}$ $(D x+A D y)$ for all $D \in \mathbf{G D}_{\mathbf{n}}$ and $x, y \in \mathbb{F}^{n}$.
$b \rightarrow a)$ The matrix A is invertible. Thus, for every $1 \leq i \leq n$ there exists $y_{i} \in \mathbb{F}^{n}$ such that $A y_{i}=e-e_{i}$. It is trivial that $\operatorname{tr}\left(y_{i}\right)=n-1$. By hypothesis $\left(e_{i}+A y_{i}\right) \succ_{g s}\left(D e_{i}+A D y_{i}\right), \forall D \in \mathbf{G D}_{\mathbf{n}}$, and hence $e \succ_{g s}\left(D e_{i}+A D y_{i}\right), \forall D \in \mathbf{G D} \mathbf{D}_{\mathbf{n}}$. Thus,

$$
\begin{equation*}
\left(D e_{i}+A D y_{i}\right)=e, \forall D \in \mathbf{G D}_{\mathbf{n}} . \tag{3.1}
\end{equation*}
$$

It is clear that $[J-(n-1) A] \in \mathbf{G D}_{\mathbf{n}}$ and hence $D[J-(n-1) A] \in \mathbf{G D}_{\mathbf{n}}$, $\forall D \in \mathbf{G D}_{\mathbf{n}}$. Therefore, by (3.1),

$$
\begin{gather*}
D[J-(n-1) A] e_{i}+A D[J-(n-1) A] y_{i}=e \\
\quad \Rightarrow(D A-A D) e_{i}=0, \text { for all } 1 \leq i \leq n \\
\Rightarrow A D=D A, \forall D \in \mathbf{G D}_{\mathbf{n}} . \tag{3.2}
\end{gather*}
$$

Put $D=P_{i j}$ in (3.2), where $P_{i j}$ is the permutation that interchanges the $i^{\text {th }}$ and $j^{\text {th }}$ rows of the identity matrix. Then, $P_{i j} A=A P_{i j}$, so $A=\alpha I+\beta J$, for some $\alpha, \beta \in \mathbb{F}$.

Lemma 3.2. Let $T_{1}, T_{2}: \mathbb{F}^{n} \rightarrow \mathbb{F}^{n}$ satisfy $T_{1}(x)=\alpha A x+\beta J x$ and $T_{2}(x)=\operatorname{tr}(x)$ a, for some $\alpha, \beta \in \mathbb{F}, \alpha \neq 0$, invertible matrix $A \in \mathbf{G D}_{\mathbf{n}}$ and $a \in\left(\mathbb{F}^{n} \backslash \operatorname{span}\{e\}\right)$. Then there exist a $g$-doubly stochastic matrix $D$ and a vector $x \in \mathbb{F}^{n}$ such that $T_{1}(x)+T_{2}(x)$.

Proof. Assume that, if possible, $T_{1}(x)+T_{2}(x) \succ_{g s} T_{1}(D x)+T_{2}(D x)$, $\forall D \in \mathbf{G D}_{\mathbf{n}}, \forall x \in \mathbb{F}^{n}$. Then by elementary properties of gs-majorization, $\alpha A x+\operatorname{tr}(x) a \succ_{g s} \alpha A D x+\operatorname{tr}(x) a, \forall D \in \mathbf{G D}_{\mathbf{n}}, \forall x \in \mathbb{F}^{n}$.
Put $b=\frac{1}{\alpha} a$. Then

$$
\begin{equation*}
A x+\operatorname{tr}(x) b \succ_{g s} A D x+\operatorname{tr}(x) b, \quad \forall D \in \mathbf{G D}_{\mathbf{n}}, \forall x \in \mathbb{F}^{n} \tag{3.3}
\end{equation*}
$$

The matrix $A$ is invertible thus there exists $x_{0} \in \mathbb{F}^{n}$ such that $A x_{0}=$ $(e-b)$. Put $x=x_{0}-\left(\frac{n-1-\operatorname{tr}(b)}{n}\right) e$ in (3.3). Then $A D x_{0}=(e-b)$, $\forall D \in \mathbf{G D}_{\mathbf{n}}$, so $b \in \operatorname{span}\{e\}$ and hence $a \in \operatorname{span}\{e\}$, which is a contradiction.

Now, we state the main theorem of this section.
Theorem 3.3. Let $T: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \rightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ be a linear operator that preserves gs-majorization. Then one the following holds:
(i) There exist $A_{1}, \cdots, A_{m} \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ such that $T(X)=\sum_{j=1}{ }^{m} \operatorname{tr}\left(x_{j}\right) A_{j}$, where $X=\left[x_{1}|\cdots| x_{m}\right]$.
(ii) There exist $S \in \mathbf{M}_{\mathbf{m}}, a_{1}, \cdots, a_{m} \in \mathbb{F}^{m}$ and invertible matrices $A_{1}, \cdots, A_{m} \in \mathbf{G D}_{\mathbf{n}}$, such that $T(X)=\left[A_{1} X a_{1}|\cdots| A_{m} X a_{m}\right]+J X S$.

Proof. Define the embedding $E^{j}: \mathbb{F}^{n} \rightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ by $E^{j}(x)=x e_{j}^{t}$ and projection $E_{i}: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \rightarrow \mathbb{F}^{n}$ by $E_{i}(A)=A e_{i}$ for $1 \leq i, j \leq m$. Put $T_{i}{ }^{j}=E_{i} T E^{j}$. Then,

$$
\begin{equation*}
T(X)=T\left[x_{1}|\cdots| x_{m}\right]=\left[\sum_{j=1}^{m} T_{1}^{j}\left(x_{j}\right)|\cdots| \sum_{j=1}^{m} T_{m}^{j}\left(x_{j}\right)\right] . \tag{3.4}
\end{equation*}
$$

It is easy to show that $T_{i}^{j}: \mathbb{F}^{n} \rightarrow \mathbb{F}^{n}$ preserves gs-majorizaton. Then each $T_{i}{ }^{j}$ is of the form (a) or (b) in Theorem 2.4. Now, we consider two cases:

Case 1; Let $T_{i}{ }^{j}(x)=\operatorname{tr}(x) a_{i}{ }^{j}$, for some $a_{i}{ }^{j} \in \mathbb{F}^{n}, \forall 1 \leq i, j \leq$ $m$. Define $A_{j}=\left[a_{1}{ }^{j}|\cdots| a_{m}{ }^{j}\right]$. By (3.4), it is clear that $T(X)=$ $\sum_{j=1}^{m} \operatorname{tr}\left(x_{j}\right) A_{j}$. Hence the condition (i) holds.

Case 2; Let there exist $1 \leq p, q \leq m$, such that $T_{p}{ }^{q}(x)=\gamma_{p}{ }^{q} B_{p}{ }^{q} x+$ $\delta_{p}^{q} J x$, for some $\gamma_{p}^{q}, \delta_{p}^{q} \in \mathbb{F}, \gamma_{p}^{q} \neq 0$, and invertible matrix $B_{p}{ }^{q} \in \mathbf{G D}_{\mathbf{n}}$.

Step 1. We show that for all $1 \leq j \leq m, T_{p}^{j}(x)=\alpha_{p}{ }^{j} A_{p} x+\beta_{p}{ }^{j} J x$, for some $\alpha_{p}^{j}, \beta_{p}{ }^{j} \in \mathbb{F}$, and invertible matrix $A_{p} \in \mathbf{G D}_{\mathbf{n}}$.

For every $x, y \in \mathbb{F}^{n}$, define $B_{x, y}=E^{j}(x)+E^{q}(y) \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$. Then for every $D \in \mathbf{G D}_{\mathbf{n}}$,

$$
\begin{align*}
B_{x, y} \succ_{g s} D B_{x, y} \Rightarrow & T\left(B_{x, y}\right) \succ_{g s} T\left(D B_{x, y}\right) \\
\Rightarrow & {\left[T_{1}^{j}(x)+T_{1}^{q}(y)|\cdots| T_{m}^{j}(x)+T_{m}^{q}(y)\right] } \\
& \succ_{g s}\left[T_{1}^{j}(D x)+T_{1}^{q}(D y)|\cdots| T_{m}^{j}(D x)+T_{m}^{q}(D y)\right] \\
\Rightarrow & T_{p}^{j}(x)+T_{p}^{q}(y) \succ_{g s} T_{p}^{j}(D x)+T_{p}^{q}(D y) \\
& \forall D \in \mathbf{G D}_{\mathbf{n}} . \tag{3.5}
\end{align*}
$$

Then by Lemma 3.2, $T_{p}^{j}(x)=\gamma_{p}^{j} B_{p}^{j} x+\delta_{p}^{j} J x$, for some $\gamma_{p}^{j}, \delta_{p}^{j} \in \mathbb{F}$, and invertible matrix $B_{p}^{j} \in \mathbf{G D}_{\mathbf{n}}$. Put $T_{p}^{j}(x)=\gamma_{p}^{j} B_{p}^{j} x+\delta_{p}^{j} J x$ as in (3.5). Thus

$$
\begin{align*}
& \gamma_{p}^{j} B_{p}^{j} x+\delta_{p}^{j} J x+\gamma_{p}^{q} B_{p}^{q} y+\delta_{p}^{q} J y \\
& \succ_{g s} \gamma_{p}^{j} B_{p}^{j} D x+\delta_{p}^{j} J D x+\gamma_{p}^{q} B_{p}^{q} D y+\delta_{p}^{q} J D y \\
& \Rightarrow \gamma_{p}^{j} B_{p}^{j} x+\gamma_{p}^{q} B_{p}^{q} y \succ_{g s} \gamma_{p}^{j} B_{p}^{j} D x+\gamma_{p}^{q} B_{p}^{q} D y \\
& \Rightarrow\left(\gamma_{p}^{q} B_{p}^{q}\right)^{-1} \gamma_{p}^{j} B_{p}^{j} x+y \succ_{g s}\left(\gamma_{p}^{q} B_{p}^{q}\right)^{-1} \gamma_{p}^{j} B_{p}^{j} D x+D y \\
& \quad \forall D \in \mathbf{G D}_{\mathbf{n}} . \tag{3.6}
\end{align*}
$$

If $\gamma_{p}^{j}=0$, we choose $B_{p}^{j}=0$. Let $\gamma_{p}^{j} \neq 0$ then, by Lemma 3.1, $B_{p}^{j}=$ $r_{p}^{j} B_{p}^{q}+s_{p}^{j} J$, for some $r_{p}^{j}, s_{p}^{j} \in \mathbb{F}$. Define $A_{p}=B_{p}^{q}$, then $T_{p}^{j}(x)=\alpha_{p}^{j} A_{p} x+$ $\beta_{p}^{j} J x$, for some $\alpha_{p}^{j}, \beta_{p}^{j} \in \mathbb{F}$.

Step 2. Now, we show that, for all $1 \leq i, j \leq m, T_{i}^{j}$ are of the form (b) of Theorem 2.4. By Step $1, T_{p}^{j}(x)=\alpha_{p}^{j} A_{p} x+\beta_{p}^{j} J x$. For every $x \in \mathbb{F}^{n}$, define $B_{x}=E^{j}(x)$. Then for all $D \in \mathbf{G D}_{\mathbf{n}}$,

$$
\begin{aligned}
B_{x} \succ_{g s} D B_{x} & \Rightarrow T\left(B_{x}\right) \succ_{g s} T\left(D B_{x}\right) \\
& \Rightarrow\left[T_{1}^{j}(x)|\cdots| T_{m}^{j}(x)\right] \succ_{g s}\left[T_{1}^{j}(D x)|\cdots| T_{m}^{j}(D x)\right] \\
& \Rightarrow T_{p}^{j}(x)+T_{i}^{j}(x) \succ_{g s} T_{p}^{j}(D x)+T_{i}^{j}(D x), \forall D \in \mathbf{G D}_{\mathbf{n}}
\end{aligned}
$$

Thus by Lemma $3.2, T_{i}^{j}(x)=\gamma_{i}^{j} B_{i}^{j} x+\delta_{i}^{j} J x$, for some $\gamma_{i}^{j}, \delta_{i}^{j} \in \mathbb{F}$ and invertible matrix $B_{i}^{j} \in \mathbf{G D}_{\mathbf{n}}$. Now again by Step $1, T_{i}^{j}(x)=\alpha_{i}^{j} A_{i} x+$ $\beta_{i}^{j} J x$, for some $\alpha_{i}^{j}, \beta_{i}^{j} \in \mathbb{F}$ and invertible matrix $A_{i} \in \mathbf{G D}_{\mathbf{n}}$. For every $1 \leq i \leq m$, define

$$
a_{i}=\left(\begin{array}{c}
\alpha_{i}^{1} \\
\vdots \\
\alpha_{i}^{m}
\end{array}\right) \quad \text { and } \quad S=\left(\begin{array}{ccc}
\beta_{1}^{1} & \cdots & \beta_{m}^{1} \\
\vdots & \vdots & \vdots \\
\beta_{1}^{m} & \cdots & \beta_{m}^{m}
\end{array}\right)
$$

Then,

$$
\begin{aligned}
& T(X)=T\left[x_{1}|\cdots| x_{m}\right]=\left[\sum_{j=1}^{m} T_{1}^{j}\left(x_{j}\right)|\cdots| \sum_{j=1}^{m} T_{m}^{j}\left(x_{j}\right)\right] \\
& =\left[A_{1} \sum_{j=1}^{m} \alpha_{1}^{j} x_{j}|\cdots| A_{m} \sum_{j=1}^{m} \alpha_{m}^{j} x_{j}\right]+J\left[\sum_{j=1}^{m} \beta_{1}^{j} x_{j}|\cdots| \sum_{j=1}^{m} \beta_{m}^{j} x_{j}\right] \\
& =\left[A_{1} X a_{1}|\cdots| A_{m} X a_{m}\right]+J X S .
\end{aligned}
$$

Hence the condition (ii) holds.
Corollary 3.4. Let $T$ satisfy the condition (ii) of Theorem 3.3 and let $\operatorname{rank}\left[a_{1}|\cdots| a_{m}\right] \geq 2$. Then $T(X)=A X R+J X S$, for some $R, S \in \mathbf{M}_{\mathbf{m}}$ and invertible matrix $A \in \mathbf{G D}_{\mathbf{n}}$.

Proof. Without loss of generality, let $\left\{a_{1}, a_{2}\right\}$ be a linearly independent set. Let $X \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}, D \in \mathbf{G D}_{\mathbf{n}}$ be arbitrary. Then

$$
\begin{align*}
& X \succ_{g s} D X \Rightarrow T(X) \succ_{g s} T(D X) \\
& \Rightarrow\left[A_{1} X a_{1}|\ldots| A_{m} X a_{m}\right] \succ_{g s}\left[A_{1} D X a_{1}|\ldots| A_{m} D X a_{m}\right] \\
& \Rightarrow A_{1} X a_{1}+A_{2} X a_{2} \succ_{g s} A_{1} D X a_{1}+A_{2} D X a_{2} \\
& \Rightarrow X a_{1}+\left(A_{1}^{-1} A_{2}\right) X a_{2} \succ_{g s} D X a_{1}+\left(A_{1}^{-1} A_{2}\right) D X a_{2} \tag{3.7}
\end{align*}
$$

Since $\left\{a_{1}, a_{2}\right\}$ is linearly independent, for every $x, y \in \mathbb{F}^{n}$, there exists $B_{x, y} \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ such that $B_{x, y} a_{1}=x$ and $B_{x, y} a_{2}=y$. Put $X=B_{x, y}$ as in (3.7). Thus,

$$
\begin{array}{rll}
B_{x, y} a_{1}+\left(A_{1}^{-1} A_{2}\right) B_{x, y} a_{2} & \succ_{g s} & D B_{x, y} a_{2}+\left(A_{1}^{-1} A_{2}\right) D B_{x, y} a_{2} \Rightarrow \\
x+\left(A_{1}^{-1} A_{2}\right) y & \succ_{g s} & D x+\left(A_{1}^{-1} A_{2}\right) D y, \forall D \in \mathbf{G D}_{\mathbf{n}}
\end{array}
$$

Then by Lemma 3.1, $A_{1}^{-1} A_{2}=\alpha I+\beta J$ and hence $A_{2}=\alpha A_{1}+\beta J$, for some $\alpha, \beta \in \mathbb{F}, \alpha \neq 0$.

For every $i \geq 3$, if $a_{i}=0$ we can choose $A_{i}=A_{1}$. If $a_{i} \neq 0$ then $\left\{a_{1}, a_{i}\right\}$ or $\left\{a_{2}, a_{i}\right\}$ is linearly independent. Then by the same argument as above, $A_{i}=\gamma_{i} A_{1}+\delta_{i} J$, for some $\gamma_{i}, \delta_{i} \in \mathbb{F}, \gamma_{i} \neq 0$, or $A_{i}=\lambda_{i} A_{2}+\mu_{i} J$, for some $\lambda_{i}, \mu_{i} \in \mathbb{F}, \lambda_{i} \neq 0$.

Define $A=A_{1}$. Then for every $i \geq 2, A_{i}=\alpha_{i} A+\beta_{i} J$, for some $\alpha_{i}, \beta_{i} \in \mathbb{F}$ and hence

$$
T(X)=\left[A X a_{1}\left|A X\left(r_{2} a_{2}\right)\right| \cdots \mid A X\left(r_{m} a_{m}\right)\right]+J X S=A X R+J X S
$$ where, $R=\left[a_{1}\left|r_{2} a_{2}\right| \cdots \mid r_{m} a_{m}\right]$, for some $r_{2}, \cdots, r_{m} \in \mathbb{F}$ and $S$ is the same as in Theorem 3.3.

The following example shows that if $\operatorname{rank}\left[a_{1}|\cdots| a_{m}\right]=1$, the above corollary does not hold when $\mathbb{F}=\mathbb{R}$.

Example 3.5. Let $T: \mathbf{M}_{\mathbf{3}, \mathbf{2}} \rightarrow \mathbf{M}_{\mathbf{3 , 2}}$ be defined by $T(X)=\left[X e_{1} \mid\right.$ $\left.P X e_{1}\right]$ where $P=\left(\begin{array}{lll}0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{array}\right)$. We show that $T$ preserves gs-majorization, and $T$ is not of the form $T(X)=A X R+J X S$ in Corollary 3.4. Let $X=\left[x \mid x^{\prime}\right], Y=\left[y \mid y^{\prime}\right] \in \mathbf{M}_{\mathbf{3 , 2}}$ and $X \succ_{g s} Y$. Now we consider two cases:

Case 1; Let $x \in \operatorname{span}\{e\}$, then $y=x$, therefore, $T(X)=T(Y)$.
Case 2; Let $x \notin \operatorname{span}\{e\}$. Put $x=\left(\begin{array}{c}x_{1} \\ x_{2} \\ x_{3}\end{array}\right), y=\left(\begin{array}{c}y_{1} \\ y_{2} \\ y_{3}\end{array}\right)$, then $x_{1}+x_{2}+x_{3}=y_{1}+y_{2}+y_{3}$. Set

$$
R=\left(\begin{array}{cc}
x_{1}-x_{3} & x_{2}-x_{3} \\
x_{3}-x_{2} & x_{1}-x_{2}
\end{array}\right)
$$

It is clear that $R$ is invertible. Define,

$$
D:=\left(\begin{array}{ccc}
r_{1} & r_{2} & 1-\left(r_{1}+r_{2}\right) \\
s_{1} & s_{2} & 1-\left(s_{1}+s_{2}\right) \\
1-\left(r_{1}+s_{1}\right) & 1-\left(r_{2}+s_{2}\right) & r_{1}+r_{2}+s_{1}+s_{2}-1
\end{array}\right)
$$

where

$$
\binom{r_{1}}{r_{2}}=R^{-1}\binom{y_{1}-x_{3}}{y_{3}-x_{2}} \quad \text { and } \quad\binom{s_{1}}{s_{2}}=R^{-1}\binom{y_{2}-x_{3}}{y_{1}-x_{2}}
$$

It is easy to check that $D \in \mathbf{G D}_{\mathbf{3}}, D x=y$ and $D P x=P y$. Therefore, $T$ preserves gs-majorization.

Finally we show that $T$ is not of the form $T(X)=A X R+J X S$ as in Corollary 3.4. Assume that, if possible, $T(X)=A X R+J X S$, for some $R, S \in \mathbf{M}_{\mathbf{m}}$ and invertible matrix $A \in \mathbf{G D}_{\mathbf{n}}$. Then $A X R+J X S=$ $\left[X e_{1} \mid P X e_{1}\right], \forall X \in \mathbf{M}_{\mathbf{3 , 2}}$, so
$\left[A X R_{1}+J X S_{1} \mid A X R_{2}+J X S_{2}\right]=\left[X e_{1} \mid P X e_{1}\right] \forall X \in \mathbf{M}_{\mathbf{3} \mathbf{2}}$,
where $R_{i}$ and $S_{i}$ are $i^{\text {th }}$ column of R and S respectively. Let $x \in \mathbb{F}^{3}$ be arbitrary. Put $X=[x \mid 0]$ in the above equation, then $\left[r_{11} A x+s_{11} J x \mid\right.$ $\left.r_{12} A x+s_{12} J x\right]=[x \mid P x], \forall x \in \mathbb{F}^{3}$. Therefore, $P=\alpha I+\beta J$, for some $\alpha, \beta \in \mathbb{F}$, which is a contradiction.

Lemma 3.6. Let $T: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \rightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ be a linear operator, such that, $T(X)=D X R+J X S$ for some $R, S \in \mathbf{M}_{\mathbf{m}}$ and invertible matrix $D \in$ $\mathbf{G D}_{\mathbf{n}}$. Then $T$ is invertible if and only if $R$ and $(R+n S)$ are invertible.

Proof. Without loss of generality, we can assume that $D=I$. Let $A$ be the matrix representation of $T$ with respect to the standard basis of $\mathbf{M}_{\mathbf{n}, \mathbf{m}}$. Then, it is easy to show that $A$ is similar to the following block matrix,

$$
\left(\begin{array}{cccc}
R+n S & S & \cdots & S \\
0 & R & \cdots & S \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R
\end{array}\right) .
$$

Therefore, $T$ is invertible if and only if $R$ and $(R+n S)$ are invertible.
Theorem 3.7. Let $T: \mathbf{M}_{\mathbf{n}, \mathbf{m}} \rightarrow \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ be a linear operator. Then $T$ strongly preserves gs-majorization if and only if $T(X)=A X R+J X S$ for some $R, S \in \mathbf{M}_{\mathbf{m}}$ and invertible matrix $A \in \mathbf{G D}_{\mathbf{n}}$, such that, $R$ and $R+n S$ are invertible.

Proof. If $m=1$, the result holds by Corollary 2.5 . So let $m \geq 2$. Let $T(X)=A X R+J X S$, such that $R$ and $(R+n S)$ are invertible. Let $X \succ_{g s} Y$. It is easy to show that $T(X) \succ_{g s} T(Y)$. Now, let
$T(X) \succ_{g s} T(Y)$. Then, $D T(X)=T(Y)$, for some $D \in \mathbf{G D}_{\mathbf{n}}$. Thus,

$$
\begin{aligned}
D T(X)=T(Y) & \Rightarrow D A X R+J X S=A Y R+J Y S \\
& \Rightarrow J[D A X R+J X S]=J[A Y R+J Y S] \\
& \Rightarrow(J X)(R+n S)=(J Y)(R+n S) \\
& \Rightarrow J X=J Y
\end{aligned}
$$

Then, $\left(A^{-1} D A\right) X=Y$. Therefore, $X \succ_{g s} Y$.
Conversely, let $T$ strongly preserve gs-majorization. Then $T$ is invertible and by Theorem 3.3, $T(X)=\left[A_{1} X a_{1}|\ldots| A_{m} X a_{m}\right]+J X S^{\prime}$, for some $S^{\prime} \in M_{m}, a_{1}, \cdots, a_{m} \in \mathbb{F}^{m}$ and invertible matrices $A_{1}, \ldots, A_{m} \in$ $\mathbf{G D}_{\mathbf{n}}$. We show that $\operatorname{rank}\left[a_{1}|\ldots| a_{m}\right] \geq 2$. Assume that, if possible, $\left\{a_{1}, \ldots, a_{m}\right\} \subseteq \operatorname{span}\{a\}$, for some $a \in \mathbb{F}^{n}$. Since $m \geq 2$, we choose $0 \neq b \in(\operatorname{span}\{a\})^{\perp}$. Define, $X_{0} \in \mathbf{M}_{\mathbf{n}, \mathbf{m}}$ such that the first and the second rows are $b^{t}$ and $-b^{t}$, respectively, and the other rows are zero. It is clear that $X_{0} \neq 0$ and $T\left(X_{0}\right)=0$, which is a contradiction. Then by Corollary 3.4, there exist $R, S \in \mathbf{M}_{\mathbf{m}}$ and invertible matrix $A \in \mathbf{G D}_{\mathbf{n}}$ such that $T(X)=A X R+J X S$. Hence by Lemma $3.6, R$ and $R+n S$ are invertible.

## Acknowledgements

We would like to thank the referee for some useful comments. This research was supported by the Mahani Mathematical Research Center.

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[^0]:    $\operatorname{MSC}(2000): 15 \mathrm{~A} 03,15 \mathrm{~A} 04,15 \mathrm{~A} 51$
    Keywords: Strong preserver, g-doubly stochastic matrices, gs-majorization Received: 11 Jully 2006, Revised: 2 September 2006
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