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FORCED OSCILLATIONS OF A DAMPED KORTEWEG-DE VRIES EQUATION ON A PERIODIC DOMAIN

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Abstract. In this paper, we investigate a damped Korteweg-de Vries equation with forcing on a periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. We can obtain that if the forcing is periodic with small amplitude, then the solution becomes eventually time-periodic.

Keywords: Forced oscillation, Korteweg-de Vries equation, stability, time-periodic solution.
MSC(2010): Primary: 35Q53; Secondary: 35B40.

1. Introduction

The Korteweg-de Vries (KdV) equation with damping effect posed on $\mathbb{T}$

$$(1.1) \qquad \begin{cases} u_t + u_{xxx} + uu_x + GG^*u = 0, & x \in \mathbb{T}, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{T} \end{cases}$$

has been investigated by many authors [4, 6, 7], where GG^* is an operator defined in [4], which is sketched here just for the sake of completeness. Suppose that g is a given nonnegative smooth function such that $\{g > 0\} = \omega \subset \mathbb{T}$ and

$$2\pi[g] = \int_{\mathbb{T}} g(x)dx = 1,$$

where $[\cdot]$ denotes the mean value of the function g over $\mathbb{T}$. Let

$$(G\phi)(x) = g(x)\Big(\phi(x) - \int_{\mathbb{T}} g(y)\phi(y)dy\Big), \ \forall \ \phi \in L^2(\mathbb{T}),$$

and G^* denotes its adjoint operator.

In this paper, we consider (1.1) with periodic forcing f,

$$(1.2) \qquad \begin{cases} u_t + u_{xxx} + uu_x + GG^*u = f, & x \in \mathbb{T}, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases}$$

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where $f = f(x,t)$ is a time-periodic function of period τ. In order to keep volume or mass conserved, i.e.,

$$I(t) = \int_{\mathbb{T}} u(x,t)dx$$

be invariant under motion, we assume $f = Gh$, where $h = h(x,t)$ is a time-periodic function of period τ. With this assumption, it is easy to see that

$$\frac{d}{dt}\int_{\mathbb{T}} u(x,t)dt = 0. \tag{1.3}$$

There have been many studies concerned with time-periodic solutions of partial differential equations in the literature (see [3,5,9]). In recent years, the asymptotically time-periodic solutions of the KdV type equation attracted the attention of many authors.

First, Zhang [10] considered a KdV equation on the finite interval $(0,1)$:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_{xx} - \gamma u = 0, & 0 < x < 1,\ t > 0, \\ u(x,0) = 0, & 0 \le x \le 1, \\ u(0,t) = h(t),\ u(1,t) = 0,\ u_x(1,t) = 0, & t \ge 0. \end{cases} \tag{1.4}$$

Assuming either $\alpha > 0$ or $\gamma > 0$, Zhang showed that if the boundary forcing h is a periodic function of period τ with small amplitude, then the solution u of (1.4) is asymptotically time-periodic (of periodic τ), i.e.,

$$\lim_{t\to\infty} \|u(\cdot,t+\tau) - u(\cdot.t)\|_{L^2(0,1)} = 0.$$

Then, in [1], Bona, Sun and Zhang studied the KdV type equation posed in a quarter plane

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} - \alpha u_{xx} - \gamma u = 0, & x > 0, t > 0, \\ u(x,0) = 0,\ u(0,t) = h(t), & x \ge 0, t \ge 0. \end{cases} \tag{1.5}$$

They obtained that if $\gamma > 0$ and h is a periodic function of period τ with small amplitude, then the solution of (1.5) is asymptotically time-periodic satisfying

$$\|u(\cdot,t+\tau) - u(\cdot.t)\|_{L^2(\mathbb{R}^+)} \le Ce^{-\beta t} \text{ for any } t \ge 0, \tag{1.6}$$

where C and β are two positive constants.

Later, Usman and Zhang [8] considered an initial-boundary problem of the KdV equation without damping effect posed on the finite interval $(0,1)$, namely,

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & 0 < x < 1,\ t > 0, \\ u(0,t) = h(t),\ u(1,t) = 0,\ u_x(1,t) = 0, & t > 0, \\ u(x,0) = \phi(x), & 0 < x < 1. \end{cases} \tag{1.7}$$

They proved that if $h \in C_b^1(\mathbb{R}^+)$ is a periodic function of period τ, and if there exist $\beta > 0$ and $\delta > 0$ such that if $\|\phi\|_{L^2(0,1)} + \|h\|_{C^1(0,\tau)} \le \delta$, then the corresponding solution u of (1.7) satisfies (1.6), where $C > 0$ is a constant depending only on δ.

Motivated by these results, it is natural to ask: Does the solution of (1.2) have the similar property in some suitable space? Our main result in this paper is a positive answer to this question.

Theorem 1.1 (Main Theorem)**.** *Let $s \geq 0$ and $\theta \in \mathbb{R}$ be given. Assume that $h \in C(\mathbb{R}^+; H^s(\mathbb{T}))$ is a time-periodic function of period τ, $u_0 \in H^s(\mathbb{T})$ with $[u_0] = \theta$. Then there exist $\beta = \beta(s,\theta) > 0$, $\delta_1 = \delta_1(s,\theta) > 0$ and $\delta_2 = \delta_2(s,\theta) > 0$ such that if $\|u_0\|_s \leq \delta_1$ and $\|h\|_{C([0,\tau];H^s(\mathbb{T}))} < \delta_2$, the corresponding solution u of*

$$\begin{cases} u_t + u_{xxx} + uu_x + GG^*u = Gh, & x \in \mathbb{T}, t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{T} \end{cases} \tag{1.8}$$

satisfies

$$\|u(\cdot, t+\tau) - u(\cdot,t)\|_s \leq Ce^{-\beta t}, \text{ for any } t \geq 0$$

where $C > 0$ is a constant depending only on s, θ, δ_1 and δ_2.

Throughout this paper, we assume that $[u_0] = 0$. Then we can deduce that the solution u of (1.8) satisfies

$$[u] = [u_0] = 0.$$

For the case $[u_0] = \theta \neq 0$, let $v(x,t) = u(x,t) - \theta$. It is easily seen that v solves

$$\begin{cases} v_t + \theta v_x + v_{xxx} + vv_x + GG^*v = Gh, & x \in \mathbb{T},\ t \in \mathbb{R}^+, \\ v(x,0) = u_0(x) - \theta, & x \in \mathbb{T}. \end{cases} \tag{1.9}$$

The basic idea of the following proof in this case is similar to the case $[u_0] = 0$ with minor change.

The rest of this paper is outlined as follows: In Section 2, we investigate the linear system and provide some preliminary results in Bourgain spaces; Section 3 is devoted to the well-posedness of (1.8). The proof of our main result is given in Section 4.

2. **Preliminaries**

2.1. **The linear system.** In this subsection, we consider the system

$$\begin{cases} u_t + u_{xxx} + GG^*u = 0, & x \in \mathbb{T},\ t \in \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{2.1}$$

First, we introduce the space $H^s(\mathbb{T})$.

For any $s \geq 0$, $H^s(\mathbb{T})$ denotes the Sobolev space

$$H^s(\mathbb{T}) = \{u : \mathbb{T} \to \mathbb{R};\ \|u\|_s := \|(1-\partial_x^2)^{\frac{s}{2}} u\|_{L^2(\mathbb{T})} < \infty\}.$$

Its dual is denoted by $H^{-s}(\mathbb{T})$. Set

$$H_0^s(\mathbb{T}) = \{u \in H^s(\mathbb{T}) :\ [u] = 0\}$$

let A_G denote the operator

$$A_G w = -w''' - GG^* w$$

on the domain $\mathcal{D}(A_G) = H_0^3(\mathbb{T})$.

Clearly, A_G is a densely defined closed operator in $L_0^2(\mathbb{T}) = H_0^0(\mathbb{T})$. It is easy to deduce that

$$(A_G w, w)_{L^2(\mathbb{T})} = -\|G^* u\|_0^2 \leq 0 \qquad \forall\ w \in \mathcal{D}(A_G).$$

Similarly, for any $v \in \mathcal{D}(A_G^*)$, $(A_G^* v, v)_{L^2(\mathbb{T})} \leq 0$, where $A_G^* v = v''' - GG^* v$ and $\mathcal{D}(A_G^*) = H_0^3(\mathbb{T})$. This implies that both A_G and its adjoint A_G^* are dissipative. Thus the operator A_G generates a strongly continuous semigroup $\{S_G(t)\}_{t\in\mathbb{R}}$ on the space $L_0^2(\mathbb{T})$.

The following result is due to [4].

Proposition 2.1. *([4, Proposition 2.3]) Let $s \geq 0$ be given. There exists a number $\alpha > 0$ independent of s such that for any $u_0 \in H_0^s(\mathbb{T})$, the corresponding solution of (2.1) satisfies*

$$\|u(\cdot, t)\|_s = \|S_G(t)u_0\|_s \leq Ce^{-\alpha t}\|u_0\|_s$$

for any $t \geq 0$, where $C > 0$ is a constant depending only on s.

2.2. **The Bourgain spaces and their properties.** In this subsection, we introduce the Bourgain space which was introduced in [2] briefly.

For given $b, s \in \mathbb{R}$ and a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$, we define the norms

$$\|u\|_{X_{b,s}} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \langle k\rangle^{2s} \langle \xi - k^3\rangle^{2b} |\widehat{\widehat{u}}(k,\xi)|^2 d\xi \right)^{\frac{1}{2}},$$

$$\|u\|_{Y_{b,s}} = \left(\sum_{k\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \langle k\rangle^{s} \langle \xi - k^3\rangle^{b} |\widehat{\widehat{u}}(k,\xi)| d\xi \Big)^2 \right)^{\frac{1}{2}},$$

where $\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}$, and $\widehat{\widehat{u}}(k,\xi)$ denotes the Fourier transform of u with respect to the space variable x and the time variable t. The Bourgain space $X_{b,s}$ (resp. $Y_{b,s}$) associated to the KdV equation on $\mathbb{T}$ is the completion of the space $\mathcal{S}(\mathbb{T} \times \mathbb{R})$ under the norm $\|u\|_{X_{b,s}}$ (resp. $\|u\|_{Y_{b,s}}$).

For given $b, s \in \mathbb{R}$, let

$$Z_{b,s} = X_{b,s} \cap Y_{b-\frac{1}{2},s}$$

be endowed with the norm

$$\|u\|_{Z_{b,s}} = \|u\|_{X_{b,s}} + \|u\|_{Y_{b-\frac{1}{2},s}}.$$

For a given interval I, let $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) be the restriction space of $X_{b,s}$ to the interval I with the norm

$$\|u\|_{X_{b,s}(I)} = \inf\{\ \|\tilde{u}\|_{X_{b,s}} |\ \tilde{u} = u \text{ on } \mathbb{T} \times I\ \}$$

$$\Big(\text{ resp. } \|u\|_{Z_{b,s}(I)} = \inf\{ \ \|\tilde{u}\|_{Z_{b,s}} | \ \tilde{u} = u \text{ on } \mathbb{T} \times I \ \} \Big).$$

For simplicity, we denote $X_{b,s}(I)$ (resp. $Z_{b,s}(I)$) by $X_{b,s}^T$ (resp. $Z_{b,s}^T$) if $I = (0, T)$.

Now we state some lemmas which can be found in [4].

Lemma 2.2. *If $b_1 \leq b_2$ and $s_1 \leq s_2$, then the space X_{b_2,s_2} is continuously embedded in the space X_{b_1,s_1}.*

Lemma 2.3. *$Z_{\frac{1}{2},s}(I) \hookrightarrow C(\overline{I}; H^s(\mathbb{T}))$ for any $s \in \mathbb{R}$.*

Lemma 2.4. *let $s \geq 0$, $T > 0$ be given. Then there exists a constant $C > 0$ such that*

(1) For any $\phi \in H^s(\mathbb{T})$,

$$\|S_G(t)\phi\|_{Z^T_{\frac{1}{2},s}} \leq C\|\phi\|_s.$$

(2) For any $f \in Z^T_{-\frac{1}{2},s}$,

$$\left\| \int_0^t S_G(t-\xi) f(\xi) d\xi \right\|_{Z^T_{\frac{1}{2},s}} \leq C\|f\|_{Z^T_{-\frac{1}{2},s}}.$$

(3) For any $u, v \in Z^T_{\frac{1}{2},s}$, $[u] = [v] = 0$,

$$\left\| \int_0^t S_G(t-\xi)(uv)_x(\xi) d\xi \right\|_{Z^T_{\frac{1}{2},s}} \leq C\|u\|_{Z^T_{\frac{1}{2},s}} \|v\|_{Z^T_{\frac{1}{2},s}}.$$

3. Well-posedness of (1.8)

First, we need a proposition.

Proposition 3.1. *Assume that $h \in C(\mathbb{R}^+; H^s(\mathbb{T}))$ is a time-periodic function of period τ, then*

$$\left\| \int_0^t S_G(t-\sigma)(Gh)(\sigma) d\sigma \right\|_{Z^T_{\frac{1}{2},s}} \leq C\|h\|_{C([0,\tau];H^s(\mathbb{T}))},$$

here (and elsewhere) C is a generic positive constant that may vary from place to place.

Proof. According to Lemma 2.4,

$$\left\| \int_0^t S_G(t-\sigma)(Gh)(\sigma) d\sigma \right\|_{Z^T_{\frac{1}{2},s}} \leq C\|Gh\|_{Z^T_{-\frac{1}{2},s}}.$$

Let $\bar{h}$ be the zero extension of $h\chi_{[0,T]}$, where $\chi_{[0,T]}$ is the characteristic function of $[0, T]$.

Then by definition of the space $Z^T_{-\frac{1}{2},s}$,

$$\|Gh\|_{Z^T_{-\frac{1}{2},s}} \leq \|G\bar{h}\|_{Z_{-\frac{1}{2},s}} = \|G\bar{h}\|_{X_{-\frac{1}{2},s}} + \|G\bar{h}\|_{Y_{-1,s}}.$$

It follows from Lemma 2.2 and Hölder inequality that

$$\begin{aligned}
\|G\bar{h}\|_{X_{-\frac{1}{2},s}} &\leq C\|G\bar{h}\|_{X_{0,s}}, \\
\|G\bar{h}\|_{Y_{-1,s}} &= \left(\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{R}}\langle k\rangle^s \frac{1}{\langle \xi - k^3\rangle}|\widehat{\widehat{G\bar{h}}}(k,\xi)|d\xi\right)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{1+|\xi-k^3|^2}d\xi\int_{\mathbb{R}}\langle k\rangle^{2s}|\widehat{\widehat{G\bar{h}}}(k,\xi)|^2d\xi\right)^{\frac{1}{2}} \\
&\leq C\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\langle k\rangle^{2s}|\widehat{\widehat{G\bar{h}}}(k,\xi)|^2d\xi\right)^{\frac{1}{2}} \\
&= C\|G\bar{h}\|_{X_{0,s}}.
\end{aligned}$$

Now it is sufficient to estimate $\|G\bar{h}\|_{X_{0,s}}$.

Since it is not difficult to prove that G is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$, we have

$$\begin{aligned}
\|G\bar{h}\|_{X_{0,s}} &= \|G\bar{h}\|_{L^2(\mathbb{R};H^s(\mathbb{T}))} = \left(\int_0^T \|(Gh)(t)\|_s^2dt\right)^{\frac{1}{2}} \\
&\leq C\left(\int_0^T \|h(t)\|_s^2dt\right)^{\frac{1}{2}} \leq C\|h\|_{C([0,\tau];H^s(\mathbb{T}))}.
\end{aligned}$$

Thus, we obtain

$$\left\|\int_0^t S_G(t-\sigma)(Gh)(\sigma)d\sigma\right\|_{Z^T_{\frac{1}{2},s}} \leq C\|h\|_{C([0,\tau];H^s(\mathbb{T}))}.$$

□

Now we can get the well-posednees of (1.8).

Theorem 3.2. *Let $s \geq 0$ be given, $u_0 \in H^s_0(\mathbb{T})$, and let $h \in C(\mathbb{R}^+;H^s(\mathbb{T}))$ be a time-periodic function of period τ. Then there exist constants $\delta_1' > 0$ and $\delta_2' > 0$ such that if*

$$\|u_0\|_s \leq \delta_1' \quad \textit{and} \quad \|h\|_{C([0,\tau];H^s(\mathbb{T}))} \leq \delta_2',$$

the system (1.8) admits a unique solution $u \in Z^T_{\frac{1}{2},s} \cap C([0,T],L^2_0(\mathbb{T}))$ for any $T > 0$. Moreover, there exists a constant $C_0 > 0$ independent of δ_1' and δ_2' such *that*

$$\|u(\cdot,t)\|_s \le C_0\delta_1', \qquad \forall\ t>0.$$

Proof. First, we establish the existence and uniqueness of a solution $u \in Z^T_{\frac{1}{2},s}$, where $T>0$ will be determined later. Rewrite the system (1.8) in its integral form

$$u(t) = S_G(t)u_0 - \int_0^t S_G(t-\xi)(uu_x)(\xi)d\xi + \int_0^t S_G(t-\xi)(Gh)(\xi)d\xi.$$

Define the map

$$\Gamma(u)(t) = S_G(t)u_0 - \int_0^t S_G(t-\xi)(uu_x)(\xi)d\xi + \int_0^t S_G(t-\xi)(Gh)(\xi)d\xi.$$

Define the closed ball B_R in $Z^T_{\frac{1}{2},s} \cap C([0,T];L^2_0(\mathbb{T}))$:

$$B_R = \{u \in Z^T_{\frac{1}{2},s} \mid [u]=0, \|u\|_{Z^T_{\frac{1}{2},s}} \le R\},$$

where $R>0$ is a constant to be determined later.

According to Lemma 2.3, Lemma 2.4 and Proposition 2.1, we can find constants $C_1,\cdots,C_6$ such that

$$\begin{aligned}
\|\Gamma(u)\|_{Z^T_{\frac{1}{2},s}} &\le C_1\|u_0\|_s + C_2\|h\|_{C([0,\tau];H^s(\mathbb{T}))} + C_3\|u\|^2_{Z^T_{\frac{1}{2},s}} \\
&\le C_1\delta_1' + C_2\delta_2' + C_3R^2, \\
\|\Gamma(u_1)-\Gamma(u_2)\|_{Z^T_{\frac{1}{2},s}} &\le C_3(\|u_1\|_{Z^T_{\frac{1}{2},s}} + \|u_2\|_{Z^T_{\frac{1}{2},s}})\|u_1-u_2\|_{Z^T_{\frac{1}{2},s}} \\
&\le 2C_3R\|u_1-u_2\|_{Z^T_{\frac{1}{2},s}}, \\
\|\Gamma(u)(T)\|_s &\le C_4e^{-\alpha T}\|u_0\|_s + C_5\|h\|_{C([0,\tau];H^s(\mathbb{T}))} + C_6\|u\|^2_{Z^T_{\frac{1}{2},s}} \\
&\le C_4e^{-\alpha T}\delta_1' + C_5\delta_2' + C_6R^2
\end{aligned}$$

for any $u,u_1,u_2 \in B_R$, where C_4 is independent of T.

Pick $R = 2C_1\delta_1'$ and $T>0$ such that $2C_4e^{-\alpha T} \le 1$. Let

$$\delta_1' = \min\Big\{\frac{1}{12C_1C_3}, \frac{C_4e^{-\alpha T}}{8C_1^2C_6}\Big\}, \tag{3.1}$$

then, we have

$$2C_3R \le \frac{1}{3} \quad \text{and} \quad C_6R^2 \le \frac{1}{2}C_4e^{-\alpha T}\delta_1'.$$

Let

$$\delta_2' = \min\Big\{\frac{C_4e^{-\alpha T}\delta_1'}{2C_5}, \frac{2C_1\delta_1'}{3C_2}\Big\}, \tag{3.2}$$

then, we have

$$C_5\delta_2' \le \frac{1}{2}C_4e^{-\alpha T}\delta_1' \quad \text{and} \quad C_2\delta_2' \le \frac{1}{3}R.$$

Consequently, we can deduce that for any $u, u_1, u_2 \in B_R$,

$$\|\Gamma(u)\|_{Z^T_{\frac{1}{2},s}} \le R,$$
$$\|\Gamma(u_1) - \Gamma(u_2)\|_{Z^T_{\frac{1}{2},s}} \le \frac{1}{3}\|u_1 - u_2\|_{Z^T_{\frac{1}{2},s}},$$
$$\|\Gamma(u)(T)\|_s \le 2C_4 e^{-\alpha T}\delta_1' \le \delta_1'.$$

Therefore, Γ is a contraction in B_R. Its unique fixed point u is the desired solution of (1.8) in $Z^T_{\frac{1}{2},s} \cap C([0,T]; L^2_0(\mathbb{T}))$ which fulfills

$$\|u\|_{Z^T_{\frac{1}{2},s}} \le 2C_1\delta_1' \quad \text{and} \quad \|u(\cdot,T)\|_s \le \delta_1'.$$

Proceeding as above on the intervals $[T,2T],[2T,3T],\cdots$, we can obtain that (1.8) admits a solution u in $Z_{\frac{1}{2},s}(nT,(n+1)T) \cap C([nT,(n+1)T]; L^2_0(\mathbb{T}))$ and

$$\|u\|_{Z_{\frac{1}{2},s}(nT,(n+1)T)} \le 2C_1\delta_1', \quad \|u(\cdot,nT)\|_s \le \delta_1', \quad \forall\, n \in \mathbb{N}^+, \tag{3.3}$$

provided δ_1' and δ_2' are chosen according to (3.1) and (3.2).

For any $t \ge 0$, there exists an integer $k \in \mathbb{N}^+$ such that $kT \le t < (k+1)T$, it follows from Lemma 2.3 and (3.3) that

$$\|u(\cdot,t)\|_s \le \|u\|_{C([kT,(k+1)T];H^s(\mathbb{T}))} \le C_7\|u\|_{Z_{\frac{1}{2},s}(kT,(k+1)T)} \le 2C_1C_7\delta_1'.$$

This completes the proof of Theorem 3.2. □

Next, we give a proposition which will be used in the next section.

Proposition 3.3. *Let $s \ge 0$, $0 \le a_1 < a_2$ be given, $u_0 \in H^s_0(\mathbb{T})$, and let $h \in C(\mathbb{R}^+; H^s(\mathbb{T}))$ be a time-periodic function of period τ. For any $\varepsilon > 0$, there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that if*

$$\|u_0\|_s \le \delta_1 \quad \textit{and} \quad \|h\|_{C([0,\tau];H^s(\mathbb{T}))} \le \delta_2,$$

the solution u of (1.8) satisfies

$$\|u\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le \varepsilon,$$

where δ_1, δ_2 depend only on ε, s and $|a_2 - a_1|$.

Proof. Let us consider the map Γ_1,

$$\Gamma_1(u)(t) = S_G(t-a_1)u(\cdot,a_1) - \int_{a_1}^t S_G(t-\xi)(uu_x)(\xi)d\xi + \int_{a_1}^t S_G(t-\xi)(Gh)(\xi)d\xi.$$

It follows from Lemma 2.3 and Lemma 2.4 that

$$\|\Gamma_1(u)\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le C_8\|u(\cdot,a_1)\|_s + C_9\|h\|_{C([0,\tau];H^s(\mathbb{T}))} + C_{10}\|u\|^2_{Z_{\frac{1}{2},s}(a_1,a_2)},$$
$$\|\Gamma_1(u_1)-\Gamma_1(u_2)\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le C_{10}(\|u_1\|_{Z_{\frac{1}{2},s}(a_1,a_2)} + \|u_2\|_{Z_{\frac{1}{2},s}(a_1,a_2)})\|u_1-u_2\|_{Z_{\frac{1}{2},s}(a_1,a_2)},$$

where C_8, C_9 and C_{10} are positive constants depending only on s and $|a_2-a_1|$.

According to the proof of Theorem 3.2, for any $\delta_1 \le \delta_1'$, there exists a constant $\delta_2'(\delta_1) \le \delta_2'$ such that if $\delta_2 \le \delta_2'(\delta_1)$, we have

$$\|u(\cdot,t)\|_s \le C_0\delta_1, \quad \forall\ t>0.$$

Define the closed ball $\widetilde{B}_{R_1}$ in $Z_{\frac{1}{2},s}(a_1,a_2)\cap C([a_1,a_2];L^2_0(\mathbb{T}))$:

$$\widetilde{B}_{R_1} = \{u\in Z_{\frac{1}{2},s}(a_1,a_2) \mid [u]=0, \|u\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le R_1\},$$

where $R_1>0$ will be determined later.

Then for any $u,u_1,u_2\in \widetilde{B}_{R_1}$, if $\delta_1\le\delta_1'$ and $\delta_2\le\delta_2'(\delta_1)$,

$$\|\Gamma_1(u)\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le C_0C_8\delta_1 + C_9\delta_2 + C_{10}R_1^2,$$
$$\|\Gamma_1(u_1)-\Gamma_1(u_2)\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le 2C_{10}R_1\|u_1-u_2\|_{Z_{\frac{1}{2},s}(a_1,a_2)},$$

Assume that $R_1 = 2C_8C_0\delta_1$ and let

$$\delta_1 \le \frac{1}{12C_0C_8C_{10}} \quad \text{and} \quad \delta_2 \le \frac{2C_8C_0\delta_1}{3C_9}, \tag{3.4}$$

then we can obtain that

$$\|\Gamma_1(u)\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le R_1,$$
$$\|\Gamma_1(u_1)-\Gamma_1(u_2)\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le \frac{1}{3}\|u_1-u_2\|_{Z_{\frac{1}{2},s}(a_1,a_2)}.$$

Thus the map Γ_1 is a contraction on $\widetilde{B}_{R_1}$ provided δ_1 and δ_2 are chosen according to (3.4). Let

$$\delta_1 = \min\left\{\delta_1', \frac{1}{12C_0C_8C_{10}}, \frac{\varepsilon}{2C_0C_8}\right\},$$
$$\delta_2 = \min\left\{\delta_2'(\delta_1), \frac{2C_8C_0\delta_1}{3C_9}\right\}.$$

If $\|u_0\|_s\le\delta_1$ and $\|h\|_{C([0,\tau];H^s(\mathbb{T}))}\le\delta_2$, we have

$$\|u\|_{Z_{\frac{1}{2},s}(a_1,a_2)} \le R_1 = 2C_8C_0\delta_1 \le \varepsilon.$$

□

4. **Proof of Theorem 1.1**

For a given initial value $u_0 \in H_0^s(\mathbb{T})$, let $u(x,t)$ be the corresponding solution of (1.8) and $w(x,t) = u(x,t+\tau) - u(x,t)$. Then $w(x,t)$ solves the following the system

$$\begin{cases} w_t + w_{xxx} + (aw)_x + GG^*w = 0, & x \in \mathbb{T},\ t \in \mathbb{R}^+, \\ w(x,0) = w_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.1}$$

where $a(x,t) = \frac{1}{2}(u(x,t+\tau) + u(x,t))$ and $w_0(x) = u(x,\tau) - u_0(x)$.

We first check the well-posedness of the system (4.1).

Proposition 4.1. *Let $s \geq 0$, $T > 0$ be given, and there exists a constant $\mu_1 = \mu_1(s,T) > 0$ such that if $a \in Z^T_{\frac{1}{2},s}$, $[a] = 0$ and $\|a\|_{Z^T_{\frac{1}{2},s}} \leq \mu_1$, then there exists a unique solution $w \in Z^T_{\frac{1}{2},s} \cap C([0,T], L^2_0(\mathbb{T}))$. Moreover, there exists a constant C independent of a and w_0 such that*

$$\|w\|_{Z^T_{\frac{1}{2},s}} \leq C\|w_0\|_s.$$

Proof. The system (4.1) can be rewritten in an equivalent integral form

$$w(t) = S_G(t)w_0 - \int_0^t S_G(t-\xi)(aw)_x(\xi)d\xi. \tag{4.2}$$

We seek a solution w to (4.2) as a fixed point of the map

$$\Gamma_2(w)(t) = S_G(t)w_0 - \int_0^t S_G(t-\xi)(aw)_x(\xi)d\xi$$

in some closed ball B_{R_2} in the space $Z^T_{\frac{1}{2},s} \cap C([0,T]; L^2_0(\mathbb{T}))$. It is easy to deduce that for any $w, z \in B_{R_2}$, there exist constants C_{11}, C_{12} such that

$$\begin{aligned} \|\Gamma_2(w)\|_{Z^T_{\frac{1}{2},s}} &\leq C_{11}\|w_0\|_s + C_{12}\|a\|_{Z^T_{\frac{1}{2},s}}\|w\|_{Z^T_{\frac{1}{2},s}}, \\ \|\Gamma_2(w) - \Gamma_2(z)\|_{Z^T_{\frac{1}{2},s}} &\leq C_{12}\|a\|_{Z^T_{\frac{1}{2},s}}\|w - z\|_{Z^T_{\frac{1}{2},s}}. \end{aligned}$$

Choose $R_2 = 2C_{11}\|w_0\|_s$ and $C_{12}\|a\|_{Z^T_{\frac{1}{2},s}} \leq \frac{1}{2}$, then Γ_2 is a contraction in B_{R_2}. Furthermore, its fixed point w satisfies

$$\|w\|_{Z^T_{\frac{1}{2},s}} \leq R_2 = 2C_{11}\|w_0\|_s.$$

□

Lemma 4.2. *Let $s \geq 0$, and there exist $T_0 > 0$, $0 < \gamma < 1$ and $\mu_2 > 0$ such that if $\|a\|_{Z^{T_0}_{\frac{1}{2},s}} \leq \mu_2$, then the solution w of the system (4.1) satisfies*

$$\|w(\cdot, T_0)\|_s \leq \gamma\|w_0\|_s.$$

Proof. We proceed as in the proof of Proposition 4.1 ($T = T_0$) to obtain a solution w of (4.1) in $Z^{T_0}_{\frac{1}{2},s}$ provided $\|a\|_{Z^{T_0}_{\frac{1}{2},s}} \leq \mu_1(T_0)$, where $\mu_1(T_0)$ is μ_1 in Proposition 4.1 when $T = T_0$. Moreover, there exist constants C_{13}, C_{14} such that

$$\begin{aligned}\|w(\cdot,T_0)\|_s &\leq \|S_G(T_0)w_0\|_s + \left\|\int_0^{T_0} S_G(T_0-\xi)(aw)(\xi)d\xi\right\|_s \\ &\leq C_{13}e^{-\alpha T_0}\|w_0\|_s + C_{14}\|a\|_{Z^{T_0}_{\frac{1}{2},s}}\|w_0\|_s,\end{aligned}$$

here we have used Proposition 2.1, Lemma 2.3, Lemma 2.4 and Proposition 4.1.

Fix $T_0 > 0$ such that $0 < 2C_{13}e^{-\alpha T_0} = \gamma < 1$, and set

$$\|a\|_{Z^{T_0}_{\frac{1}{2},s}} \leq \mu_2 := \min\Big\{\frac{C_{13}C_{14}}{e}^{-\alpha T_0}, \mu_1(T_0)\Big\}.$$

We can obtain

$$\|w(\cdot,T_0)\|_s \leq \gamma\|w_0\|_s.$$

□

Now, we can prove our main result.

Proof of Theorem 1.1. For any $t \geq 0$, there exists an integer $k \in \mathbb{N}$ such that $kT_0 \leq t < (k+1)T_0$.

Proceeding as in the proof of Proposition 4.1 on the interval $[kT_0, (k+1)T_0]$, we can obtain that

$$\|w\|_{Z_{\frac{1}{2},s}(kT_0,(k+1)T_0)} \leq C(T_0)\|w(\cdot,kT_0)\|_s, \tag{4.3}$$

when $\|a\|_{Z_{\frac{1}{2},s}(kT_0,(k+1)T_0)} \leq \mu_1(T_0)$, where $C(T_0)$ and $\mu_1(T_0)$ are C and μ_1 in Proposition 4.1.

Then proceed as in the proof of Lemma 4.2 on $[0,T_0], [T_0,2T_0], \cdots, [(k-1)T_0, kT_0]$. We can deduce that for $\|a\|_{Z_{\frac{1}{2},s}(nT_0,(n+1)T_0)} \leq \mu_2,\ \forall\, n \in \mathbb{N}^+$,

$$\|w(\cdot,kT_0)\|_s \leq \gamma^k\|w_0\|_s. \tag{4.4}$$

Since $a = a(x,t) = \frac{1}{2}(u(x,t+\tau)+u(x,t))$, according to Proposition 3.3, there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ depending only on s, T_0 and $\min\{\mu_1(T_0), \mu_2\}$ such that if $\|u_0\|_s \leq \delta_1$ and $\|h\|_{C([0,\tau];H^s(\mathbb{T}))} < \delta_2$, we can obtain that

$$\|a\|_{Z_{\frac{1}{2},s}(nT_0,(n+1)T_0)} \leq \min\{\mu_1(T_0), \mu_2\},\ \forall\, n \in \mathbb{N}^+.$$

It follows from Lemma 2.3, (4.3) and (4.4) that

$$\begin{aligned}\|w(\cdot,t)\|_s &\leq \|w\|_{C([kT_0,(k+1)T_0];H^s(\mathbb{T}))} \leq C\|w\|_{Z_{\frac{1}{2},s}(kT_0,(k+1)T_0)} \\ &\leq C\|w(\cdot,kT_0)\|_s \leq C\gamma^k\|w_0\|_s \leq C\gamma^{\frac{t}{T_0}-1}\|w_0\|_s \\ &\leq \frac{C}{\gamma}e^{\frac{t}{T_0}\ln\gamma}\|w_0\|_s = Ce^{-\beta t}\|w_0\|_s \\ &\leq Ce^{-\beta t}.\end{aligned}$$

This ends the proof of Theorem 1.1. □

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